INSTITUTE OF INFORMATICS AND AUTOMATION PROBLEMS

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THE DEVELOPMENT OF THE METHODS AND ALGORITHMS FOR THE DESIGN OF OPTIMAL SCHEMES FOR INFORMATION DISSEMINATION IN COMPUTER NETWORKS

PhD Thesis

For obtaining a candidate degree in technical sciences in specialty 05.13.05 “Mathematical modeling, numerical methods and program complexes”

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Yerevan 2018

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Introduction

Actuality of the research

During the last decades, the application of medium and large-scale network computing systems (clusters, grids) has become widespread in different areas (research, economics, social, etc.). Parallel to their dissemination, the construction of optimal network structures has become actual among the network nodes to ensure fast information exchange. For their investigation, new approaches have been developed based on discrete mathematics, the graph theory, parallel programming and the probability theory.

The previous century was revolutionary in the development of communication facilities ([1]). We become able to ensure an efficient communication at very long distances, which in turn had its crucial impact on the way we live currently. The development of reliable and high-performance large communication networks, such as the telephone networks and the Internet, were the driving power of this revolution. The main goal when constructing these network was the need of providing efficient communication using the given available resources. Apart from these networks, we can observe in recent decades a big interest in different types of large networks, such as sensor networks, peer-to-peer (P2P) networks, mobile ad-hoc networks, and social networks, where the main goal is not that of providing efficient communication between any two components.

As a good example of such networks can serve a sensor network, which consists of a large number of unreliable, low cost, wireless sensors and, generally, is used for the purpose of 'detecting' and 'monitoring' certain events. For instance, smoke sensors with wireless transmission support deployed in a large building, or a set of interconnected wireless sensors deployed for surveillance in a secure facility.

The peer-to-peer networks made by connecting big number of participants of different nature (e.g., computers or handheld devices) over an already existing network such as the Internet, are another good example of current generation networks. Usually such networks require minimal infrastructural support, since they operate over already existing networks and, as a result, a significant benefit arises in cases like information dissemination. For example, usually Internet content provider needs to maintain servers with high bandwidth streaming support to be able to broadcast a popular TV show to a multiple users simultaneously in case of there is no supporting P2P network. Contrary, in the presence of a P2P network there is a big possibility that a user will obtain popular content from the nearby peer. Thus, P2P based architectures for content dissemination tend to reduce the cost for a content provider (e.g., see [2], [3]).

The research of possible applications of mobile ad-hoc network gain big popularity recently. These researches include such topics as smart cars traveling on road without human interaction, or hundreds of unmanned aerial vehicles deployed for various observations. The networks of this kind need algorithms for the purpose of coordination or consensus of the operation of their various components (e.g., see [4], [5]).

Finally, with the recent activity in the development of “social network applications”, it is obvious the appearance of massive social networks between individuals connected over a networks of various types. These online applications increased the number of connections the social network of an individual could possibly include drastically. Furthermore, the daily increasing usage of handheld devices like smartphones or smartwatches brings the possibility to construct such social networks on top of P2P networks formed by these devices in the near future (e.g., see [6], [7]).

The arrival of sensor, wireless ad-hoc and peer-to-peer networks made the design of asynchronous, distributed and fault-tolerant computation and information exchange algorithms a necessity. This is mainly because such networks are constrained by the following operational characteristics:

- usually they do not have a centralized entity for facilitating computation, communication and time synchronization

- typically the nodes of the network have only “local” knowledge about the network topology
- nodes may join or leave the network (even expire), so that the network topology itself is dynamic
- in the case of sensor networks, there is a limit on the the computational power and energy resources.

These constraints motivate the design of simple asynchronous decentralized algorithms for computation where each node exchanges information with only a few of its immediate neighbors in a time instance (or, a round). The goal in this setting is to design algorithms so that the desired computation and communication is done as quickly and efficiently as possible.

Our study of distributed asynchronous algorithms, also known as gossip algorithms, is motivated by applications to sensor, peer-to-peer and ad hoc networks for computation and information exchange in an arbitrarily connected network of nodes. Usually, in such network nodes operate under limited computational, communication and energy resources. These constraints naturally give rise to "gossip" algorithms: schemes which distribute the computational burden and in which a node communicates with only some subset of the remaining nodes and there is no any single point of coordination.

One interesting observation is that unlike the telephone network or the Internet, the main purpose of the many of the next generation networks is not providing an efficient communication between various networked entities. The above stated can be confirmed by, for instance, sensor networks, peer-to-peer networks, mobile networks of vehicles and social networks. Indeed, these networks of the new generation do require algorithms for communication, computation, or merely spreading information. For example, a sensor network do require an estimation algorithm for event detection given the sensor observations, a P2P network may require a dissemination algorithm using peer information, a network of aerial vehicles may need an algorithm to reach consensus to co-ordinate their surveillance efforts, or viral advertising in a social network. Usually this kind of networks lack any defined infrastructure; they show unpredictable dynamics and face resource constraints in the environments that they operate in. Therefore, algorithms operating within them need to be really simple, distributed, robust against networks dynamics, with good resource utilization and without any single point of failure. Gossip/broadcast algorithms, as the name suggests, are built as an unreliable, asynchronous information exchange protocol. Taking into consideration their immense simplicity and wide applicability, this class of algorithms has become a favorable choice as a canonical architectural solution for the next generation networks.

Now let us concentrate on the definition of gossip protocol and bring here the most widely accepted definition ([8]):

*"A **gossip protocol** is a style of computer-to-computer communication protocol inspired by the form of gossip seen in social networks. Modern distributed systems often use gossip protocols to solve problems that might be difficult to solve in other ways, either because the underlying network has an inconvenient structure, is extremely large, or because gossip solutions are the most efficient ones available.*

*The term **epidemic protocol** is sometimes used as a synonym for a gossip protocol, because gossip spreads information in a manner similar to the spread of a virus in a biological community."*

The concept of gossip communication can be shown by making parallels with the office situation where workers spreading rumors. Suppose each hour the random pair of office employees meet near the coffee machine to have a quick chat (along with some coffee or tea). They share a latest rumor for that day, say Emily tells Mike she believes a new general manager of the company will be Tom. At the next meeting Mike tells about this to John, while

Emily repeats the same thing to Arthur. After each coffee-break the number of informed employees roughly double. But there also can be cases when the rumor repeats, say Emily tells he same thing to Dave, who already have heard about that. This kind of communication protocol was implemented in various computer systems, where each machine or peer picks another one at random with given frequency and shares any hot rumors known to him at that moment.

One of the benefits of such a protocol is its robustness - even if Arthur fails to understand Emily, there is a big chance that he'll hear the same thing from another colleague. In more technical words, the gossip protocol satisfies the following conditions:

- it is based on periodic, peer-to-peer inter-process communications
- the size of the exchanged information during the communications is bounded
- there might be faults in communication process
- there is some randomness in the peer selection process. Peers might be selected also from small set of neighbors (nodes that are close to the given node)
- after interaction of the two agents they both come to some equivalent state
- there is some implicit redundancy of the delivered information based on the replication

There are several types of implementation/application of gossip protocol, some of which are:

- *Dissemination protocols*. They use gossip to spread information, as an example can be considered Background data dissemination protocols, that continuously gossip about information associated with the nodes. In these applications typically propagation latency does not have much importance, this is because of rare state changes or not critical requirements to operate always on most uptodate data
- *Anty-entropy protocols*. For repairing replicated data, which operate by comparing replicas and fixing differences.
- *Aggregation protocols*. These compute the aggregate value for the whole network by sampling the values of some measurement nodes are making (like sensed data in case

of sensors) and result in some value that describes the whole network (which can be composed of multiple thousands nodes). The key requirement here is that the aggregate must be computable by bounded pairwise information exchanges.

Let us assume that we want to find the object that most closely matches given search pattern, where the network is of unknown size, but where the machines are connected to each other and where each machine is running a small *agent* program that implements a gossip protocol.

- A user would trigger a search by making the local agent starting gossipping about the search keyword. (The assumption is that agents either have some knowledge about their local environment, i.e. list of neighbour peers, or retrieve this information from some kind of a shared store.)
- Periodically, at some rate (let's say ten times per second, for simplicity), each agent program picks some other agent at random, and initiates a gossip with it. Search strings known to A will now also be known to B, and vice versa. In the next "round" of gossip A and B will pick additional random peers, maybe C and D. This approach of round-by-round doubling makes the protocol very robust, even if some communications fail or some of the selected peers already know the given search pattern.
- Each agent only can check its local machine for the given search pattern, immediately upon receival of search pattern.
- Agents also gossip about the best match, to date. Thus, if A gossips with B, after the interaction, A will know of the best matches known to B, and vice versa. This way the best matches will "spread" through the network.

The messages might get large by time(for example, if many searches are active all at the same time), that is why a size limit should be introduced. Also, searches should "age out" of the network.

It follows that with high probability within logarithmic time in the size of the network (the number of nodes/agents), any new search string will have reached all the agents. Within almost the same approximate length of an additional delay, every agent will learn about the best matches that was found among all the nodes. In particular, the agent that started the search will have found the best match.

For example, in a network with 25,000 machines, we can find the best match after about 30 rounds of gossip: 15 to spread the search string and 15 more to discover the best match. Assuming that a gossip exchange could occur as often as once every tenth of a second, hence this form of network search could search a big data center in about 3 seconds.

In this scenario, searches usually have age and might age out of the network after, say, 10 seconds from being initiated. By then, the initiator almost certainly knows the answer and there is no point to continue the further gossip about that keyword.

Gossip protocols have actually been used to achieve various goals, such like for achieving and maintaining distributed database consistency, counting the number of nodes in a network of unknown size, robustly spreading news through the network, organizing nodes according to some structuring policy, building so-called overlay networks, computing aggregates, sorting the nodes in a network, electing leaders, etc.

There are also some real world software implementation of the gossip protocol to discover nodes in peer-to-peer networks, some of which are Apache Gossip ([9]), gossip-python ([10]) and Smudge ([11]).

The above described is a purely random peer-selection scheme for gossip: when an agent A decides to run a gossip round, it picks some peer B uniformly and at random within the whole network (or launches a message on a random walk that will terminate at a random agent). But usually, in the design of the gossip algorithms there is a concept of “neighbour” and the agents choose these neighbours for interaction rather then random node in a network that can be far away (in terms of network delay). These *biased* gossip protocols need to ensure a sufficient degree of connectivity to avoid the risk of complete disconnection of one side of a network from the other, but if care is taken, can be faster and more efficient than protocols that are purely random.

As already was stated one of the main challenges of efficient operation of today's typical computer networks (internet, cloud, cluster, grid) is the organization of fault-tolerant and efficient transfer of the information/data among the entities forming the network. In this context, the scientific work done in the sphere of stochastic and quasi stochastic information exchange/dissemination processes have crucial importance.

On the other hand, one of the most important problems in discrete mathematics and informatics is the construction and investigation of the behaviour of the models of distributed computing systems. The crucial challenge in the construction of multicomputer systems with multiprocessors (cloud, cluster, grid) is the construction of optimal topology/schema that will also guarantee the required level of fault-tolerance for the system. One of the common choices as a network topology to achieve this goal are Knodel graphs ([12]) or the schemes derived from them by various modifications.

In order to obtain optimal values of the main characteristics (execution time, number of channels, number of calls, etc.) of the full information exchange network it is common and actual the research of the deterministic information exchange processes that are being executed on the top of such networks. The investigation of the characteristics and attributes of constructed networks, the construction of corresponding algorithms and theorems will also have positive impact on the migration to models described by stochastic processes and their further investigation.

The information exchange problems are formally divided into 2 categories - full duplex (gossip) and half-duplex (broadcast). The initial problem definition that was proposed is the minimal telephone call problem that was solved in a different ways by Tijdeman [13], Baker and Shostak [14] and the number of minimum required calls is equal to 2n-4, n>3 (n is the number of participants). However, for the practical applications this definition was modified, new properties were added and some of the proposed categories of problem definitions are

- what is the minimal execution time (number of rounds) of complete gossip/broadcast graphs, assuming that the number of calls is the minimal
- what is the minimal number of lines of complete gossip/broadcast graphs, assuming that the gossiping time is the minimal

The first problem was addressed by Nieminen ([15]), but his results were partially refuted by Labahn ([16]). He formulated the exact results for the minimum time - $T = 2\lceil\log_2 n\rceil - 3$, refuted the claim about uniqueness of such graphs and constructed the example of such a graphs based on the NOHO graphs described by West ([17]). The research that has been done and some obtained results give us a base to not only extend, but also describe the class of all possible solutions regarding this problem.

The second category of problems are generally considered to be NP hard and was solved only for some particular definitions.

The broadcasting/gossiping type of network communication also often occur in distributed computing, e.g., in global processor synchronization and updating distributed databases. There is also an implicit use of such communication tasks in many parallel computing problems, where the source data is distributed among various processors (in matrix multiplication, parallel solving of linear systems, or parallel sorting).

In the phase of practical applications of the obtained optimal gossip/broadcast schemes a new challenge has occurred: the guarantee of the required level of fault-tolerance of the proposed schemas.

There is a strict distinction between the bounded and probabilistic models of communication in fault-tolerant gossiping/broadcasting. This distinction enforce differences in algorithm design and final goals. This domain of fault-tolerant broadcasting/gossiping is very large area of research and can vary depending on communication and fault models.

There can be various communication modes here. But all of them consists of a call, that takes place between two adjacent nodes of the network and lasts 1 unit of time. The communication mode specifies which calls can take place simultaneously during 1 unit of time and the amount of messages that can be transferred in 1 call. There are two widely studied modes of communication in such a networks - shouting and whispering. In the first mode each node can call all its neighbours during a single unit of time, while in the second mode the communication is limited by only one neighbour at a time. There is also a full-duplex (gossip) and half-duplex (broadcast) notations that appear in literature regarding the communication modes. The first one assumes that the communication link between nodes is bi-directional and messages can traverse in both direction, while the second mode assumes that node can only send or receive an information in a given call. Full-duplex mode corresponds to telephone conversations, while half-duplex is used in messaging. In the current study mainly focus on the bounded, full-duplex, whispering communication modes of fault-tolerant gossiping.

There is also a notion of the packet size in this communication, that has significant impact on the performance of underlying algorithms. This is the amount of the information that can be transmitted between nodes (in one direction) in one call. This parameter can noticeably change the performance of gossiping algorithms, since the large packet values allow us to transmit all the accumulated info in a single call. The typical size of the packets can vary from the single unit that is equal to the initial information of the node, to a potentially unbounded value that can contain all the informations that exist in the network. Since in practice we usually can achieve large bandwidth (especially optical networks) the potentially unbounded packet size is considered in our study.

Faults can also be classified into different categories which impacts the communication protocol drastically. There can be several failure types in communication - only links, only nodes or both links and nodes. Also the nature of faults can vary, it can be crash or Byzantine faults. The first one assumes that the faulty component does not take part in the communication process at all, which means the information can be lost but not changed.

This is not the case of Byzantine faults, where in the worst-case scenario the faulty components can have arbitrary behaviour, sometimes even maliciously.

Another important characteristic of faults is their duration - permanent (the status of component is not changed) or transient (the status of the communication component may change in arbitrary time). The first one is typically associated with the hardware failures while the second can correspond to communication failures with temporary reasons.

In our study we consider the only links, crash and permanent fault models.

As noted above the crucial role here plays the distribution of faults - is it probabilistic or bounded. In a bounded model we have an upper limit k on the number of faults that can possibly occur in the system, the location of faults is usually not being specified - which means the worst possible option is assumed. In the probabilistic models the probability is given for the failure of each communication component (node or link). In the bounded gossiping mode all the fault-free nodes should exchange their information, provided that there are at most k faulty components. In the probabilistic model the communication is said to be only almost safe (since it does not make sense to speak 100% fault tolerant communications here) with the given probability.

We study only bounded model of fault distribution, though recent surveys show the increasing interest in the probabilistic model.

The one last thing to say about the fault tolerant communication models is about the adaptiveness of the algorithm/model. The broadcasting/gossiping algorithms or models of communication can either be non-adoptive, where the communication flow (or calls between nodes) is scheduled in advance and does not change during the execution and adaptive, where every node has a local knowledge of the outcome of the attempts of communication it has made so far (faulty/successful) and thus it can choose to execute different calls based on this knowledge. Since the local memory and computational power of each processor have been increased, the adaptive algorithms become more realistic to implement. But in the current study we focus mainly on non-adaptive gossiping models.

Considering the emerging need of fault-tolerant schemas the new classes of open problems were proposed: construct schemas that will be able to perform k-fault-tolerant gossipping and satisfy the following conditions

- have minimal possible number of calls
- have minimal possible time
- complete in minimal possible time (number of rounds), and have minimal possible number of calls under mentioned condition
- have minimal possible number of calls, and complete in a minimal possible time (number of rounds) under that condition

It is also being considered to extend this classes with the additional condition of minimal possible number of lines.

In this work the main focus is on constructing and presenting gossip schemes with optimal gossipping properties (number of rounds) as well as constructing a new fault-tolerant gossip scheme that has improved the known upper bound on the minimum number of calls. It is also introduced in this work the operation of "local interchange" on the gossip graphs. The application of this operation on the gossip graph tends to change the structure/topology of a graph to obtain a new graph (not isomorphic to initial one) that has the same gossipping properties as the initial one. We show multiple application cases of this operation to obtain various results.

On the other hand, we have developed a software tool called "Graph Plotter" the main intention of which was to provide a researchers of this topic with possibility to construct graphs and verify their hypothesis before going into details of theoretical analysis. This tool has multiple working modes and can be useful on analysis of simple gossip graphs, fault-tolerant gossipping, as well as Messy gossip models.

The chapter 1 of this work discusses various gossipping models, definitions and formulations of the entities we are going to deal with throughout the work, known results in the area of gossip and fault-tolerant gossip problems, the comparisonal analysis of these results, the problems that are still open here. This chapter tends to give to reader a brief overview of the research that has been done in this sphere by various authors, as well as introduce to the main concepts and properties that are of big interest in this type of problems. The comparative analysis of the known and obtained results should give a reader enough level of understanding about the values of the theoretical results that was obtained by us with regards to the various types of problems in the area of gossip networks.

In chapter 2 we will be focusing on the main theoretical results obtained during our research and their analytical justifications. This chapter will be denoted to the "method of local interchange" that was defined as a specific case of [13] and was given more general interpretation by us in [18] and to the applications of this operation. This chapter also discusses the fault-tolerant gossipping and our input on this specific problem area - several new constructions had been suggested that tend to improve some of the most definitive properties of these kind of problems.

In the last chapter our main focus will be around the software package that was developed by us using the visualization libraries offered by Wolfram Mathematica and C++. The main purpose of this tool is to provide convenient way for researchers to be able to investigate gossip problem, verify their hypothesis as well as to provide visualization tools for these problems (for the sake of better presentation and visual understanding purposes). Besides from the convenient visualization tools, it has a variety of working modes for different specifications of the general gossip problem (such as NOHO, NODUP, Fault-tolerant, Messy) as well as support for generating Knodel graphs with the highlitening of all the folded paths.

The aim of the research

The aim of this research is to provide network structures for gossip communication that will have optimal or near optimal values for the main properties of gossiping process (number of calls, number of channels and gossiping time). For this purpose it was proposed and solved the following problems:

- construct a method with the help of which it will be possible to obtain new gossiping topologies – not isomorphic to the known ones, that will be equivalent in terms of the main properties of the gossiping process (number of calls and gossiping time) to the known models and would be more suitable for the given requirements
- construct the fault-tolerant gossip schemes satisfying the proposed level of fault-tolerance that will insure full information exchange between the nodes of the system with the minimum known number of calls and communication rounds
- develop a software package, that will allow to model any gossip scheme and by using them as a source, obtain new topologies, that will be more optimal for satisfying the proposed structural requirements

The practical significance of the research

The results obtained in this research, particularly, minimum gossip graphs are applicable as an underlying communication topology for the distributed systems of large scale and can provide optimal communication between the various components of such systems. From this perspective it is also worth mentioning fault-tolerant gossip schemas that can be applicable in a multicomponent decentralized systems operating in a non-stable environment.

On the other hand the designed Graph Plotter software package has a practical value in terms of the modeling of the above mentioned schemas. It allows us to construct such networks (for small number of vertices), make some structural changes to them, verify the level of fault-tolerance, etc.

The scientific novelty of the research

1 The new construction method of the minimum gossip graphs for some subsets of the number of vertices of the system was obtained.

2 New method of the proof of minimum possible number of the calls for performing complete gossiping among n participants was obtained, using the method of “local interchange”.
3 New method of construction of the NOHO gossiping schemes that allow to perform complete information exchange between its nodes (gossiping) in the minimum possible time.
4 New method of the construction of k-fault-tolerant gossip graphs based on the Wheel graphs.
5 It was proofed the resistance of the gossiping properties of the Knodel graph against the cyclic permutation of the weights of its edges.
6 New hypothesis about the upper bound value of the minimum possible number of calls of a k-fault-tolerant gossiping scheme and the minimum possible k-fault-tolerant gossiping time were proposed and verified experimentally.

The practical application of the results

The obtained minimum gossip schemes were applied in the new software project of the “Effortis” Inc. The project is a multi-agent distributed system designed for storing and serving data in a P2P networks. It was implemented based on Apache Gossip open source library and as a communication topology between various nodes forming the P2P network was chosen the MGG proposed by us. As a result, the application is able to accomplish the synchronization of an internal state of the network within optimal time and minimum number of remote procedure calls.

The main results proposed for defense

1 New method of construction of the minimum gossip graphs for some subsets of the number of nodes of the system.

2 New NOHO gossip schemes, that allow full information exchange in minimum possible time.
3 New fault-tolerant gossip schemes based on Wheel graphs with near-optimal number of calls for the relatively small level of required fault-tolerance, as well as new method of construction of optimal fault-tolerant gossip graphs based on Knodel graphs.
4 Software package that allows to model any gossip scheme and obtain new topologies with better structural correspondence with the given requirements and equivalent gossiping properties, as well as to verify experimentally the gossiping properties of the constructed schemes and the level of fault-tolerance

The reports of the results of research

- International conference: "Computer Science and Information Technology'' (CSIT-2013), Yerevan, Armenia, 2013 թ. (2 reports)
- International conference: "Computer Science and Information Technology'' (CSIT-2015), Yerevan, Armenia, 2015 թ.
- International conference: "Computer Science and Information Technology'' (CSIT-2017), Yerevan, Armenia, 2017 թ.

Publications

The main results of the research were published in 8 scientific articles: [18], [19], [20], [21], [22], [23], [24], [25].

The structure and volume of the research

This research includes an introduction, 3 chapters, conclusion, and the list of literature with 85 references. The total size of this work is 107 pages, which includes 26 figures.

Specification and main issues related to information dissemination process in various networks, that can be modeled as gossip process

1.1 The comparison analysis of the known results about gossip schemes

Gossiping is one of the basic problems of information dissemination in communication networks (along with broadcasting).

In 1950, Bavelas [26] studied various communication patterns that can help small people groups solve common tasks. Typically he was studying the following task: each of the persons was given five playing cards; these cards should remain with their initial owners, but the owners can communicate with each other following the rules of given communication pattern by written messages; the task was considered completed when each of the players selects one of their cards in a way that the selected cards make the highest-ranking poker hand that can be possibly made by selecting a card from each player. He considered such metrics as the number of messages and the required time to complete the task and showed that for the communication pattern (that were under his consideration) among n people, 2(n-1) messages were required for this task. Also he showed that, in case each message takes a unit of time and any communication pattern is allowed, then the required time to complete the task is no more than $\lceil \log_2 n \rceil$ time units.

Shimbel (in 1951, [27]) proposed the following problem in group communication: each of the *n* members of the group initially knowns to whom they can communicate with, but has no any knowledge about the same for other members. The goal of the communication pattern was for the whole group to learn “the complete structure of their communication system”. Shimbel raised several questions: what are the communication patterns that allow to solve the problem? What is the connection between the number of channels and the time required?; etc.

Landau (in 1954, [28]) studied task-oriented groups in which the initial information of each individual has to be transmitted to all the others for the task to be completed. Each individual also sends all the information that has been known to him during the communication to one another individual. He considered this typical task: "each individual is given a set of colored marbles, only one color being common to all the group. The members of the group must exchange messages about their own colors and what they have learned about the colors of the others, until finally everybody knows the common color. Messages are sent only after everyone has indicated readiness to transmit; the transmissions then take place simultaneously, each individual sending to just one of their possible recipients. Each member of the group knows initially to whom they can send messages, but has no knowledge to whom others can send". The major condition put on communication pattern by Landau was that each individual picks a recipient at random from the list of permitted recipients with equal probability. He has determined the time required for completion of such a task for various communication patterns among 3 or 4 people.

The gossip problem (also known as a telephone problem) is attributed to A. Boyd (see [29] for review), although to the best knowledge of the reviewers, it was first formulated by R. Chesters and S. Silverman (Univ. of Witwatersrand, unpublished, 1970). The simple definition is as follows:

"Consider a set of n persons (nodes) each of which initially knows some unique piece of information that is unknown to the others, and they can make a sequence of telephone calls to spread the information. During a call between the given two nodes, they exchange the whole information known to them at that moment. The problem is to find a sequence of calls with minimum length (minimal gossip scheme), by which all the nodes will know all pieces of a information (complete gossiping)".

We are now going to review the history of the Gossip Problem, the papers in which the minimum number of calls in a complete gossip graphs with *n* nodes (persons) were determined. Then we will focus on some known variations of the gossip problem and known results here. We will conclude this section with complete overview of fault-tolerant gossipping problem and corresponding results.

Let f(n) be the minimum number of calls that is required to complete gossipping among n people in a classical gossiping model. It has been shown in numerous works ([14], [30], [29], [13]) that

$$f(1) = 0; f(2) = 1; f(3) = 3,$$
$$f(n) = 2n - 4, \ for\ n >= 4$$

Each of the papers above provided varying approach to prove the above result, some of which we will discuss below.

In the proof provided by Tijdeman ([13]) the following 2 statements were used to determine the f(n):

1. *If two people in a complete calling sequence **L** are interchanged at a point in time when they have precisely the same information, the new calling sequence **L'** is also complete.*
2. *If in a sequence of calls **L**, persons **A** and **B** were identified (merged) in order for form a new calling sequence **L'**, then at any point in L' the person AB knows at least all the information known to **A** and **B** at the corresponding point in **L**, which is also true about the amount of information known to other persons in **L'** and corresponding person in **L**.*

Apart from these statements in Tijdemans' work it was proved that the reverse of complete gossiping calling sequence is also complete.

The proof presented by Baker and Shostak ([14]) was very short, but this paper was the first one that considered also the amount of duplication of the information that is being transmitted during calling sequence. They introduced the idea that the calls can be made in a way that no one hears his own (NOHO) information from another member.

Additionally, Tijdeman and Hajnal, Milner and Szemeredi ([29]) were the first who noticed that f(n) satisfies the following recursive relation:

$$f(n+1) \leq f(n) + 2, \; for\, n \geq 4.$$

This was conducted based on the obvious observation that in case we have a new member *g* join our initial group of n persons. If this member will call to one of the remaining members first, then this additional information will be transmitted in f(n) calls among the *n* initial gossips. In the end we will need one additional call for *g* to obtain all the information.

Since then many variations of gossip problem have been introduced and investigated (see e.g. [31], [32], [16], [26], [33], [34], [35], [36], [37]]).

One of the variations of the Gossip Problem that was studied was formulated by restricting the communication pattern between people. This restriction was considered in the studies of Harary and Schwenk [38] and Golumbic [39]. They considered a situation when an individual can only possibly call to some subset of all the persons taking part in communication. According to the original Gossip Problem definition there were no any restriction on possible calls between individuals (i.e. underlying communication graph is complete), which is not the case here. By placing such a restriction on communication graph the following result was obtained:

If the gossip graph among a set of ***n*** *people is a tree, then f*$(n) = 2n - 3$*, for* $n \geq 2$.

The following important results were obtained in the mentioned papers (each of them):

For any connected gossip graph with n vertices, $2n - 4 \leq f(n) \leq 2n - 3, for\, n \geq 4$.

For any connected gossip graph with n vertices which contains a cycle of length 4,

$$f(n) = 2n - 4, for\, n \geq 4.$$

So, based on the above statements the conjecture was made in both papers that if the graph (connected) is not containing a 4-cycle then f(n)=2n-3. This was proved by Bumby [30] later, which means

*For any connected gossip graph **G** with **n** vertices,* $f(n) = 2n - 4$ *only if **G** contains a cycle of the length 4.*

Another known variation of the Gossip Problem that has been of interest for researchers is the gossipping in Grid graphs. These kind of graphs are very popular and have variety of applications, such as to model network of the streets of the city, telephone networks, matrix manipulations and parallel computing architectures. Thus the challenge of transmitting the information between nodes of the grid was the topic of a lot of studies ([40], [41]). The following 3 metrics were particularly of interest in these studies:

- $T(G)$: the minimum time required to complete full-duplex gossiping in a graph G;
- $M(G)$: the minimum number of calls required to complete gossiping in a graph G;
- $N(G)$: the minimum number of calls necessary to complete gossiping in the minimum amount of time in a graph G.

The following results regarding these functions were obtained for any grid $G_{m,n}$:

- $T(G_{m,n})$ = the diameter of $G_{m,n}$
- $N(G_{m,n}) = 2\text{mn} - 4$, in case m and n are even
- $M(G_{m,n}) = 2\text{mn} - 4$, follows from [38]

Some other assumption that results to another class of generalization of the Gossip Problem is that the information is transmitted by conference calls that include exactly *k* vertices. This means that on each round there are *k* individuals involved in a call and after that call all the information between them is exchanged. In such case as an underlying communication graph was considered a k-uniform hypergraph with *n* vertices. In this circumstances the following questions naturally become of interest:

- what is the minimum number of calls (denoted f(n,k)) required to complete gossipping in a *k*-uniform hypergraph with *n* vertices?
- what are the corresponding connected hypergraphs that allow gossipping in f(n,k) calls?
- how many calls (f(H)) are required to perform complete gossipping in an arbitrary connected k-uniform hypergraph?

The first question was answered by multiple authors ([42], [43], [44]) and the following values for f(n,k) were proposed:

$$f(n,k) = \left\lceil \frac{n-k}{k-1} \right\rceil + \lceil n/k \rceil, 1 \leq n \leq k^2$$

$$f(n,k) = 2\left\lceil \frac{n-k}{k-1} \right\rceil, n = k^2$$

The 3rd question was partially answered by Liestman in 1984 ([45]). The following was conducted for f(H):

$$f(n,k) \leq f(H) \leq 2n - 2k + 1.$$

They also considered 2nd question and give the solution/class of solutions for it (depending on the result of f(n,k) mod 2).

Another topic for the research is gossiping in directed graphs. It was considered by Harary and Schwenk (in [38]), i.e. in this model the communication is one-way (half-duplex) and thus the communication graph is directed. The following observation was made:

*For any **strongly connected** communication graph with **n** vertices,* $f(n) = 2n - 2$.

The topic was also of interest for Golumbic ([39]), but he was mainly concentrated on weakly connected digraphs with *k* strong components.

Some new variations of initial Gossip Problem definition was obtained (and also was considered by us in our works [18]) by placing restriction on the amount of repeating information that one can hear during the calling sequence.

Particularly Cot ([46]) was interested in gossip schemes that permit gossiping with 2n-4 calls and are "a-redundant" - neither of its vertices known all the information before the final calls made by them. He found schemes that allow this for n=4 and n=8 cases.

Other similar class of gossip problems can be obtained by requiring that no one hears their own initial information from the other (Baker and Shostak [14]). The term "NOHO" (No Own Hears Own) was introduced to refer to these kind of problems/graphs. West studied this kind of graphs (in 1982, [17]) and showed that

NOHO gossip schemas with $2n-4$ calls exist for any even $n \geq 4$ (such schemas do not exist for odd $n > 1$).

A more strict restriction on the gossipping model is that duplicate transmissions should not exist in a calling sequence at all. This class of problems has been widely known as NODUP (NO DUPlication) gossip problem. It was first studied by Lenstra et al. ([47]) who showed the existence of such NODUP gossip schemes for all n divisible by 4 and that the number of required calls satisfies

$$f(n) \leq \frac{1}{2} n * \log_2 n \; + \; O(1), \;\; for\ 4|n.$$

West ([17], [48]) found the following lower bound for f(n) and the following upper bound (by construction):

$$\mathrm{f(n) \geq 2n - 3, for\ n > 8,}$$

$$\mathrm{f(n) \leq 9n/4 \;\; - \; 6, for\ 4|n}$$

Seress ([49], [50]) showed that:

*NODUP schemes exist for all even **n** with the exception of 6, 10, 14, 18.*

$$\frac{9\mathrm{n}}{4} - 4.5 \leq \mathrm{f(n)} \leq \frac{9\mathrm{n}}{4} - \; 3.5 \;\; \mathrm{for\ n = 2(mod\ 4), n \geq 22}$$

$$f(n) = \frac{9n}{4} - \; 6\, for\ 4|n, n \geq 8$$

For both results Seress provided also construction of corresponding graph satisfying the conditions. Apart from this Seress was interested in NODUP gossiping by conference calls with *k* person involved ([51]). He found the values of *n* and *k* for which such schemes exist. Additionally, Seress observed that it is not possible to implement NOHO or NODUP gossipping on top of directed graph (one-way calls), since if initially first individual calls second, he will learn his own information in some of the following call along with seconds information.

Another common variant of investigation of the gossip problem is to find the minimum amount of time required for the gossiping among n individuals to be completed assuming that each call takes exactly one time unit and may involve $k \geq 2$ people. The metrics of interest here was denoted by $t_0(n,k)$ and was obtained by many researchers ([26], [33] and [52]) for $k = 2$ case and by Schmitt ([53]) for $k \geq 2$ as follows:

$$\mathrm{t_0(n,2) = \lceil \log_2 n \rceil, \qquad for\ n\ even}$$

$$\mathrm{t_0(n,2) = \lceil \log_2 n \rceil + 1, \qquad for\ n\ odd}$$

$$\mathrm{t_0(n,k) = \lceil \log_k n \rceil, \qquad for\ k|n}$$

$$\mathrm{t_0(n,k) = \lceil \log_k (n/(k-1)) \rceil + 1, \qquad otherwise}$$

The above results were obtained assuming the underlying communication model has no any restriction on the possible pairs of calling sequence (complete graph). Labahn (in 1986, [54]) considered the same in a scope of connected trees with *n* vertices. In case $t_L(T)$ is the time required for complete gossiping when the underlying structure is a tree T he showed that:

$$2\lceil \log_2 n \rceil - 1 \leq t_L(T) \leq 2n - 3,$$

and managed to describe such trees with $2\lceil \log_2 n \rceil - 1$ time and calling sequence with minimum possible time.

Landau ([52]) was researching the one-way gossiping, where during each call an individual could have sent own information to up to *k* receivers. So the metrics of interest here was the same minimum time required to complete such a gossiping process, which was shown to be

$$t_k(n) = \lceil \log_{k+1} n \rceil.$$

Several other researchs about the minimum time were performed by Shimbel ([27]) and Entringer and Slater ([55]). Their work and results was about the minimum time required in gossip schemes under various constraints in digraphs. For the time required to gossip in hypergraphs with n vertices and each edge containing less than or equal k vertices (note, not necessarily k-uniform) Schmitt ([53]) proved that:

$$t(n,k) = \lceil \log_k n \rceil, \qquad if\ k|n$$

$$t(n,k) = \left\lceil \log_k \left(\frac{n}{k-1}\right) \right\rceil + 1, \text{otherwise.}$$

Among other variations of Gossip Problem (which are much more) we think also worth mentioning the work done by Cot (in [46]) and Richards and Liestman (in [45]). Cot considered a model in which each call is being made by some cost that depends on the distance between the two vertices or the number of exchanged messages. The goal was to find a calling sequence that minimizes overall gossiping cost. While Richards and Liestman was interested in finding the number of calls required for each individual to learn at least *k* pieces of the information in a gossip graph with *n* vertices. They found a possible solution for some of the values of *n* and *k*. For $k \leq 3$ they have found an exact solution and found upper bound (by construction) for $k \geq 4$.

This section was a brief overview of all the work and obtained results that are of interest for us in Gossip Problem. In the next section we will consider another variation of the gossip problem - fault-tolerant gossiping, since this topic is especially important in the context of work done during our research.

1.2 Fault-tolerant gossiping and the known results

One of the natural generalizations of this problem is the k-fault-tolerant gossip problem, which assumes that some of the components (nodes or calls) in the communication scheme can fail to function properly and the communication flows that include these components will be disturbed. There is a wide range of fault classifications in a full-duplex whispering (and shouting) mode of gossiping (see the introduction of this work or [56] for a complete reference) which is a direct consequence of the popularity and actuality of this issue, but we are going to focus only on bounded, crash type, non-adaptive permanent call faults. Some of the work done in this specific area ([34]– [37]) serve as a solid baseline for our research. So in our model there are only links (calls) that might be faulty and the nodes cannot change the sequence of their future calls depending on the current failed calls. Here the aim is to find a minimal gossip scheme, which guarantees the full exchange of the information in the case of at most k arbitrary fails, regardless of which of the calls failed. The gossip schemes, which provide k-fault-tolerance, are called k-fault-tolerant gossip schemes and are typically expressed as multigraphs with the nodes representing commonicating individuals and the edges - the calls made between them. Denote the minimal number of calls in the k-fault-tolerant minimal gossip scheme by $\tau(n,k)$.

Berman and Hawrylycz [35] obtained the lower and upper bounds for $\tau(n,k)$:

$$\left\lceil\frac{k+4}{2}(n-1)\right\rceil - 2\left\lceil\sqrt{n}\right\rceil + 1 \leq \tau(n,k) \leq \left\lfloor(k+\frac{3}{2})(n-1)\right\rfloor$$

for $k \leq n-2$, and

$$\left\lceil\frac{k+3}{2}(n-1)\right\rceil - 2\left\lceil\sqrt{n}\right\rceil \leq \tau(n,k) \leq \left\lfloor(k+\frac{3}{2})(n-1)\right\rfloor$$

for k ≥ n − 2. They have also conjectured that the exact value of the seeking property is very close to the upper bound and differs only with some constant expression: $\tau(n,k) = (k+3)n/2 - const$. But this conjecture was disproved letter in [37].

Afterwards, Hadded, Roy and Schaffer [37] proved that

$$\tau(n,k) \leq (\frac{k}{2}+2p)(n-1+\frac{n-1}{2^p-1}+2^p)$$

where p is any integer between 1 and $\log_2 n$ inclusive. By choosing p appropriately, this result improves the upper bounds obtained by Berman and Hawrylycz for almost al k. Particularly, by choosing $p = \lceil \log_2 n / 2 \rceil$, the following bound is obtained: $\tau(n,k) \le nk/2 + O(k\sqrt{n} + n\log_2 n)$. They also conducted that the gossiping time of the their scheme differs from the optimal one just with small multiplicative factor.

For the special case when $n = 2^p$ for some integer p, Haddad, Roy and Schaffer [37] also showed that

$$\tau(n,k) \le min\left\{\begin{matrix} \left(\left\lceil \frac{k+1}{\log_2 n} \right\rceil + 1\right) \frac{n\log_2 n}{2}, \\ \left(\left\lfloor \frac{k+1}{\log_2 n} \right\rfloor + 1\right) \frac{n\log_2 n}{2} + \left((k+1)\ mod\ \log_2 n\right)(2n-4) \end{matrix}\right\}.$$

Thus, $\tau(n,k) \le nk/2 + O(n\log_2 n)$, when n is a power of 2.

Later on, Ho and Shigeno [34] showed that

$$\left\lfloor \frac{n(k+2)}{2} \right\rfloor \le \tau(n,k) \le \frac{n(n-1)}{2} + \left\lceil \frac{nk}{2} \right\rceil.$$

Thus, it holds that $nk/2 + O(n) \le \tau(n,k) \le nk/2 + O(n^2)$. These bounds improve the previous bounds for small n and sufficiently large k.

Recently, Hasunama and Nagamochi [36] showed that

$$\tau(n,k) \le \begin{cases} \frac{n\log_2 n}{2} + \frac{nk}{2}, & if\ n\ is\ a\ power\ of\ 2 \\ 2n\lfloor \log_2 n \rfloor + n\left\lceil \frac{k-1}{2} \right\rceil, & otherwise \end{cases}$$

And

$$\tau(n,k) \ge \left\lceil \frac{3n-5}{2} \right\rceil + \left\lceil \frac{1}{2}\left(nk + \left\lfloor \frac{n+1}{2} \right\rfloor - \lfloor \log_2 n \rfloor\right) \right\rceil$$

Their results were based on constructions of gossip schemes using Hypercubes and Circulant graphs as an underlying network topology. From their results it holds that $\tau(n,k) \le nk/2 + O(n\log_2 n)$. Particularly, their upper bound improves the upper bound by Hou and

Shigeno for all $n \geq 13$. They also improve the upper bound by Haddad et al. by showing that the factor $(k/2 + 2p)$ in their upper bound can be replaced with a smaller factor $k/2 + p$:

$$\tau(n,k) \leq \left(\frac{k}{2} + p\right)\left(n - 1 + \frac{n-1}{2^p - 1} + 2^p\right),$$

where p is any integer between 1 and $\log_2 n$ inclusive. They have also improved the lower bound suggested by Berman and Paul for $k > n/2$ and Ho and Shigeno's lower bound when $n \geq 5$.

In our work we construct two classes of k-fault-tolerant gossip schemes based on Knodel graphs([20], [57]) and wheel graph [21], which improve the previously known results on the upper bound for the number of calls. The obtained expressions for n and k are:

$$\tau(n,k) \leq n/2 \lceil \log_2 n \rceil + nk/2,$$

for even n, and

$$\tau(n,k) \leq \begin{cases} \frac{2}{3}(n-1)k + \frac{5}{2}(n-1), & \text{if} \quad (k \bmod 3) = 0, \\ \frac{2}{3}(n-1)(k-1) + \frac{7}{2}(n-1), & \text{if} \quad (k \bmod 3) = 1, \\ \frac{2}{3}(n-1)(k-2) + 4(n-1), & \text{if} \quad (k \bmod 3) = 2, \end{cases}$$

for odd n. Particularly, for large odd n we have the below result which improved upper bounds for some large values of n and small values of k

$$\tau(n,k) \leq \frac{2}{3}nk + O(n).$$

For the even *n*, our result is a generalization of the one obtained by Hasunama and Nagamochi for hypercubes and is based on the construction on Knodel graphs. As a side effect our construction based on Knodel graphs also yields the minimal gossiping and broadcasting time of the fault-tolerant gossip/broadcast scheme to be equal $T(n,k) = \lceil log_2 n \rceil + k$ (the problem proposed by Gargano in [58]).

1.3 Open problems in the gossip/broadcast schemas

In the current section we are going to give brief overview of the current state of the research in the specific area of deterministic type of gossiping/broadcasting problems and are of interest for us. Some of these problems were already addressed in our study and some complete solutions or bounds were obtained, while the others were raised by us (see [22] for reference).

Problem 1. First, speaking of broadcasting in any connected graph G. As was mentioned above the process of broadcasting can be viewed as an a bit simple variation of gossiping, where instead of all-to-all information exchange we consider the one-to-all information broadcasting. The broadcasting from any vertex u in G determines the spanning tree rooted at u and the minimum time required to complete broadcasting is denoted by $b(u)$. The broadcast time of any graph G is defined as follows: $b(G) = max\,\{b(u) \mid u \in V(G)\}$. It is obvious that the lower bound for $b(G)$ is $\lceil log_2 n \rceil$ and the graphs that can perform broadcasting in minimum time are called broadcast graphs. The complete graph K_n satisfies the following: $b(K_n) = \lceil log_2 n \rceil$, thus is a broadcast graph. But the research is interested more on the graphs with minimal possible number of edges that are still broadcast graphs. These kind of graphs are called *minimum broadcast graphs*. Speaking more formally these are graphs G (|G|=n) for which $b(G) = \lceil log_2 n \rceil$ and for every spanning subgraph $G' \subset G$, $b(G') > \lceil log_2 n \rceil$. The number of edges in minimum broadcast graphs is denoted by B(n). There is no much knowledge about B(n), particularly the research that was carried out on this ([59], [60] and [61]) yield results for B(n) only for some particular values of n ($n \le 18$ in general). Later Peleg ([62]) showed that $B(n) \in \Theta(L(n-1)n)$, where $L(k)$ denotes the exact number of consecutive leading 1's in the binary representation of k. Only this much is known about the behaviour of $B(n)$ and we find it one of the most important topics to investigate.

Problem 2. Another problem of big interest ([63], [64]) is identifying the minimum number of rounds (or minimum time) of the optimal broadcast and gossip (both directed and undirected modes) algorithm, which is denoted by T(G). There are only lower and upper bounds known for this entity. For example for deBruijn network of the size n the known results are $1.3171n \leq T(G) \leq 1.5n + 1.5$. The most recent survey by Perennes ([65]) gives a complete overview of the most recent obtained bounds for T(G) (that improve the one provided above).

Problem 3. According to definition proposed by R. Labahn a graph G=(V,E) with |V|=n even is considered to be minimum gossip graph (n-mgg) iff it is possible to complete gossiping in $\lceil log_2 n \rceil$ rounds using G as a communication graph and |E| has the smallest possible value G(n) among all such graphs with n vertices. Considering the above said he proposed the following problem that is still actual: the value of G(n) is known for $n = 2^p, n = 2^p - 2$ and $n = 2^p - 4$. Determine the value of G(n) for all the other or (at least) suggest lower/upper bounds.

Problem 4. There were a lot of work done by West ([17], [66], [67], [48]) and Seress ([49], [50]) in the specific area of NOHO and NODUP gossip schemes. Particularly they were interested in finding such schemes for broadcasting and gossiping with minimum possible number of calls and minimum time required to gossip. Let T(n) be the minimum time required to gossip on any graph G with n vertices. Thus $T(n) = \lceil log_2 n \rceil$ for any even n and $T(n) = \lceil log_2 n \rceil + 1$ for odd n. However, the NOHO and NODUP restrictions generally were not applied (or considered) when proving the above results. So, the following questions were naturally raised by these authors (and many others):

- how does NOHO/NODUP restriction change the above results?
- what is the minimum time required to complete NOHO/NODUP gossiping on the graph with n vertices?

- what are the values of n permitting NOHO/NODUP gossiping in minimum time and what are the corresponding graphs?

Problem 5. This problem is very actual and we had a chance to visit this topic already in this current study. It was raised one more time by Douglas West in [68] and is about "fault-tolerant" gossiping. As already was discussed in section 1.2 of the current study the problem of finding the minimum number of edges in a fault-tolerant gossip multigraph is still open. Haddad, Roy and Schaffer come up with the constructions that are asymptotic to the lower bound in case of k being a growing function of n. But in practice we often need the network to have some fixed small level of edge fault-tolerance, even if the number of vertices can be comparably big. And in practice, there is no known construction that can yield fault-tolerant gossip scheme with the number of vertices close to the known lower bound of $\lfloor nk/2 \rfloor + \Theta(n)$. Additionally very little is known in the sphere of vertex fault-tolerant gossiping where the remaining part of the communication network should be able to communicate in case of at most k vertices fail to transfer any information. As a side note, West offered to focus also on bounded maximum degree graphs, i.e. how many edges are needed for k fault-tolerant gossiping in case we bound the maximum degree of a communication graph by some function f(n). This additional restriction tends to limit the number of required lines in communication model, so that the vertices will reuse an existing infrastructure as much as possible and the cost overall communication topology can be reduced.

Problem 6. Let $B_k(n)$be the k-tolerant broadcast function that was defined and studied first in [69]. This number is the minimum number of links in a network supporting k-tolerant broadcasting in minimum possible time T. Note in this model we consider not call failures but link failures, i.e. no call using the failed communication channel can be accomplished. The first results regarding the functions T and $B_k(n)$ was obtained in [69] for some particular values of *n* and *k* and are the followings:

$$T = \lceil log_2 n \rceil + 1, k = 1 \; and \; n > 2,$$

$$T = \lceil log_2 n \rceil + 2, k = 2, n > 4 \; and \; n \neq 2^i - 1,$$

$$T = \lceil log_2 n \rceil + 3, k = 2, n > 6 \; and \; n = 2^i - 1,$$

$$B_1(n) \leq \lceil n log_2 n \rceil - n/2, n = 2^m,$$

$$B_1(n) \leq \lceil n log_2 n \rceil - n, n = 2^m + 2^{\lfloor log_2 n \rfloor}$$

$$B_1(n) = \lceil n log_2 n \rceil, otherwise.$$

After that a lot of research was done towards obtaining the values T and $B_k(n)$ (see [61], [70], [71] and [72] for complete overview). The current state of the research for the exact values of T had been established in case $n = 2^m$ and the bounds had been proposed for other values of n (suppose $m = \lfloor log_2 n \rfloor \; and \; n = 2^m + j$):

$$T = \lceil log_2 n \rceil + k, if \; k \leq n - log_2 n - 1 \; and,$$

$$T = \lceil log_2 n \rceil + k + 1, if \; n - log_2 \; n \leq k \leq n - 2$$

$$T \leq m + k + 1 + \lfloor k/(m - \lceil log_2 j \rceil - 1) \rfloor, if \; k \leq 2^m - m - 1,$$

$$T \leq m + k + 2 + \lfloor (k-1)/(m - \lceil log_2 j \rceil - 1) \rfloor, if \; 2^m - m - 1 < k \leq 2^m - 2.$$

The tightness of these bounds depends on the values of k and j, for small j when n is close to $2^{\lfloor log_2 n \rfloor}$ and small values of k the upper bound is quite tight to $log_2 n + k$. This is not true for large values of j and the obtained result is far from lower bound when k is large. The exact values for T for arbitrary n and k remained unknown and are subject for our future research.

The exact values for $B_k(n)$ were obtained only for some particular values of n and k. Particularly, it was shown that (see [73], [58])

$$B_1(2^m - 2) = m2^{m-1} - 1,$$

$$B_1(2^m - 6) = (m-1)(2^m - 6)/2, \text{ when m is even}$$

$$B_k(2^m) = (m+k)2^m/2$$

Also in [71] the construction has been proposed that supports k-tolerant broadcasting in minimum time, and there is only a constant factor difference between the number of links of those schemes and the minimum value of $B_k(n)$for small values of k ($k < \lfloor log_2 n \rfloor$). The exact value of $B_k(n)$ is still open problem to solve for the researchers.

The next two problems had been raised by us throughout various stages of our research. They both are connected with the applications of the "method of local interchange" defined in the section 2.1 .

Open Problem 1. What are the sufficient and necessary conditions for the application of the methods of local interchange on the k fault-tolerant gossip graph to not affect the level of the fault-tolerance of the graph.

Remark 1: We have shown in [57] that in the case of k = 0 (complete gossip graph) the application of this method on the graph still yields another gossip graph. On the other side by using software tool designed by us and described in [23] we have noted that it often does not affect the level of fault- tolerance of Knodel based k fault-tolerant graph. But it does affect the level of fault-tolerance of the fault-tolerant graphs that are constructed by graph combination method (see [20]). So we find this problem the problem of interest as it would yield different variations for the structure of fault-tolerant gossip graph.

Open Problem 2: Is it possible to use the method of local interchange to obtain k fault-tolerant minimal broadcast schemes.

Remark 2: With the help of Graph Plotter (see [23]) we have noted that the application of A_2^- on the $G = W_{\lceil log_2 n \rceil, n} + W_{1,n}$(see the sections 1.4 and 2.1 for definitions of mentioned operations) yields minimal broadcast 1 fault-tolerant scheme with $T = \lceil log_2 n \rceil + 1$.

Assuming that the calls between non-overlapping pairs of nodes can take place simultaneously, the minimum amount of time T (n) required to complete gossiping is $\lceil log_2 n \rceil$ for even n and $\lceil log_2 n \rceil + 1$ for odd n. As it is shown in [32], any gossip scheme on n ≥ 4 vertices with 2n – 4 calls has at least 2$\lceil log_2 n \rceil - 3$ rounds.

Open Problem 3: Is it possible by the help of the method of local interchange find the minimum number of calls of a gossip scheme with time (rounds) T(n).

1.4 Some common definitions and notations of gossip problem

In the current section we are going to focus on providing the definitions of the entities that are of importance for the our study. We are going to refer to the definitions and notations of this section from the rest of the work. This section is also somehow complements the definitions that were already presented throughout the work. Let us start with the basics, the first thing we want to focus is the gossip graph definition.

Definition. A gossip scheme (a sequence of calls between n nodes) can be represented by an undirected edge-labeled graph $G = (V, E)$ with n vertices ($|V| \equiv |V(G)| = n$). The vertices and edges of G represent correspondingly the nodes and the calls between the pairs of nodes of a gossip scheme. An edge-labeling of G is a mapping $t_G: E(G) \rightarrow Z_+$. The label $t_G(e)$ of the given edge $e \in E(G)$ represents the moment of the time, when the corresponding call was occurred.

A sequence $P = (v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$ with vertices $v_i \in V(G)$ for $0 \leq i \leq k$ and edges $e_i \in E(G)$ for $1 \leq i \leq k$ is called a walk from a vertex v_0 to a vertex v_k in G with length k, if each edge e_i joins two vertices v_{i-1} and v_i for $1 \leq i \leq k$. A walk, in which all the vertices are distinct is called a path. If $t_G(e_i) < t_G(e_j)$ for $1 \leq i < j \leq k$, then P is an ascending path from v_0 to v_k in G. Given two vertices u and v, if there is an ascending path from u to v, then v receives the information of u. Note that two different edges can have the same label only in case they are not adjacent. Since we consider only (strictly) ascending paths, then such edges (i.e. calls) are independent, which means that the edges with the same label can be reordered arbitrarily but for any $t_1 < t_2$ all the edges with the label t_1 are ordered before any of the edges with the label t_2.

Definition. Let $P = (e_1, e_2, \ldots, e_k)$ be a path with edges $e_i \in E(G)$ for $1 \le i \le k$ in a labeled graph G. If P is divided into $s+1$ subpaths $P_{(1)} = (e_1, e_2, \ldots, e_{p_1})$, $P_{(2)} = (e_{p_1+1}, \ldots, e_{p_2})$, …, $P_{(s+1)} = (e_{p_s+1}, \ldots, e_k)$, then we write $P = P_{(1)} \odot P_{(2)} \odot \cdots \odot P_{(s+1)}$, where $\odot$ is the concatenation operation on two paths for which the last vertex of one path is the first vertex of the other. If $P = P_{(1)} \odot P_{(2)} \odot \cdots \odot P_{(s+1)}$ such that $P_{(j)}$ is an ascending path for $1 \le j \le s+1$ and $P_{(j)} \odot P_{(j+1)}$ is not an ascending path for $1 \le j \le s$, then P is a s-folded ascending path in G. For an s-folded ascending path P, the folded number of P is defined to be s.

Definition. Consider two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with the same set of vertices V and labeled edge sets E_1 and E_1, respectively. The edge sum of these graphs is a graph $G_1 + G_2 = G = (V, E)$ with $E = E_1 \cup E_2$, whose edges $e \in E$ are labeled by the following rules:

$$t_G(e) = \begin{cases} t_{G_1}(e), & if\ e \in E_1 \\ t_{G_2}(e) + \max\limits_{e\prime \in E_1} t_{G_1}(e'), & if\ e \in E_2 \end{cases},$$

The edge sum $G_1 + G_2 + \cdots + G_h$ of h identical graphs ($G_1 = G_2 = \ldots = G_h \equiv G$) is denoted by hG. Each set $E(G_i)$ in hG is denoted by $E_i(hG)$, i.e. $E(hG) = \cup_{1 \le i \le h} E_i(hg)$. Note that the labels of the edges in $E_i(hG)$ are greater than the corresponding edges in $E(G)$ by $(i - 1) * max_{e \in E(G)}\, t_G(e)$. Given a subset of edges $A \subseteq E(G)$, denote its copy in the set $E_i(hG)$ by A_i. By this analogy, a path P in G as a subset of $E(G)$ has a copy in $E_i(hG)$, which we denote by P_i.

The Knodel graphs along with Hypercube and Circulant graph is one of the most popular network topologies for implementing gossip problem. They have been introduced about 43 years ago ([33]) and are defined only for even order n and for some degree $1 \le \Delta \le \lfloor \log_2 n \rfloor$. More formally the family of Knodel graphs were defined They have been very widely studied as an interconnection network, because of their good properties in terms of broadcasting and gossiping. Particularly, the Knodel graph of order 2^k and the degree k are equally popular interconnection networks in the literature as the hypercube H_k and the recursive circulant graph $G(2^k, 4)$, since all of them can perform gossiping (also broadcasting) in minimum possible time. We think we should denote an emphasized attention to these graphs since we use them a lot in our studies.

Definition. The Knodel graph on $n \geq 2$ vertices (n even) and of degree $\Delta \geq 1$ is denoted by $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the pairs (i, j) with $i = 1, 2$ and $0 \leq j \leq n/2 - 1$. For every j, $0 \leq j \leq n/2 - 1$ and $l = 1, \dots, \Delta$, there is an edge with the label l between the vertex $(1, j)$ and $(2, (j + 2^{l-1} - 1) \; mod \; n/2)$.

An example of Knodel graph is shown in FIg. 1. Note that $W_{1,n}$ consists of $n/2$ disconnected edges. For $\Delta \geq 1$, $W_{\Delta,n}$ is connected.

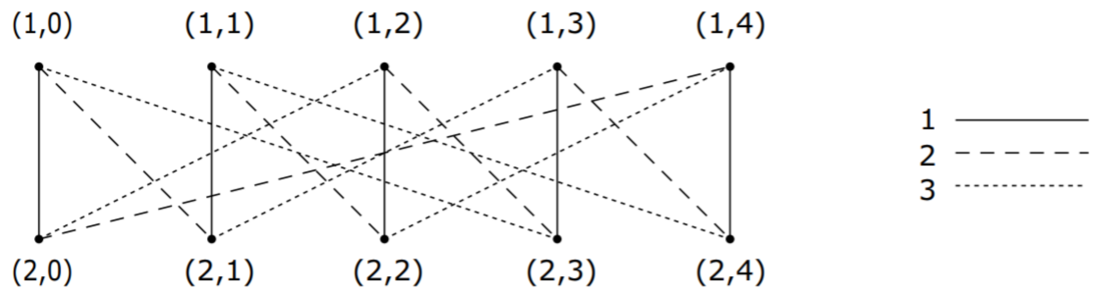

Fig. 1) The Knodel graph $W_{3,10}$ with a number of vertices n = 10 and degree Δ = 3.

The solid, dashed and dotted edges are labeled correspondingly 1, 2 and 3 in the above graph.

Definition. The interval between two vertices (α, β) and (γ, δ) in the Knodel graph $W_{\lfloor \log_2 n \rfloor, n}$ is defined to be

$$
R\big((\propto,\beta);(\gamma,\delta)\big)=\begin{cases}\delta-\beta, & if\ \delta\geq\beta\ and\ \propto=1,\\ \dfrac{n}{2}-|\delta-\beta|, & if\ \delta<\beta\ and\ \propto=1,\\ |\delta-\beta|, & if\ \delta\leq\beta\ and\ \propto=2,\\ \dfrac{n}{2}-(\delta-\beta), & if\ \delta>\beta\ and\ \propto=2.\end{cases}
$$

Throughout the study we often refer to the concept of the isomorphism of the gossip graphs. Since these are weighted graphs and there is no commonly accepted definition of this concept for weighted graphs, we will bring the definition here and will refer to it when using this concept.

Definition. Two weighted graphs G and H with the order n are considered to be isomorphic to each other, if and only if there exists a bijection between the vertex set of these graphs - $\mathrm{f}: \mathrm{V(G)} \rightarrow \mathrm{V(H)}$, so that if the vertices u and v are adjacent in G then the vertices $\mathrm{f(u)}$ and $\mathrm{f(v)}$ are also adjacent in H and $\mathrm{t_G(u,v)} = \mathrm{t_H(f(u),f(v))}$.

Further let us focus on the definition of fault-tolerance in communication graphs, since we are going to (and already did) refer to this concept a lot throughout this study.

Definition. The communication between two vertices of G is called k-failure safe if an ascending path between them remains, even if arbitrary k edges of G are deleted (the corresponding calls are failed). The graph G is called a k-fault-tolerant gossip graph if the communication between all the pairs of its vertices is k-failure safe.

From the Menger theorem [74] about edge connectivity between nodes, it follows that a k-fault-tolerant gossip graph is a graph whose edges are labeled in such a way that there are at least $k + 1$ edge-disjoint ascending paths between two arbitrary vertices. A 0-fault-tolerant gossip graph is simply called a gossip graph.

One important note, since we speak about multigraphs in case of fault-tolerant gossiping/broadcasting, we should make clear distinction between the edges and links of the graph. The link is a communication channel that connects two vertices to each other, while the edge of the fault-tolerant graph only represents the one single communication event between the two vertices. From the above said, we can conclude that the link can contain multiple edges and we should refer to the terms “minimum number of edges” and “minimum number of links” as a subject of different discussions.

Now that we have provided definitions for some common terms used in our study we can proceed with the next sections in this chapter, where our main focus will be on listing the theoretical results obtained by us with corresponding analytical proofs.

Construction and investigation of the optimal gossip and fault-tolerant gossip schemes

2.1 Description of the method of local interchange

As we have already mentioned, the gossip process can be easily modeled as a graph, whose vertices represent people in gossip scheme and edges represent calls between them (each of them has weight which represents the moment when communication took place). So the graph is called a complete gossip graph if there are ascending paths between all pairs of vertices of this graph.

In the current section we are going to define the method of local interchange which we use throughout our research extensively.

Denote the set of edges adjacent to a given vertex v by $E_v(G)$. Given an edge e and one of its two endpoints v, we consider the following two subsets of the set $E_v(G)$:

$$\rho_v^{+}(e, G) = \{e' \in E_v(G) \mid t_G(e') \geq t_G(e)\},$$

$$\rho_v^{-}(e, G) = \{e' \in E_v(G) \mid t_G(e') \leq t_G(e)\}.$$

Sometimes we omit the argument G in notations.

Definition. An identification of two vertices v_1 and v_2 [13] in a gossip graph G is a gossip graph G' , which is defined as follows: The edges between the vertices v_1 and v_2 are deleted and these two vertices are replaced by a vertex u, whose set of incident edges is $E_u(G') = E_{v_1}(G) \cup E_{v_2}(G)$.

An interchange of two vertices in a calling scheme is defined as follows: started from the indicated moment of a time to the end of the calling scheme two vertices (and the edges adjacent to them) are replaced by each other. In this work we leverage the concept of local interchange, i.e. an interchange that is defined only for adjacent vertices.

Definition. The permute higher operation $P^+(e)$ on a selected edge $e \in E(G)$connecting vertices u and v is called a modification of G, which moves the edges of G adjacent to e as follows:

$$E_u(P^+(e)G) = \rho_u{}^-(e,G) \cup \rho_v{}^+(e,G),$$
$$E_v(P^+(e)G) = \rho_v{}^-(e,G) \cup \rho_u{}^+(e,G).$$

Correspondingly, the permute lower operation $P^-(e)$ moves the edges of G adjacent to e as follows:

$$E_u(P^-(e)G) = \rho_u{}^+(e,G) \cup \rho_v{}^-(e,G),$$
$$E_v(P^-(e)G) = \rho_v{}^+(e,G) \cup \rho_u{}^-(e,G).$$

The operators $P^+$and $P^-$are called the operators of local interchange.

Lemma. The result of the application of the operators of local interchange on a complete gossip graph is also a complete gossip graph.

Proof. Actually, if a call between two participants takes place at the moment t_0, started from that point they both have the same information and are equivalent in the context of gossiping. Hence, if we interchange all the calls that were initiated or participated by these two vertices afterwards (permute higher), our new gossip scheme would also be complete. At the same time neither the number of edges, nor the number of rounds required to perform gossiping were changed. The same assumptions could be made in case of permute lower operation.

Let us define an operation on gossip graphs $A^+ = P^+(e_1)P^+(e_2)\dots P^+(e_p)$ $(A^- = P^-(e_1)P^-(e_2)\dots P^-(e_p))$ is the sequence of permute higher (lower) operations on edges $e_i, i = 1,\dots,p$.

Lemma. The result of the operation $A^+(A^-)$ does not depend on the order of the edges e_i.

Proof. Without loss of generality we can only consider the permute higher operation A^+ with $p = 2$. Assume that $t_G(e_1) < t_G(e_2)$, where t_G is edge-ordering of gossip graph G. There are two possible cases:

1) Edges e_1 and e_2 are adjacent. In this case, if $A^+ = P^+(e_1)P^+(e_2)$ after $P^+(e_1)$ all the edges which are adjacent to e_1 including e_2) will change their endpoint from one endpoint of e_1 to another and after $P^+(e_2)$ all the edges e for which $t_G(e) > t_G(e_2)$ also will be permuted. If $A^+ = P^+(e_2)P^+(e_1)$, then after the first phase of A^+ all the edges with $t_G(e) > t_G(e_2)$ will be permuted and after the second phase all the edges which are adjacent to e_1 and have $t_G(e) > t_G(e_1)$ (including e_2) will also be permuted. These operations are demonstrated in the following figure. It is obvious that the result of these two variations of A^+ is the same.

2) Edges e_1 and e_2 are not adjacent. This is a simpler case in which the operation $P^+(e)$ on one of them does not affect directly the other. So, it is obvious that all the variations of A^+ are equivalent. ■

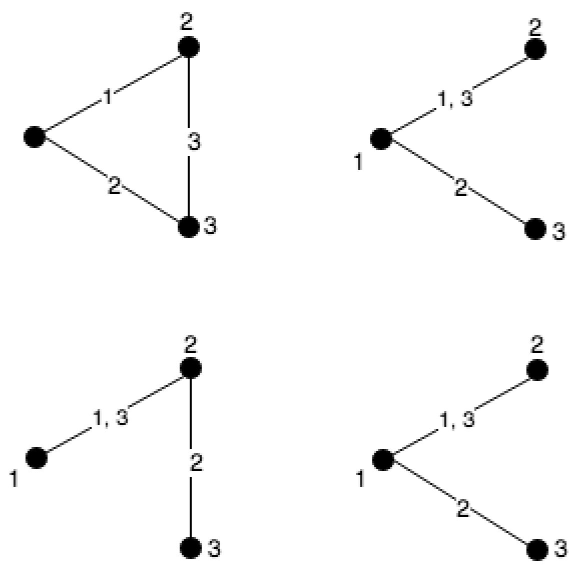

Fig. 2) Equivalence of different variations of A^+

So, in this section the operations of local interchange and their main properties were described. In the next section we give some application of the operations described here.

2.1.1 Application of the method of local interchange in determination of minimum number of calls of gossip graphs

Theorem. The minimum required number of calls in complete gossip schemes is $\boldsymbol{2n-4, n \geq 4}$***.***

Proof. As already mentioned in the introduction it has been shown in numerous works ([14], [13], [30], [29]) that the minimal number of calls in gossip schemes is 2n-4, where n is the number of vertices. In this section we will prove this statement in a completely new way, which is based on the operations of local interchange.

Let us denote by f(n) the minimum number of edges required to construct a gossip graph on n vertices. There are many solutions of this problem that give the number of edges (calls) equal to 2n - 4. Therefore, $f(n) \leq 2n-4$is valid. So, the important part is to prove that $f(n) \geq 2n-4$ also.

Consider a gossip graph with n vertices and the number of edges equal to f(n) . Our goal is to prove that $f(n) \geq f(n-1)+2$. After that, taking into account that f(4) = 4 we will get that $f(n) \geq 2n-4$.

Suppose we have a graph G with f(n) calls. There are 3 possible kinds of graphs depending on the repetition of information.

1) It contains a cycle. Suppose a $C = (e_1, e_2, \ldots, e_p)$is a cycle with length p and the edges e_1 and e_p are adjacent to the vertex v (see below).

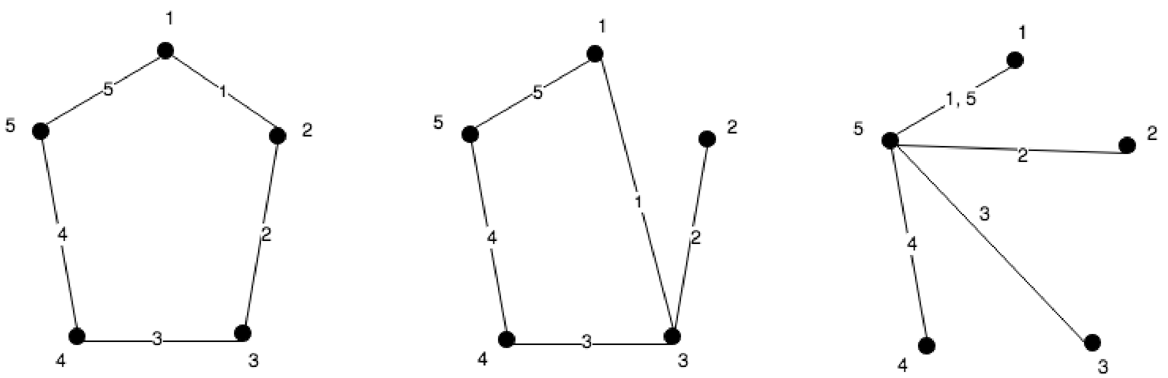

Fig. 3) Cycle C in gossip graph

The simplest case is if p = 2. If this holds, we can apply an operation of the identification of two vertices connected by this multiple edge which is described in the section above. Obviously, after this operation on complete gossip graphs, the obtained new gossip graphs are also complete (it does not affect on ascending paths between the pairs of vertices). Hence, our new graph G' with n - 1 vertices has $|E(G')| \geq f(n-1)$ edges. On the other hand G_0 is obtained from G by removing two edges and merging two vertices, thus $|E(G')| = f(n) - 2$. Hence, we obtain $f(n) \geq f(n-1) + 2$.

Now consider the case of $p \neq 2$and an operation $A^- = (P^-(e_2), P^-(e_3)\dots P^-(e_{p-1}))$. As it was proved in the previous section the result of application of this operation on a complete gossip graph is also a complete gossip graph. In the result the edges $e_1$and $e_2$will join and form a double-edge. Thus, we came to the case of p = 2. Further steps are described above.

2) There are no cycles (NOHO graphs). In this case we can refer to [17], [66], but let us also consider this case. Let $L_1 = (e_1, e_2, \dots, e_n)$ and $L_2 = (l_1, l_2, \dots, l_m)$ be ascending paths from the vertex v to the vertex u.

Let us define the operations

$$A_e{}^- = (P^-(e_2)P^-(e_3)\dots P^-(e_{n-1}))$$
$$A_l{}^- = (P^-(l_2)P^-(l_3)\dots P^-(l_{m-1}))$$

on gossip graph. After the application of these operations we will obtain a tetragon which involves the vertices u and v and the edges e_1, l_1, e_n, l_m. Then we choose the greatest from the pair e_1, l_1 and apply "permute lower" operation on it. We will obtain a triangle that is also a cycle (case 1) . This process is demonstrated in the following figure.

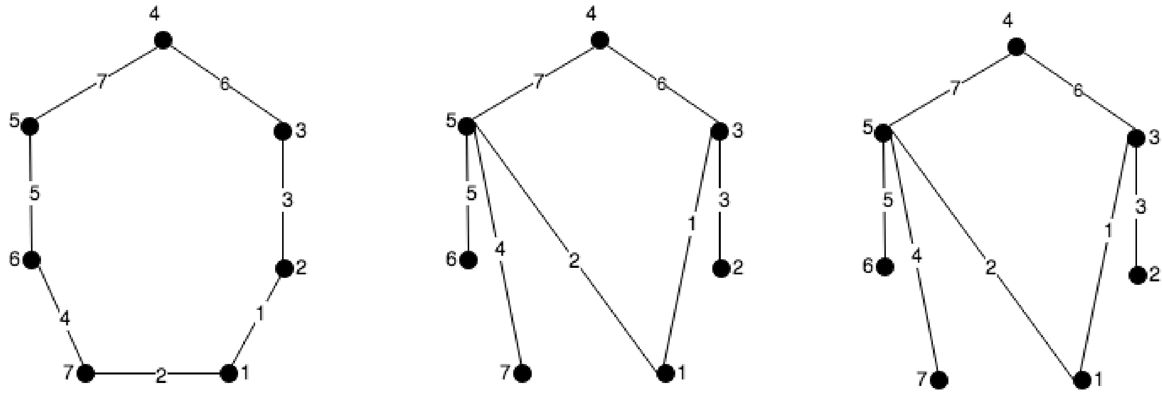

Fig. 4) Ascending paths in gossip graphs

3) There are no duplicates of information (NODUP graphs). In this case there is exactly one ascending path between any pair of vertices. It has been shown in [21], that in this case $f'(n) = 2.25n - 6, n = 4k, k \geq 2$, where $f'(n)$ is the minimum number of edges of NODUP graph and it intersects with 2n-4 only when n = 8. So, if it is a NODUP graph, it always has more edges than 2n - 4 and the only exception is the case when n = 8. Case of n = 4 was not considered in [50], but in this case $f(n)$ and $f'(n)$ also coincide.

2.2 NOHO graphs based on Knodel graphs

In [68] it was mentioned that the minimum time needed to complete gossiping is $\lceil log_2 n \rceil$. In [75] a problem of finding minimum time of gossiping in case of NOHO graphs (see "Open Problems" section) was raised. In this section we will show a new method of construction of NOHO graphs with minimum gossiping time by applying an operation of local interchange on Knodel graphs ([12]).

As per definition brought in previous chapter the Knodel graph is connected only if $\Delta \geq 2$.We will consider Knodel graphs with $\Delta = \lceil log_2 n \rceil$.

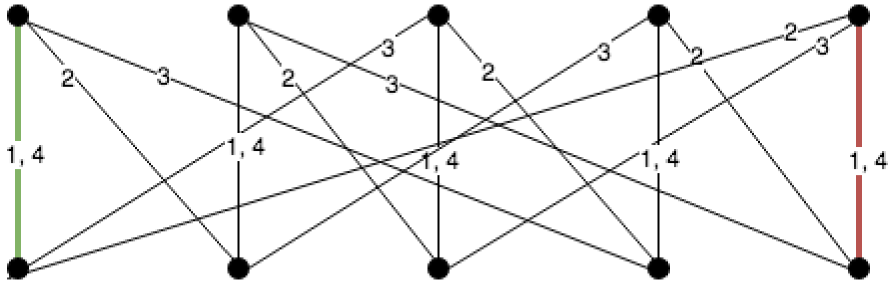

Fig. 5) Gossip graphs constructed based on Knodel graphs ($G = W_{[log_2 n],n} + W_{1,n}$)

It is common knowledge (for example proof see [57]) that the combination of two Knodel graphs with $\Delta_1 = \lceil log_2 n \rceil$ and $\Delta_2 = 1$ is a complete gossip graph (see Fig. 5 above). Let us define the operation $A_l^- = \{P^-(e) | t_G(e) < l\}$ as an operation of permutation of all edges that have lower weight value (or label) then l. After application of A_2^- on Knodel graphs all edges with label 1 will be permuted (see Fig. 6 below).

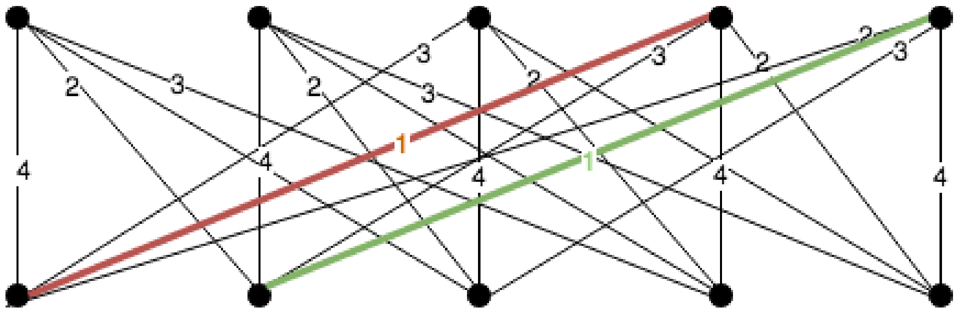

Fig. 6) NOHO graph based on Knodel graph

Lemma: The result of application of A_2^- on complete gossip graph based on Knodel graph is a NOHO gossip graph with minimum gossiping time $T = \lceil log_2 n \rceil$.

Proof. Without loss of generality we can consider only the permutation of an edge that connects vertices (1, 0) and (2, 0). This edge has two adjacent edges with label 2, hence, after A_2^- it will change its position twice. So, after this operation the vertices that connect this edge are (2, 1) and (1, n/2 - 1). These vertices were not connected in our initial graph (according to the definition of Knodel graph), hence, after A_2^- there will be no duplicate edges. On the other hand, here we apply an inverse operation of the operation described in the previous section of case 2. Really, after permutation of this edge the vertices (1, 0), (2, 0), (2, 1) and (1, n/2 - 1) and edges with labels $t_G(e_1) = 1$, $t_G(e_2) = 2$, $t_G(e_3) = 2$ and $t_G(e_4) = \lceil log_2 n \rceil$ will form a tetragon which is not a cycle. Thus, we exclude the possibility of existence of cycles after this operation. Hence, after application of $\boldsymbol{A_2}^-$ we obtain a NOHO graph which is equivalent to our initial complete gossip graph. Moreover, the gossiping time of the obtained graph is obviously $\lceil log_2 n \rceil$, which is the smallest possible gossiping time among all the types of gossip graph, thus the NOHO restriction had no impact on this property.

2.3 Minimum gossip graphs

In the current section the method of construction of Gossip graphs providing a full information exchange with minimal number of calls (2n-4) in minimum possible time is described. The basis for the graphs of the presented class is the subgraph of canonical form obtained from NOHO graphs by applying the operation of local interchange on them (that was already describe in previous sections).

In [66] the class of NOHO graphs was described, which were gossip graphs with n vertices and 2n-4 calls, where no one from participates listens to its own information. Further, in [15], this problem was considered and a wrong inference was made about the structure of the minimum gossip scheme. According to it, all minimum gossip schemes consist of a cube and eight minimum broadcast trees attached to its corner points. Later in [32], this statement was negated and it was shown that the proposed schema was only a particular case of the structure of minimum gossip graphs. In general, the inner kernel of the gossip graph with 2n-4 calls, which were presented in [32] as a poset of the edges of graph and represents their relations (lateral graph) , could be isomorphic to cube, "twisted" cube and p-grid-kernel. In the current section we are going to present different ways of construction of the minimum gossip schemes. In [15] and [32] it was shown that the time required to perform gossiping with minimum number of calls and minimum time (in other words - with minimum number of rounds) is at least $2\lceil log_2 n \rceil - 3$. The methods that allow to construct minimum gossip schemes were also considered that have an underlying basis (a minimum gossip graph) for gossiping, and new vertices participate in gossiping by forming attached trees which are connected to basis. It is obvious, that if the basis has a minimum number of edges, then the full graph would also have the same. Hence, the main problem here is to perform gossiping in minimum time. In other words, the time in which the basis performs gossiping should be minimal and the corresponding attached trees should not affect it. So, our goal is to construct such gossip schemes with variety of size of the basis.

As was mentioned above, we will start from the application of operation A^+ defined in [18] (and described in the previous sections of this chapter) on the NOHO graphs. Let us bring the definition of this operation again.

Definition. Let us define an operation on gossip graphs $A^+(e_1, e_2, \ldots, e_p) = (P^+(e_1)P^+(e_2)\ldots P^+(e_p))$, $A^-(e_1, e_2, \ldots, e_p) = (P^-(e_1)P^-(e_2)\ldots P^-(e_p))$, is the sequence of permute higher (lower) operations on edges $e_i, i = 1, \ldots, p$.

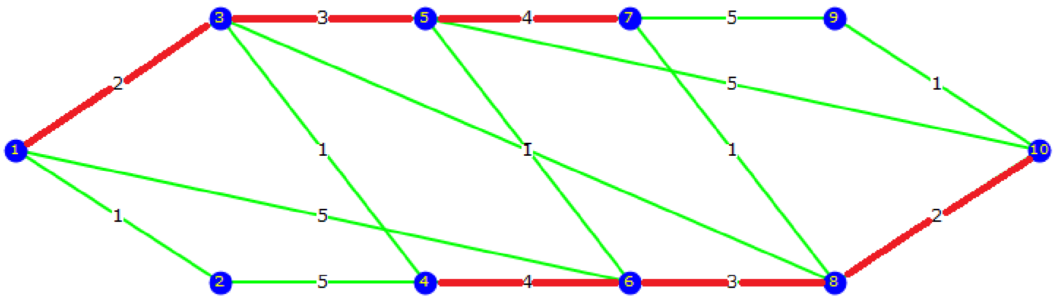

Fig. 7) NOHO graph

An application of the operations $\boldsymbol{A}^{+}(\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_p})$ and $\boldsymbol{A}^{+}(\boldsymbol{e_1}', \boldsymbol{e_2}', \ldots, \boldsymbol{e_p}')$ on NOHO graphs (see Fig. 7 above) gives as new "canonical" form of minimum gossip graph, which is a more convenient form of basis (see Fig. 8 below) for our constructions. Here p = 3 and the edges $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ $(\boldsymbol{e_1}', \boldsymbol{e_2}', \boldsymbol{e_3}')$ are highlighted in red.

After this transformation the obtained graph will be used as a basis with size $k = n'/2$, where n' is the number of vertices of the basis. Now consider the new set of vertices with the same size that communicates with the given set by incoming and outgoing calls. Let the number of rounds by which such communication takes place be denoted by r.

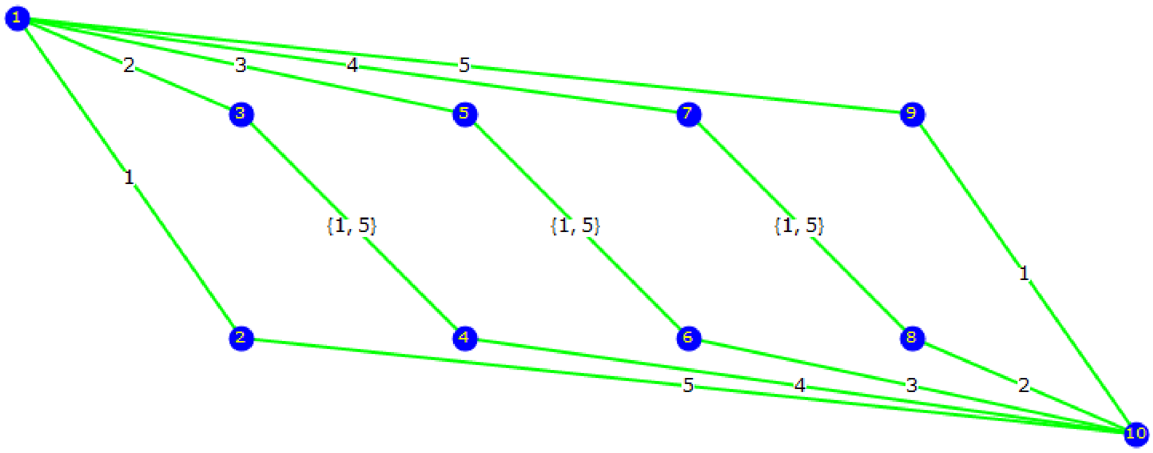

Fig. 8) Minimum gossip graph in “canonical” form

Obviously, the number of rounds to perform gossiping will be at least $2r + k$. This estimate will be reached only if the attached trees would not affect gossiping process of the basis. It means, that the process of involving new vertices in gossiping should be in parallel with the main gossiping process of the basis.

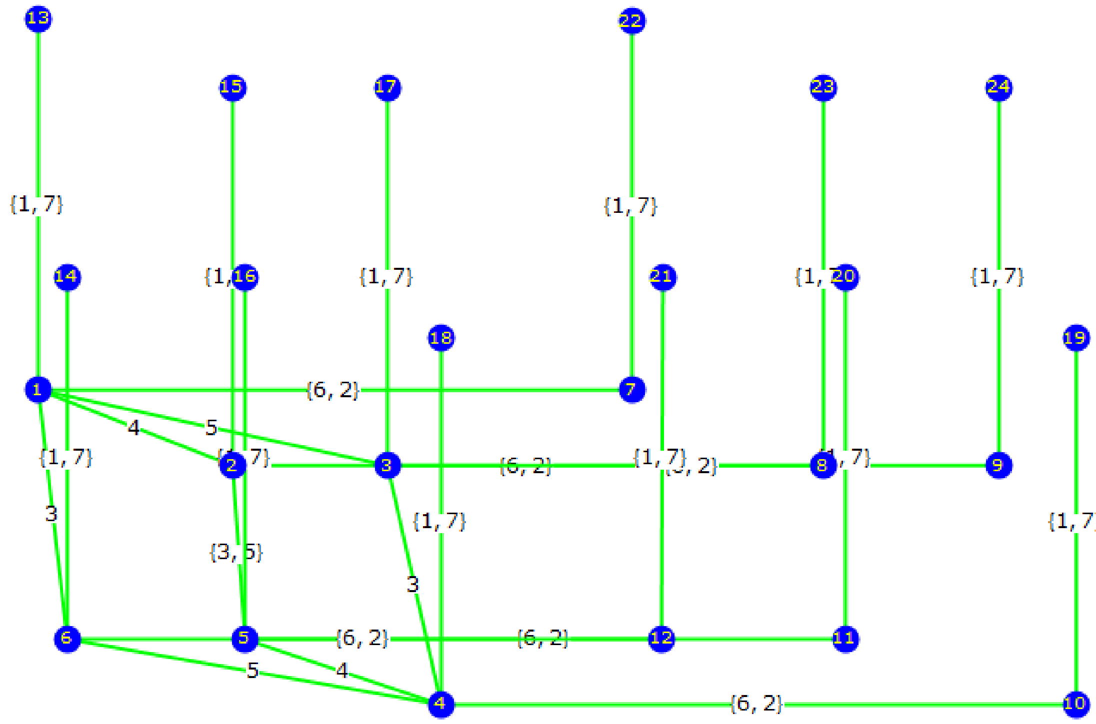

Fig. 9) Minimum gossip graph (n=24, k=3, r=2)

Hence, we have to modify the form of the basis such that we could make in-calls and out-calls with the attached trees in process of gossiping of the basis. For that, we should reorder the calls of the basis, in such a way that it will be possible to make such in-calls and out-calls. In other words, the calls between two parts of gossip scheme should have an increasing and decreasing order. An additional note is that depending on r the number of attached vertices increases exponentially. Such gossiping process for $n = 24$ is demonstrated in the above figure, where $r = 2$ and $k = 3$.

Obviously, the number of rounds required to perform gossiping is $T' = 2r + k$ which coincides with the minimum possible number of rounds for some values of n, but this scheme is not the only possible one to perform gossiping in minimum time. By changing the size of the basis (k) and number of rounds of the "batch" calls (r) a new minimum gossiping scheme will be obtained (see the Fig. 10 below).

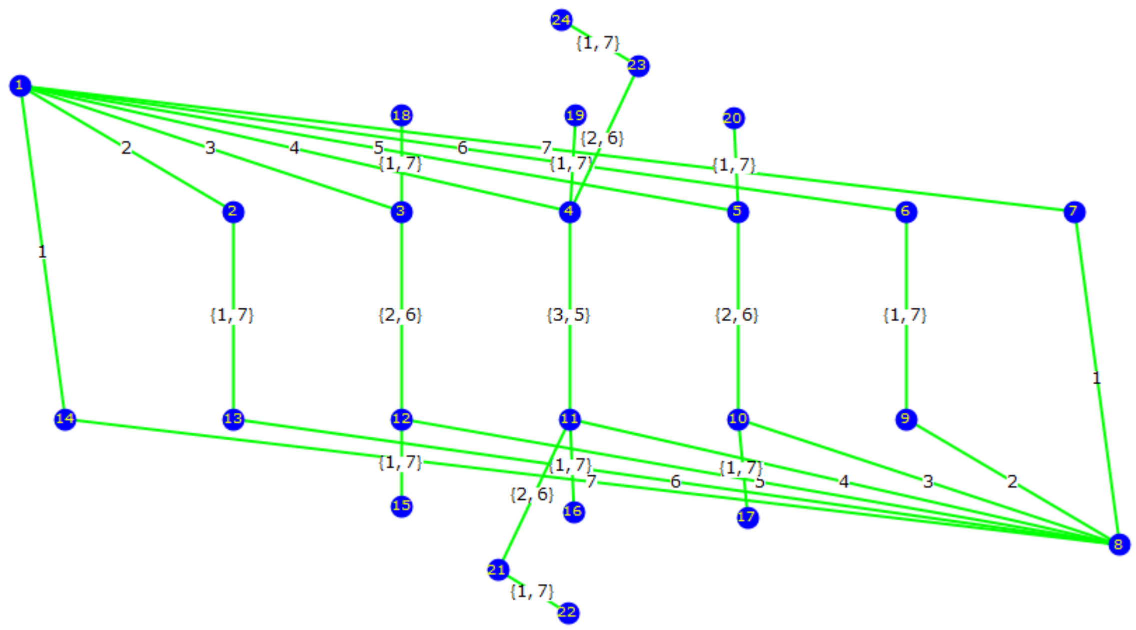

Fig. 10) Minimum gossip graph (n=24, k=7, r=0)

From these examples it is obvious that in each round new vertices are included in communication, but the number of these vertices is limited. So, taking these into account the following lemma could be formulated.

Lemma: The number of vertices in minimum gossip graph with the size of basis k and the number of rounds of the "batch" calls r is limited by the following estimates:

$$n \le 2^{k/2+r+1}, if\ k\ is\ even,$$

$$n \le 3 * 2^{(k-1)/2+r}, if\ k\ is\ odd.$$

Proof. To prove these estimates let us consider the process of gossiping more detailed. Let us divide the number of vertices that could be involved in gossiping into two components n_1 and n_2, where n_1 is the number of vertices which are involved in gossiping process via "batch" calls and n_2 is the amount of vertices involved in gossiping through the attached trees. The dependency between the number of rounds of the "batch" calls r and the n_1 number of vertices is obvious - $n_1 \le 2^r * 2k$. On the other side, it is easy to note that the number of vertices involved in the attached trees is $n_2 = 2^r * (2\sum_{i=2}^{\lceil k/2 \rceil -1}(k-2i)2^{i-2})$

and it also depends on r in exponential rule. Altogether, by considering that $n = n_1 + n_2$ for the number of vertices of the minimum gossip graph we get the following estimate:

$$n \quad \leq 2^r * (2k \ + \ 2\sum_{i=2}^{\lceil k/2 \rceil - 1}(k - 2i)2^{i-2}).$$

So, from this expression the estimates mentioned in lemma immediately follow. □

However, in case of even k it is possible by changing the structure and gossiping time of basis to reach more vertices involved in gossiping. The example given in figure below demonstrates this for $k = 8$. According to lemma the number of vertices in this case is $n \leq 32$, but we increase the number of rounds of basis by one and change the direction of the "middle" calls and it gives us the opportunity to involve 8 new vertices in this modified “twisted” gossiping scheme. In fact, our modification has changed the structure of the standard p-grid-kernel to one of its linear extension (see. [32]) and the number of new vertices involved in gossiping process after this modification is $2^{k/2+r-1}$.

Now let us consider the number of rounds which are necessary to perform gossiping with minimum number of calls. Let it be denoted by T. As it was already mentioned $T = 2\lceil log_2 n \rceil - 3$. Our goal is to show the dependency between T and n depending on the construction of gossiping scheme. In other words, we offer the method of construction of minimum gossip graphs for the particular values of n, where the values of k and r can vary. Let T' be the minimum number of rounds of the gossip scheme with $2n - 4$ calls, the size of the basis k and the number of rounds of "batch" calls r.

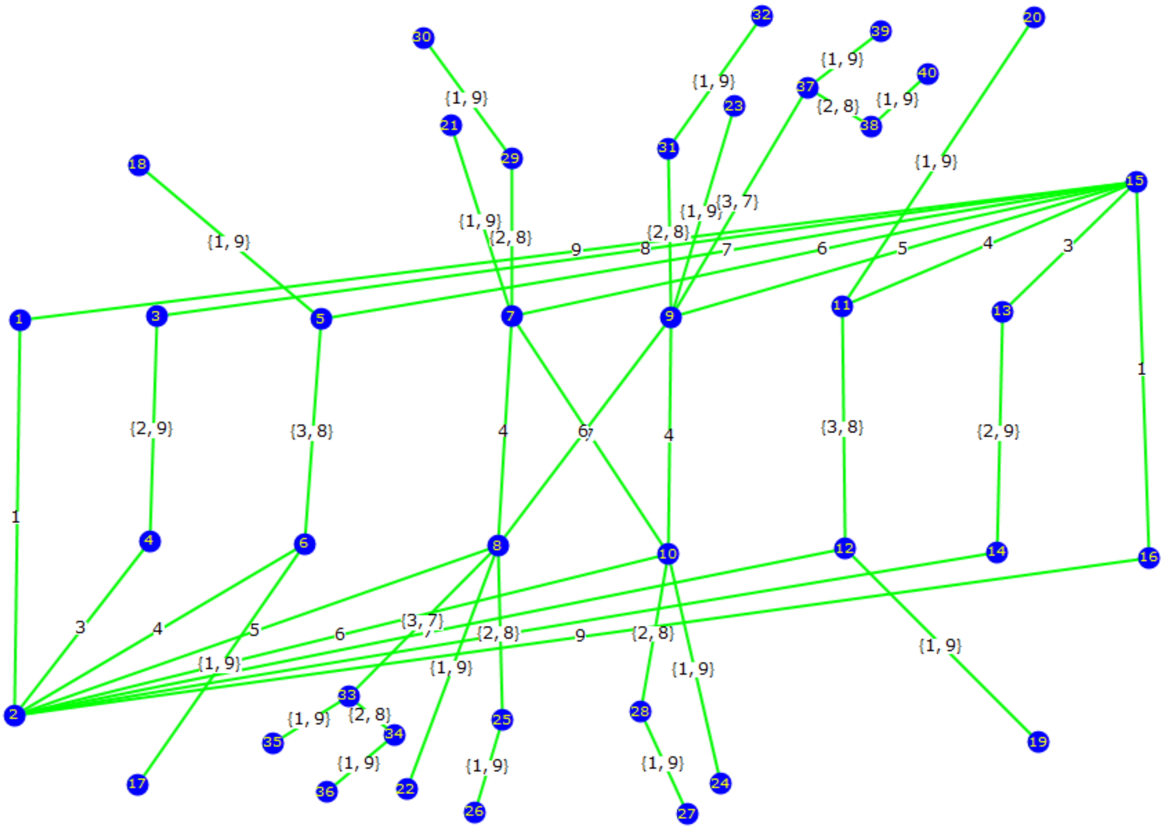

Fig. 11) Minimum gossip graph (n=40, k=8, r=0)

Theorem: In gossiping scheme with the size of the basis k and the number of rounds of the "batch" calls r the number of rounds necessary to complete gossiping among n nodes is:

$$T' = 2\lceil log_2 n \rceil - 2, if\ k\ is\ even,$$

$$T' = 2\lceil log_2 \frac{n}{3} \rceil + 1, if\ k\ is\ odd.$$

Proof. In case of even k we have $n \leq 2^{\frac{k}{2}+r+1} = 2^{\frac{T'}{2}+1}$, since $T' = 2r + k$, it follows that when n has the form as above then $T' = 2\lceil log_2 n \rceil - 2$. When k is odd $n \leq 3 * 2^{\frac{k-1}{2}+r} = 3 * 2^{\frac{T'-1}{2}}$, hence $T' = 2\lceil log_2 \frac{n}{3} \rceil + 1$. Moreover, by considering the "extended" values of n obtained after modification described above (when it is even) we have $n \leq 2^{\frac{k}{2}+r-1} + 2^{\frac{k}{2}+r-1}$.

Since, here $T' = k + 1 + 2r$ and $T' = 2\lceil log_2 \frac{n}{5} \rceil + 3$ follows immediately. It is obvious, that for some values of n, T' coincides with minimum possible number of rounds $2\lceil log_2 n \rceil - 3$. Hence, in these ranges of n the values of k and r can vary and give different construction options of minimum gossip graphs. In the below figure some of these ranges are demonstrated, the construction is minimum in the following ranges ranges: {[5-6], [9-12], [17-24], [33-48], [65-96],[129-192], [257-384], [513-768], ...}.

2.4 Gossiping properties of the edge-permuted Knodel Graphs

In this section we are discussing the Knodel graphs and their main gossiping properties. As it is known, the Knodel graphs of order 2^k, and degree k, are one of the three most known families of the interconnection networks, along with the hypercube of dimension k, H_k ([76]), and the recursive circulate graph $G(2^k, k)$ ([77]). These 3 kind of networks are commonly used as an underlying topology for computer networks, and are similar because of the same number of vertices and degree. However, in this section we are also going to present gossiping properties of Knodel graphs with the order other than 2^k.

Here we first survey a graph-theoretical results known about the Knodel graphs, then we focus on gossiping and broadcasting properties of these graphs. In the next subsections we are trying to present our research regarding the gossiping properties of the so-called "edge-permuted" Knodel graphs. As the name suggests, these are the modification of the Knodel graphs that result to a cyclic permutation of the edge weights. It appears that these graphs also have interesting properties in terms of gossiping.

Knodel graphs have been first introduced in 1975 in [33], where they were used for time optimal gossip graph construction. As a graph family they were introduced about 20 years later by Fraigniaud and Peters in [78]. As already was defined, these graphs are regular, symmetric graphs of even order n and degree $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$. It is known that for any $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$ Knodel graph $W_{\Delta,n}$ is a vertex-transitive, edge-transitive (and, thus, symmetric) and for any even n and $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$

- edge-connectivity of $W_{\Delta,n}$ is $\lambda(W_{\Delta,n}) = \Delta$,
- vertex-connectivity of $W_{\Delta,n}$ is $2\Delta/3 \leq \kappa(W_{\Delta,n}) \leq \Delta$.

It is also known that Knodel graphs have minimum possible broadcasting and gossiping time:

$$b(W_{\lfloor \log_2 n \rfloor,n}) = g(W_{\lfloor \log_2 n \rfloor,n}) = \lceil log_2 n \rceil.$$

As we have already mentioned in this work we consider only the unit cost gossiping model, where each call takes exactly 1 unit of time. But the Knodel graphs themselves serve as an optimal topology for liner-cost gossip as well. In this mode the time that the communication between two u and v vertices consists of two parts: fixed start-up time β and a propagation time L_τ. Later is proportional to the length L of the message exchanged.

In the following table we are summarizing the known gossiping and broadcasting properties of various Knodel graphs, both in a linear-cost and unit-cost modes:

$W_{k,2^k}$	Minimum Time Broadcast Graph Minimum Time Gossip Graph Minimum Time Linear Gossip Graph
$W_{k-1,2^k-2}$	Minimum Time Broadcast Graph Minimum Time Gossip Graph Minimum Time Linear Gossip Graph
$W_{k-1,2^k-4}$	Minimum Time Gossip Graph Minimum Time Linear Gossip Graph
$W_{k-1,2^k-6}$	Minimum Time Linear Gossip Graph
$W_{k-2,n}$ $2^{k-1} + 2 \leq n \leq 3 * 2^{k-2} - 4$	Broadcast Graph Linear Gossip Graph

	Gossip Graph
$W_{k-1,n}$ $3 * 2^{k-2} - 4 \leq n \leq 2^k - 2$	Broadcast Graph Linear Gossip Graph Gossip Graph

One important note here, when referring to the phrase "Minimum Time" in the above table we intend to refer to graphs with the minimum possible gossiping and broadcasting times and, in the same time ($\lceil log_2 n \rceil$), with minimum possible number of edges under the previous condition.

Additionally, apart from the above results where were only considered undirected Knodel graphs, there are also known results that proof the optimal characters of the directed Knodel graphs as an underlying gossip/broadcast topology. Remember that in this mode the links between the vertices are only one-way (half-duplex mode) and the information can only flow in one direction.

As already was mentioned our study is all about the edge-permuted Knodel graphs. In case of these graph the dimensions of the edges are used in a different way, thus let us bring here the definition of these graphs.

Definition. The edge-permuted Knodel graph with even number of vertices n and the degree $1 \leq \Delta \leq \lfloor log_2 n \rfloor$ is denoted by $M_{\Delta,n}(p)$. The vertices of $M_{\Delta,n}(p)$ are pairs (i, j); $i = 1, 2$; $0 \leq j \leq n/2 - 1$, where for each j and $l = 1, \ldots, \Delta$ there exists an edge in dimension l between the vertices $(1, j)$ and $(2, (j + 2^{p+l-1} - 1) \: mod \: n/2)$, where p is any integer in the range $p \in [0, \Delta - 1]$.

In the following subsections we will consider 3 different cases based on the number of vertices. We are going to present a new method of proof regarding the gossiping properties of these graphs for some of these cases (since they were already discussed in [73]), as well as provide the complete proof for general even number of vertices.

2.4.1 Edge-permuted Knodel Graphs with $n = 2^k - 2$ Vertices

In this section we are going to consider the modified Knodel graphs with the number of vertices equal to $n = 2^k - 2$, where k is any integer $k \geq 3$. According to the definition of modified Knodel graphs, there exist $\Delta - 1$ modifications for each $W_{\Delta,n}$. As already was mentioned in this research we consider only the case when $\Delta = \lfloor log_2 n \rfloor$, and this value is assumed for all references of Δ onwards.

Defnition. Consider two graphs $G_1 = (V; E_1)$ and $G_2 = (V; E_2)$ with the same set of vertices V and labeled edge sets E_1 and E_2, respectively. The edge sum of these graphs is a graph $G_1 + G_2 = G = (V; E)$ with $E = E_1 \cup E_2$, whose edges $e \in E$ are labeled by the following rules:

$$t_G(e) = \begin{cases} t_{G_1}(e), & if\ e \in E_1 \\ t_{G_2}(e) + \max\limits_{e' \in E_1} t_{G_1}(e'), & if\ e \in E_2 \end{cases}$$

where $t_G(\mathrm{e})$ is tha label of the edge e in the resulting graph G.

It is known that if $G = W_{\Delta,n} + W_{1,n}$, then G is a complete gossip graph. Let us first consider the graph $G' = M_{\Delta,n}(p) + M_{1,n}(p)$, $p = \Delta - 1$ and $n = 2^k - 2$.

Theorem. The graph G' is a complete gossip graph.

Proof. Since this graph is symmetric, it is enough to show that there exists an ascending path from any other vertex to the vertex $(0, 1)$. Let us consider the subgraph of G', $SUB_{G'}$, which includes the vertices (i, j), where $i = 1, 2$ and $1 \leq j \leq \Delta$. Let us particularly focus on an edge in dimension 1 between these and $(1, j')$ vertices, where $2^\Delta \leq j' \leq n/2$. The same is true about the vertices $(1, j)$ and $(0, j')$, where $1 \leq j \leq 2^\Delta$ and $2^\Delta \leq j' \leq n/2$. It is obvious that these edges connect all the remaining $2^k - 2 - 2^\Delta = 2^{\Delta+1} - 2^\Delta - 2 = 2^\Delta - 2$ vertices to the

$SUB_{G'}$. On the other hand, we have that $SUB_{G'}$ is a tree rooted at the vertex $(0,1)$. Thus, there exist ascending paths between every vertex of $SUB_{G'}$ and the vertex $(0,1)$. Considering the fact that each edge in the $SUB_{G'}$ has dimensions greater than 1, it follows that there exists an ascending path between any other vertex and the vertex $(0,1)$. The two facts that the choice of the vertex $(0,1)$ was arbitrary, also the modified Knodel graph being symmetric by construction, point out to complete gossiping of G'.

Corollary. The graph $G = M_{\Delta,n}(p) + M_{1,n}(p),\ 0 \le p \le \Delta - 1$ and $n = 2^k - 2$ is a gossip graph.

Proof. It is an easy exercise to verify that $M_{\Delta,n}(\lfloor log_2 n \rfloor - 1)$ is isomorphic to $W_{\Delta,n}$ (for detailed proof refer to [73]). Thus it is obvious that G' is isomorphic to $W_{\Delta,n} + W_{1,n}$. Hence, we can repeat this procedure, and each time we will obtain a new $M_{\Delta,n}(p) + M_{1,n}(p),\ p = \Delta - 1, \Delta - 2, \dots, 1$, that are isomorphic to each other and to initial $W_{\Delta,n} + W_{1,n}$, in case $n = 2^k - 2$.

2.4.2 Edge-permuted Knodel Graphs with $n = 2^k$ Vertices

In this section we consider modified Knodel graphs with 2^k vertices. Knodel graphs with 2^k vertices are diferent in terms of gossiping since $W_{\Delta,n}$ is actually a gossip graph. Hence, in this case there is no need to add $W_{1,n}$ to get a complete gossip graph. To start with, let us consider $M_{\Delta,n}(2)$.

Theorem. The graph $M_{\Delta,n}(2)$ is a complete gossip graph.

Proof. Let us start the proof by applying the following transformation on the vertex set of this graph V. The elements of this set are the pairs (i,j). In case of $i = 1$, if j is odd, then

$j_{new} = \lceil j/2 \rceil$ and $j_{new} = \frac{n}{2} + \lfloor j/2 \rfloor$, otherwise. If $i = 2$, the rules are as follows: $j_{new} = j/2$ in case j is even, and $j_{new} = \frac{n}{2} + \lfloor j/2 \rfloor$ in case of odd j.

We obtain two components that are Knodel graphs with the number of vertices equal to $n/2$, and these components are connected by the edge of label Δ. Therefore, if we simply remove these edges, then there will be two disjoint components which are Knodel graphs by themselves. And there is a one-to-one call mapping between the vertices of these two components with the calls labeled by $\log_2 n$ (highest labeled edges). Obviously, this graph is a complete gossip graph.

Theorem. The graph $M_{\Delta,n}(p)$, where $p \neq 2$, is not a gossip graph.

Proof. To start with the proof, let us recall our method for modifying gossip graphs (method of local interchange [18]). The operations permute higher and permute lower were introduced to modify the topology of the graph without changing its main gossiping properties (number of edges, number of rounds, etc.).

Let us denote the subset of the edge set of graph M with the weight $w_{fix} = \log_2 n - i + 2$ by E', and its elements by e_l', where $l = 1, 2, \dots, n/2$. The operation $A^{-}_{e_l'}$ should apply the "permute lower" operation on all the edges of E' resulting in a completely new graph. There would obviously be duplicate (double) edges in this graph, since the edges with weights $w_{fix} - 1$ and $w_{fix} + 1$ would connect the same vertices. Hence, taking into consideration the fact that gossiping time, i.e., the number of rounds required to complete gossiping in $W_{\Delta,n}$ was $\log_2 n$ (which is the most optimal possible value of this property), the graph $M_{\Delta,n}(p)$ is proved to be a non-gossip graph.

2.4.3 General case of edge-permuted Knodel Graphs with n not equal 2^k

In this section we are going to focus on the general case of the number of vertices being not equal to 2^k. We are going to show that, generally, the modified Knodel graphs show the same gossiping properties as the initial Knodel graphs, although in general they are not isomorphic graphs. Additionally, the results obtained in this section will let us to construct edge-disjoint ascending (sometimes, with folds) paths between any pairs of vertices in the Knodel graph.

First, let as recall some of the important definitions that were already presented in the work here, as well as give several new definitions.

Definition. A path $\mathrm{P} = (\mathrm{u}, e_1, v_1, e_2, v_2, \ldots, e_k, \mathrm{v})$ between vertices u and v in a weighted graph is said to be ascending if for each e_i and e_j $t_G(e_i) < t_G(e_j)$ for $1 \leq i < j \leq k$.

It is known that if $G = W_{\Delta,n} + W_{1,n}$, then G is a complete gossip graph.

From now on, when mentioning the vertex $\mathrm{u} = (i_1, j_1)$, we assume that $i_1 = 1$ and $j_1 = 0$. Since the Knodel graphs are symmetric, the same assumptions or statements are true for every other vertex $u' = (1, j')$.

Definition. The distance from the vertex $\mathrm{u} = (i_1, j_1)$ to the vertex $\mathrm{v} = (i_2, j_2)$ of the Knodel graph where $(j_1 + 2^{\lfloor \log_2 n \rfloor}) mod\ n/2 \geq j_2$ can be calculated by the following formula:

$$D_{u,v} = |i_1 - i_2| + |j_1 - j_2|$$

It is obvious that for each vertex u of the graph there exist two vertices $v_1 = (i_1, j_1)$ and $v_2 = (i_2, j_2)$, such that $D_{u,v_1} = D_{u,v_2}$ and $i_1 \neq i_2$. Let us consider the vertices v_1 and v_2 to be “clones” of each other. As we have already introduced the modified Knodel graph and the distance between its vertices, we can now define a “base” for this graph.

Definition. The “base” of the modified Knodel graph, relative to the vertex $\mathrm{u} = (i_1, j_1)$, are all the vertices v, such that $0 \leq D_{u,v} \leq \lfloor \log_2 n \rfloor$.

Let the vertex u be called the “root” of the base.

Let us now consider the path between the vertex u and some vertex v that belongs to the “base” of the modified Knodel graph relative to the vertex u, in the original Knodel graph. For that purpose, first we represent the distance defined above in the base-two system.

Next, we transform the base-two representation of the distance into a sum that will result to the original distance (in base-ten system), also to a path representation between two vertices.

Definition. Let $Path_{u,v}$ be equal to the sum resulting in $D_{u,v}$, the terms of which are represented by different powers of two:

$$Path_{u,v} = \sum_{l=0}^{k} s_l * (2^{l+1} - 2^l),$$

where $k = D_{u,v}$ and $s_l = 0$ or 1, depending on the base-two representation of the $D_{u,v}$.

By simplifying the above expression, we get a new expression for the Path, as follows: $Path_{u,v} = 2^{k+1} - 2^k + 2^{k-1} \dots \pm 2^0$. Note, that it can appear that the $Path_{u,v}$ presents the path between the vertex u and the clone of the vertex v. Will come back to this case and discuss it separately.

Let us now define the transformation on the $Path_{u,v}$ that will result in a completely new $Path_{u,v}$ which will show the ascending path between any pairs of vertices u' and v, where u' is the vertex different from the original destination vertex u and is located inside the "base" of the edge-permuted Knodel graph.

Definition. The operation $T(Path_{u,v}) = T(2^{k+1} - 2^k + 2^{k-1} \dots \pm 2^m) = Path_{M(u,2^m),v} = (Path_{u,v} - 2^m)$, where $Path_{u,v} - 2^m$ is a new path obtained as a result of the transformation T on the $Path_{u,v}$, $\mathrm{M(u, 2^m)}$ is a new vertex such that $D_{u,,\mathrm{M(u,2^m)}} = 2^m$, where m is the exponent of the smallest member of the $Path_{u,v}$.

The figure below illustrates the result of the application of the operation T on the highlighted path of the edge permuted Knodel graph that results in a new path between the vertices 1 and 6.

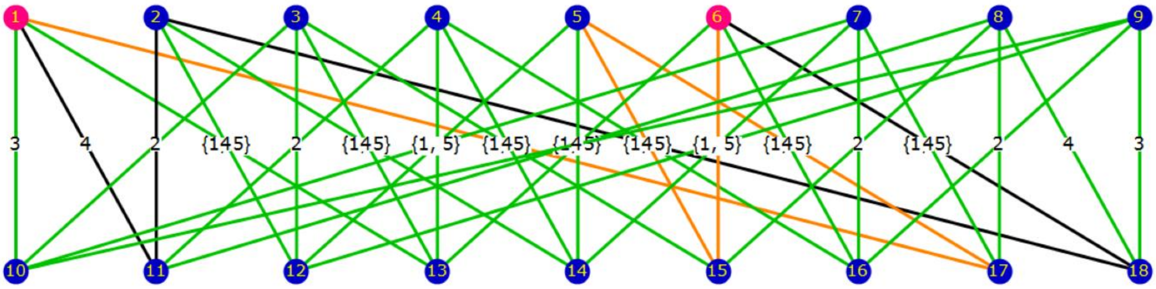

Fig. 12) Path transformation

We will apply the transformation T on the 1-folded ascending path $6 \to 1$ (original definition is as follows: $Path_{1,6} = (2^3 - 2^2) + (2^1 - 2^0)$ in order to find a new path that will have no folds at all. Since $T(Path_{1,6}) = Path_{2,6} = 2^3 - 2^2 + 2^1 - 2^0 - 2^0 = 2^3 - 2^2$, the obtained path has no folds in the edge-permuted graph, and presents the path between the vertices 6 and 2 (colored black in the above figure). Obviously, there exists an ascending path in the "base" of the edge permuted Knodel graph that connects the vertices 2 and 1 and has higher weights than that of the edges of the path $6 \to 2$. Thus, we can insist that the concatenation of these two paths is an ascending path between the vertices 6 and 1.

As already was mentioned, here we consider the edge-permuted Knodel graphs (or the Modified Knodel graphs) and show that the gossiping properties of these graphs remain the same after the edge permutation. The main goal here is to show that if there is an ascending path between the given two vertices in the original Knodel graph, then there should be such a path in the edge-permuted version of that graph. This is obtained by showing that there exists at most 1-folded ascending path from every vertex to a given vertex (1, 0) in the edge-permuted Knodel graph.

Lemma. There exists an ascending path between any two vertices u and v ($v \to u$) of the Knodel graph, in case $D_{u,v} \leq 2^{\lfloor log_2 n \rfloor}$.

Proof. It is known that Knodel graphs have descending tree rooted at each node $\mathrm{u} = (i_1, j_1)$ and containing each node $\mathrm{v} = (i_2, j_2)$, where $\left(j_1 + 2^{\lfloor \log_2 n \rfloor}\right) mod \frac{n}{2} \geq j_2$. Since $D_{u,v} \leq 2^{\lfloor \log_2 n \rfloor}$, it follows that the vertex v belongs to a descending tree rooted at u, and hence, there exists an ascending path from v to u.

Lemma. There exists at most 1-folded ascending path between any two $u = (i_1, j_1)$ and $v = (i_2, j_2)$ vertices ($v \to u$) of the edge-permuted Knodel graph in case $\left(j_1 + 2^{\lfloor log_2 n \rfloor}\right) mod \frac{n}{2} > j_2$.

Proof. There exists a path in the edge-permuted Knodel graph between any pairs of vertices $\mathrm{u} = (i_1, j_1)$ and $\mathrm{v} = (i_2, j_2)$ $(v \to u)$ in case $\left(j_1 + 2^{\lfloor \log_2 n \rfloor}\right) mod \frac{n}{2} > j_2$, which is either 0 or 1 folded, depending on the position of the vertex v. Note, the path is the same as in the original Knodel graph, only the weights of edges are different. Since we have increasing order in the new "base" of the edge-permuted Knodel graph, we also have 0 folded ascending path between any vertex of the "base" and the vertex u. If the vertex v is located outside the new "base" of the edge-permuted Knodel graph, but at the same time, $\left(j_1 + 2^{\lfloor \log_2 n \rfloor}\right) mod \frac{n}{2} > j_2$, then there exists at most 1-folded ascending path connecting the vertices v and u. The folding point of this path is located outside the "base" of the edge-permuted Knodel graph, but inside the range $\left(j_1, \ (j_1 + 2^{\lfloor \log_2 n \rfloor}\right) mod \frac{n}{2})$. Indeed, in case we have 1-folded ascending path, we can classify the edges of this path into two groups:

1) the weights of the edges belong to the "base" of the graph,

2) otherwise.

Obviously, the point where the two sub-paths of the initial path are joined is also a folding point for the whole path. In the example below, the vertex 15 is the folding point for the path $6 \rightarrow 1$. Finally, by applying the transformation T to the path, we will be able to find a corresponding ascending path (without any folds) between these two vertices.

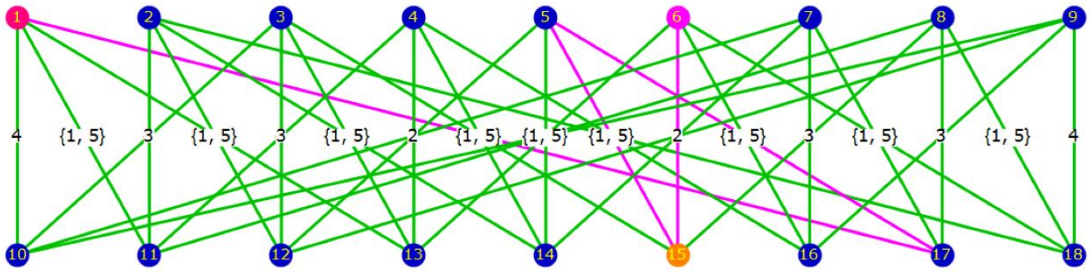

Fig. 13) Folding point of the path $6 \rightarrow 1$

Lemma. There exists an ascending path from any vertex in the base to the root vertex.

Proof. The above lemma follows from the fact that we have an ascending edge order in the scope of the “base” (like in the original Knodel graph), so there exists an ascending tree rooted at the root vertex of the base and containing all the vertices of the base.

Let us now return to the $Path$ defined in the previous section and give the modification that will make it possible to transform the $Path$ in a way to obtain the path to the “clone” vertex of the source vertex (since in this scope we consider descending paths). Let us denote the result of this modification by $Path'$. We can assume that $Path$ presents a descending path from the vertex u to the vertex v_1, so from the notation above, $Path_{u,v_1} = Path'_{u,v_2}$, where $D_{u,v_1} = D_{u,v_2}$. Firstly, assume that $Path_{u,v_1}$ has the following form: $2^{k+1} - 2^k + 2^{k-1} \ldots \pm 2^m$.

Lemma. $\boldsymbol{Path'_{u,v_2} = Path_{u,v_1} \pm (2^m - 2^m)}$ ***shows the descending path from the vertex*** $\boldsymbol{u}$ ***to the vertex*** $\boldsymbol{v_2}$***.***

Proof. The above lemma is obvious, since after applying the modification described, the last component of the $Path'_{u,v_2}$ gets an inverse sign, comparing with that of $Path_{u,v_1}$, hence the last vertex is in the opposite side of the obtained bipartite graph, also the absolute values $Path_{u,v_1}$ and $Path'_{u,v_2}$ are equal, and therefore, the distances of v_1 and v_2 from the vertex u are also equal.

Theorem. The graph $G = M_{\Delta,n}(p) + M_{1,n}(p)$, $0 \leq p \leq \Delta - 1$ and $n \neq 2^k$, is a gossip graph.

Proof. From the lemmas above it follows that there exists an ascending path between any two vertices $u = (i_1, j_1)$ and $v = (i_2, j_2)$ in the edge-permuted Knodel graph in case $\left(j_1 + 2^{\lfloor \log_2 n \rfloor}\right) mod \frac{n}{2} > j_2$. The only case, left out from the consideration, is when the expression $\left(j_1 + 2^{\lfloor \log_2 n \rfloor}\right) mod \frac{n}{2} > j_2$ does not hold true. For those cases we should consider the graph $M_{1,n}(p)$ and the vertex $u' = (i_1', j_1')$, which is connected to the vertex u by the edge $e' \in E(M_{1,n}(p))$. Since $t_{M_{\Delta,n}(p)+M_{1,n}(p)}(e') > t_{M_{\Delta,n}(p)+M_{1,n}(p)}(e)$, $e \in E(M_{\Delta,n}(p))$, it follows that e' is the final edge (call) of all the ascending paths going to u through the vertex u'. In the similar fashion as we showed above, it can be shown the existence of an ascending paths from every vertex for which $\left(j_1 + 2^{\lfloor \log_2 n \rfloor}\right) mod \frac{n}{2} < j_2$ to the vertex u'. All of these paths then will end in the vertex u through the final edge e'.

2.5 Fault-tolerant gossip graphs based on Wheel graphs

In the current section we are going to focus on fault-tolerant gossip problem. We are going to present a new method of construction of fault-tolerant gossip graphs based on so-called Wheel graphs.

We have already brought the definitions of path, ascending path, edge sum and several other concepts in the section 1.4 of this study. We also did compressional analysis of currently known results in the sphere of fault-tolerant gossiping in section 1.3. Here we will give several definitions regarding fault-tolerance and concepts used in this particular research area.

Definition. The communication between two vertices of G is called k-failure safe if an ascending path between them remains, even if arbitrary k edges of G are deleted (the corresponding calls fail). The graph G is called a k-fault-tolerant gossip graph if the communication between all the pairs of its vertices is k-failure safe.

From the Menger theorem it follows that a k-fault-tolerant gossip graph is a graph whose edges are labeled in such a way that there are at least k+1 edge-disjoint ascending paths between two arbitrary vertices. A 0-fault-tolerant gossip graph is simply called a gossip graph.

To describe the construction of k-fault-tolerant gossip graphs (schemes), we use some of the important definitions and propositions given in the section 1.4 and in the works [37], [36]. In order to simplify the discussion for edge-disjoint paths, we often omit the vertices (or edges) in the description of a path in case there is no confusion.

Let consider the multigraph hG (defined in the section 1.4). Let $P = P^{(1)} \odot P^{(2)} \odot \dots \odot P^{(s+1)}$ be an s-folded ascending path from a vertex u to a vertex v in G, where $P^{(j)}$ is an ascending sub-path for $1 \leq j \leq s+1$. Then, P_i is also an s-folded ascending path and $P_i = P_i^{(1)} \odot P_i^{(2)} \odot \dots \odot P_i^{(s+1)}$ for $1 \leq i \leq h$. Now consider the path $P(k) = P_k^{(1)} \odot P_{k+1}^{(2)} \odot \dots \odot P_{k+s}^{(s+1)}$ in hG. Then, $P(k)$ is an ascending path from u to v for $1 \leq k \leq h-s$ such that $P(k)$ and $P(k')$ are edge-disjoint if $k \neq k'$. Thus, based on P, we can construct $(h-s)$ edge-disjoint ascending paths from u to v in hG. Similarly, based on another s-folded ascending path P' from u to v, we can construct $(h-s)$ edge-disjoint ascending paths $P'(k)$ from u to

v for $1 \leq k \leq h - s$. If P and P' are edge-disjoint, then all the paths $P(1), \dots, P(h-s)$ and $P'(1), \dots, P'(h-s)$ are pairwise edge-disjoint by construction. Therefore, the following lemma holds (see [37], [36]).

Lemma. Let u and v be vertices in a labeled graph G. If there are p edge-disjoint s-folded ascending paths from u to v in G, then there are $p(h-s)$ edge-disjoint ascending paths from u to v in hG for any integer $h \geq s$.

From this lemma, if there are p edge-disjoint s-folded ascending paths from u to v in G, then there are $k+1$ edge-disjoint ascending paths from u to v in $(s + \lceil \frac{k+1}{p} \rceil)G$.

Thus, the following corollary is obtained (see [37]).

Corollary. Let G be a graph with n vertices and m edges. If there are p edge-disjoint s-folded ascending paths between every pair of vertices in a labeled graph G, then $\tau(n,k) \leq (s + \lceil \frac{k+1}{p} \rceil)m$.

In order to improve this estimation of the upper bound, a stronger proposition is formulated and proved in [36], which we will call throughout this work as “the theorem of fault tolerance”.

Theorem. Let G be a labeled graph with n vertices. Suppose that

- $E(G)$ can be decomposed into l subsets $F^{(0)}, F^{(1)}, \dots, F^{(l-1)}$ such that for any two edges $e \in F^{(i)}$ and $e' \in F^{(j)}$, $t_G(e) < t_G(e')$ if $i < j$,
- for any two vertices u and v, there are p edge-disjoint paths from u to v such that the sum of their folded numbers is at most q, and the last edges of r_i paths are in $F^{(i)}$ for $0 \leq i \leq l-1$.

Then, the minimal number of edges in a k-fault-tolerant gossip graph is bounded

$$\tau(n,k) \leq \partial(n,k),$$

with $\partial(n,k)$ defined by the expression

$$\partial(n,k) = \sum_{0\le i\le \omega} |F^{(i \bmod l)}|,$$

where ω is an integer satisfying

$$\sum_{0\le i\le \omega} r_{i \bmod l} \ge k+q+1.$$

During the proof, the graph $\tilde{G} = hG + G'$ with $h = \left\lfloor \frac{\omega}{l} \right\rfloor$ and $G' = (V, \bigcup_{0\le i\le \omega-hl} F^{(i)})$ is constructed, and showed that it is a k-fault-tolerant gossip graph. The number of edges of this graph is $|E(\tilde{G})| = \sum_{0\le i\le \omega} |F^{(i \bmod l)}|$.

Now we are going to apply the above theorem on the Wheel graphs to improve the known estimations of the upper bounds for $\tau(n,k)$.

Consider the graph $G = (V,E)$ with an odd number of vertices n, whose vertices and edges are labeled by the following rules: the label of the central vertex is u, the remaining vertices (which are located on the circle) are labeled consequently $v_1, v_1', v_2, v_2', \dots, v_m, v_m{}'$, where $n = 2m+1$. Since the periodic boundary conditions are assumed, we identify the vertices $v_{i\pm m} \equiv v_i$ and $v'_{i\pm m} \equiv v_i{}'$ for $i = 1,2,\dots,m$. The set of edges consists of three subsets

$$E(G) = F^{(0)} \cup F^{(1)} \cup F^{(2)}$$

with

$$F^{(0)} = \{(v_i, v_i'):\ t_G\big((v_i, v_i')\big) = 1, i = 1, 2, \dots, m\},$$

$$F^{(1a)} = \{(v_i{}', u):\ t_G((v_i{}', u)) = 2, i = 1, 2, \dots, m\},$$

$$F^{(1b)} = \{(v_i, u):\ t_G((v_i, u)) = 3, i = 1, 2, \dots, m\},$$

$$F^{(1)} = F^{(1a)} \cup F^{(1b)}$$

$$F^{(2)} = \{(v_i', v_{i+1}): t_G\big((v_i', v_{i+1})\big) = 4,\ i = 1,2,\dots,m\}.$$

The figure below illustrates the wheel graph G for $n = 11$ vertices. Then we apply the theorem of fault-tolerance to this graph. For all pairs of vertices in G, we first construct 3 edge-disjoint folded paths from the first vertex to the second one. For $i,j = 1,2,\dots,m$ and $j \ne i, i-1, i+1, i+2$ we have

- from v_i to u
  - $v_i \overset{3}{\rightarrow} u$

- $v_i \xrightarrow{1} v_i' \xrightarrow{2} u$
- $v_i \xrightarrow{4} v_{i-1}' \xrightarrow{2} u$

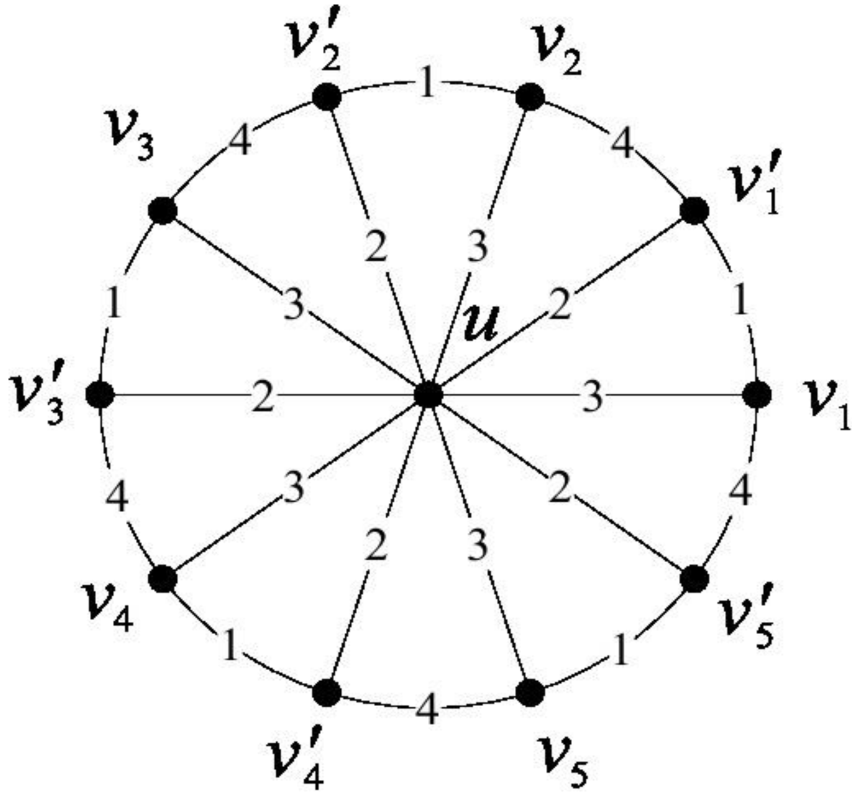

Fig. 14) Wheel graph for odd n (here n=11)

- from u to v_i'
  - $u \xrightarrow{2} v_i'$
  - $u \xrightarrow{3} v_i \xrightarrow{1} v_i'$
  - $u \xrightarrow{3} v_{i+1} \xrightarrow{4} v_i'$
- from v_i to v_j
  - $v_i \xrightarrow{3} u \xrightarrow{2} v_{j-1}' \xrightarrow{4} v_j$
  - $v_i \xrightarrow{1} v_i' \xrightarrow{2} u \xrightarrow{3} v_{j+1} \xrightarrow{4} v_j' \xrightarrow{1} v_j$
  - $v_i \xrightarrow{4} v_{i-1}' \xrightarrow{2} u \xrightarrow{3} v_j$
- from v_i to v_j'
  - $v_i \xrightarrow{3} u \xrightarrow{2} v_j'$

- $v_i \xrightarrow{1} v_i' \xrightarrow{2} u \xrightarrow{3} v_j \xrightarrow{1} v_j'$
- $v_i \xrightarrow{4} v_{i-1}' \xrightarrow{2} u \xrightarrow{3} v_{j+1} \xrightarrow{4} v_j'$

- from v_i' to v_j
  - $v_i' \xrightarrow{2} u \xrightarrow{3} v_{j+1} \xrightarrow{4} v_j' \xrightarrow{1} v_j$
  - $v_i' \xrightarrow{1} v_i \xrightarrow{3} u \xrightarrow{2} v_{j-1}' \xrightarrow{4} v_j$
  - $v_i' \xrightarrow{4} v_{i+1} \xrightarrow{1} v_{i+1}' \xrightarrow{2} u \xrightarrow{3} v_j$
- from v_i' to v_j'
  - $v_i' \xrightarrow{2} u \xrightarrow{3} v_j \xrightarrow{1} v_j'$
  - $v_i' \xrightarrow{1} v_i \xrightarrow{3} u \xrightarrow{2} v_j'$
  - $v_i' \xrightarrow{4} v_{i+1} \xrightarrow{1} v_{i+1}' \xrightarrow{2} u \xrightarrow{3} v_{j+1} \xrightarrow{4} v_j'$

The edge-disjoint folded paths for $j = i, i-1, i+1, i+2$ are shorter, they have less or equal folded numbers and are easier to construct. Therefore, they are not presented here in order to avoid an artificial growth of the text. Finally, we have

$$|F^{(0)}| = (n-1)/2, \qquad |F^{(1)}| = n-1, \qquad |F^{(2)}| = (n-1)/2,$$

$$p = 3, \qquad r_0 = r_1 = r_2 = 1, \qquad q = 3,$$

from which we obtain $\omega \geq k+3$ and

$$\sum_{i=0}^{k+3} |F^{(i \bmod 3)}| = \begin{cases} \frac{2}{3}(n-1)k + \frac{5}{2}(n-1), if\ (k \bmod 3) = 0 \\ \frac{2}{3}(n-1)(k-1) + \frac{7}{2}(n-1), if\ (k \bmod 3) = 1 \\ \frac{2}{3}(n-1)(k-2) + 4(n-1), if\ (k \bmod 3) = 2 \end{cases}$$

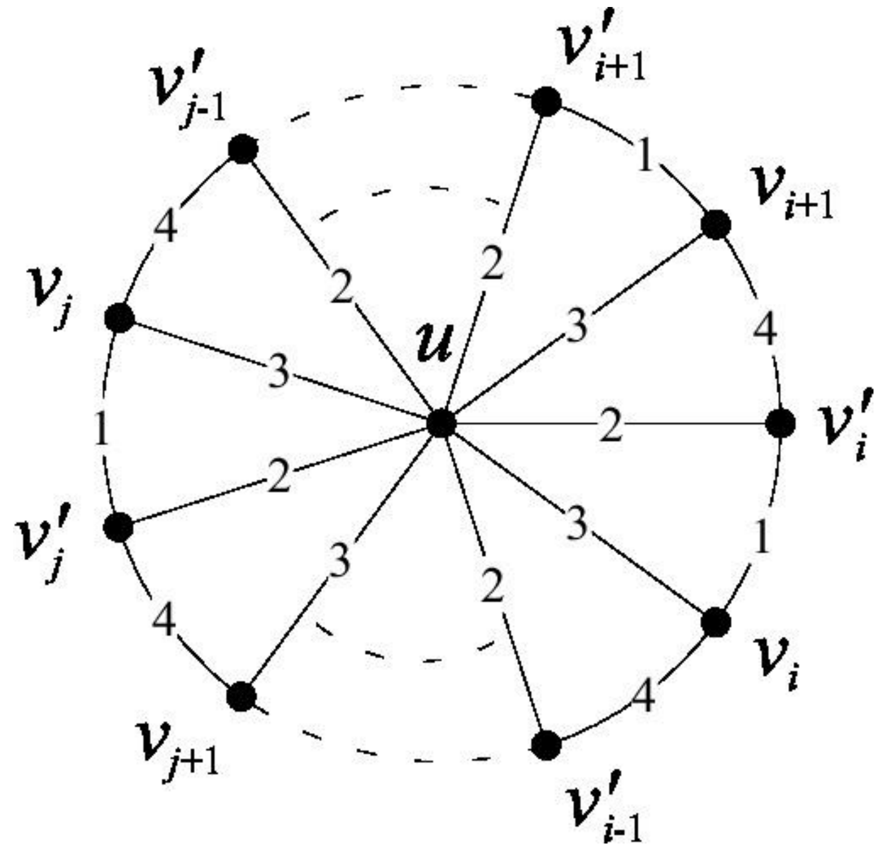

Fig. 15) the illustration of two arbitrary fixed vertices in the wheel graph.

For even n, we modify the wheel graph by adding a new vertex u' and transforming the edge set to the following expression

$$E(G) = F^{(0)} \cup F^{(1)} \cup F^{(2)}$$

with

$$F^{(0)} = \{(v_i, v_i'):\ t_G((v_i, v_i')) = 1, i = 1, 2, \ldots, m\},$$

$$F^{(1a)} = \{(v_i', u):\ t_G((v_i', u)) = 2, i = 1, 2, \ldots, m\},$$

$$e_a = (u, u'), t_G(e_a) = 3,$$

$$F^{(1b)} = \{(v_i, u'):\ t_G((v_i, u')) = 4, i = 1, 2, \ldots, m\},$$

$$e_b = (u, u'), t_G(e_b) = 5,$$

$$F^{(1)} = F^{(1a)} \cup F^{(1b)} \cup \{e_a, e_b\},$$

$$F^{(2)} = \{(v_i', v_{i+1}): t_G((v_i', v_{i+1})) = 6,\ i = 1,2,\ldots,m\}.$$

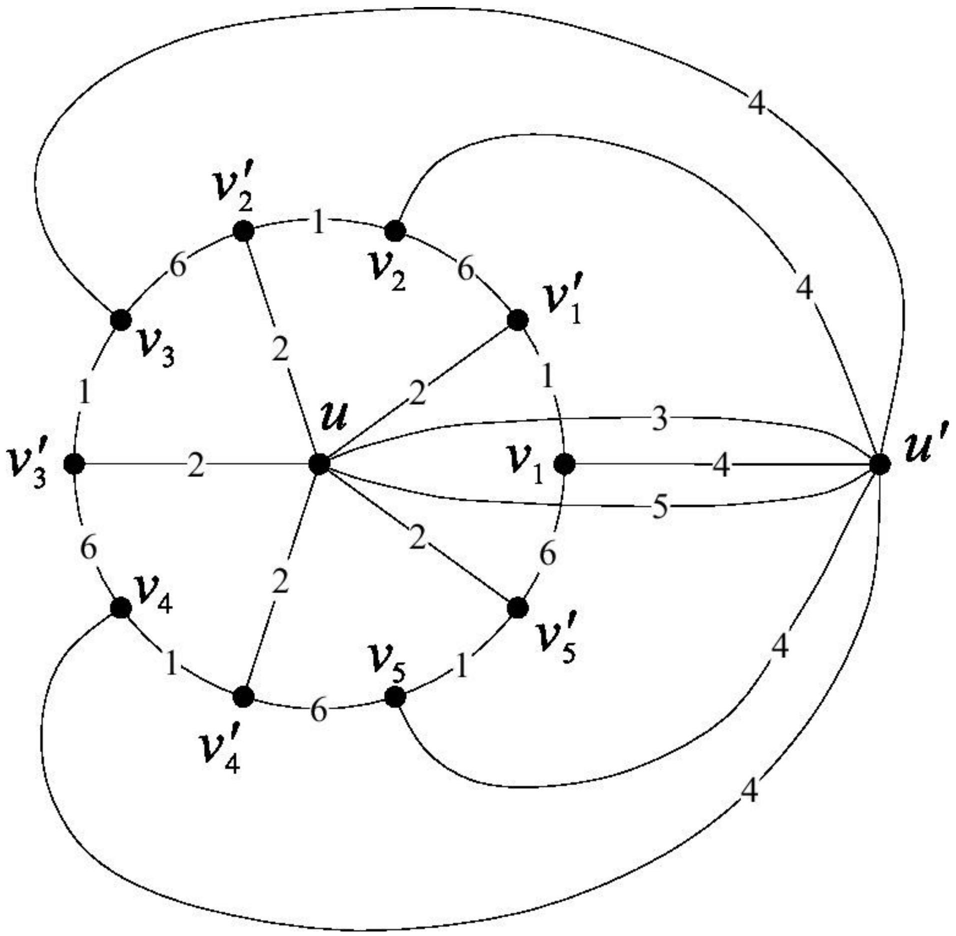

Fig. 16) Wheel graph for even n (here n=12)

Here the vertices u and u' are connected by two edges e_a and e_b. The graph G for $n = 12$ vertices is shown in the above figure. Constructing the edge-disjoint folded paths, one obtains

$$|F^{(0)}| = (n-2)/2, \qquad |F^{(1)}| = n, \qquad |F^{(2)}| = (n-2)/2,$$

$$p = 3, \qquad r_0 = r_1 = r_2 = 1, \qquad q = 3,$$

Which results $\omega \geq k + 3$ and

$$\sum_{i=0}^{k+3} |F^{(i \, mod \, 3)}| = \begin{cases} \frac{1}{3}(2n-1)k + \frac{5}{2}n - 4, & if \ (k \, mod \, 3) = 0 \\ \frac{1}{3}(2n-1)(k-1) + \frac{7}{2}n - 5, if \ (k \, mod \, 3) = 1 \\ \frac{1}{3}(2n-1)(k-2) + 4n - 5, if \ (k \, mod \, 3) = 2 \end{cases}$$

From the equation for odd n and for even n the following theorem holds.

Theorem. In k fault-tolerant gossip schemes the upper bound of $\tau(n,k)$ minimum needed calls satisfies the following condition:

$$\boldsymbol{\tau(n,k) \leq \frac{2}{3}nk + O(n).}$$

2.6 Fault-tolerant gossip graphs based on the combination of basic graphs

In this section we are going to present a new method of constructing fault-tolerant gossip graphs which is based on the unification of several graphs. This method of constructing is preferable when $k << n$ and holds for all graphs with $n > 5$ vertices. Let's denote $G = (V, E)$ as a k fault-tolerant gossip graph with the vertex set V and the edge set E. We will present it as a combination of several graphs $G = H_1 + H_2 + H_3$. Let's denote as $\rho(G)$ linear ordering of edges in G.

At first let's consider cases when n is odd and k is even. In constructing of H_1 we have to make successive calls including all vertices. If we number all vertices from 1 to n , then we have to make new edges from i to $i+1$ vertex where $i = 1, 2, \ldots, n-1$. After making these edges we have to make one more edge which will connect the 1-st and $(n-1)$-th vertices. After constructing H_1 we have to construct a new graph(H_2). The construction of H_2 should be in the following way: its edges should connect the n-th vertex to all the other vertices in the following sequence: $(n-3,\ n-5,\ n-7,\ .\ .\ .\ ,\ n-1,\ n-4,\ n-6,\ n-8,\ .\ .\ .\ ,\ n-2)$.

The last step of constructing G is the construction of H_3. In this step we have to make new successive edges that will connect the i-th and $(i-1)$-th vertices $(i = n-1, n-2, \ldots, 2)$.

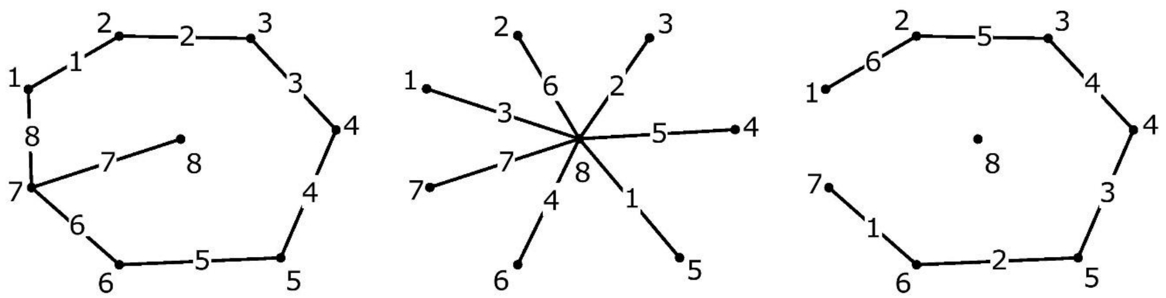

Fig. 17) H_1, H_2 and H_3 graphs

After these 3 steps we will obtain $k = 2$ fault-tolerant gossip graph.

Suppose we have to construct a k fault-tolerant gossip graph with n vertices where k is even and n is odd. To achieve our target at first we should construct a 2 fault-tolerant gossip graph and then increase the level of fault-tolerance to k. For this we need to add new H_2 and H_3 graphs to G, but with one difference - the edges of new H_2 must connect the n-th vertex to other vertices in the following sequence: $(n-5, n-7, \ldots, n-1, n-4, n-6, n-8, \ldots, n-2)$. So, repeating this action every time we increase the level of fault-tolerance with two until it reaches to k.

In case when n is even we have to change only the edge sequence in H_2 into the following: $(n-3, n-5, n-7, \ldots, n-2, n-4, n-6, n-8, \ldots, n-1)$.

Let us count how many edges we have when k was even. In H_1 we made $(n-1)+1$ edges, in H_2 graphs we made all together $k(n-2)/2+1$ edges and in H_3 graphs we made $k(n-2)/2$ edges. So, for general cases when k is even we obtain the following result: $\tau(n,k) \leq (n-2)+2+k(n-2)/2+1+k(n-2)/2 = (k+1)(n-2)+3$.

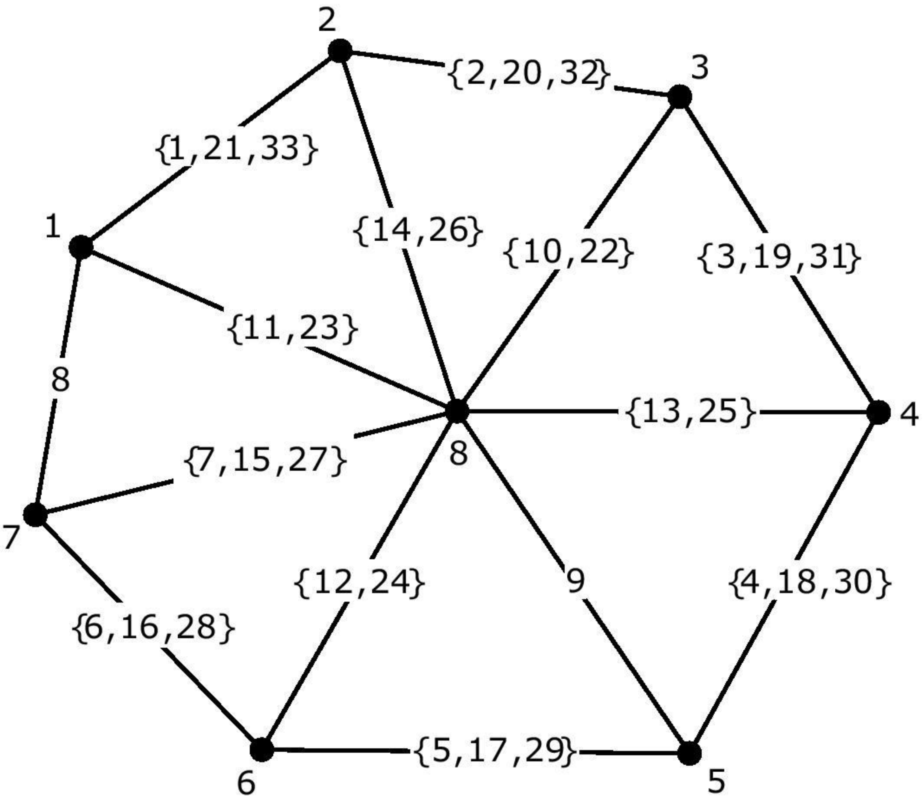

Fig. 18) k=4 fault-tolerant gossip graph

Now consider cases when k is odd ($k \geq 3$). In such cases we recommend to construct a 3 fault-tolerant gossip graph and then increase the level of fault-tolerance until achieving k. The deviation from cases when k is even is the following: after obtaining the graph $H_1 + H_2$ we have to add a new H_2 graph, whose edges must connect the n-th vertex with all others in the following sequence: $(n-3, n-5, n-7, \dots, n-1, n-4, n-6, n-8, \dots, n-2)$ in case n is odd and in the following: $(n-3, n-5, n-7, \dots, n-2, n-4, n-6, n-8, \dots, n-1)$ for even n. After this step the H_3 graph must be added and all together they will form a 3 fault-tolerant gossip graph. So, repeating adding new H_2 and H_3 graphs every time we increase the level of fault-tolerance with two until it reaches k. Note, that in repeating action the H_2 graph must change it's edge sequence into $(n-5, n-7, \dots, n-1, n-4, n-6, n-8, \dots, n-2)$ if n is odd and into $(n-5, n-7, \dots, n-2, n-4, n-6, n-8, \dots, n-1)$ in case if n is even.

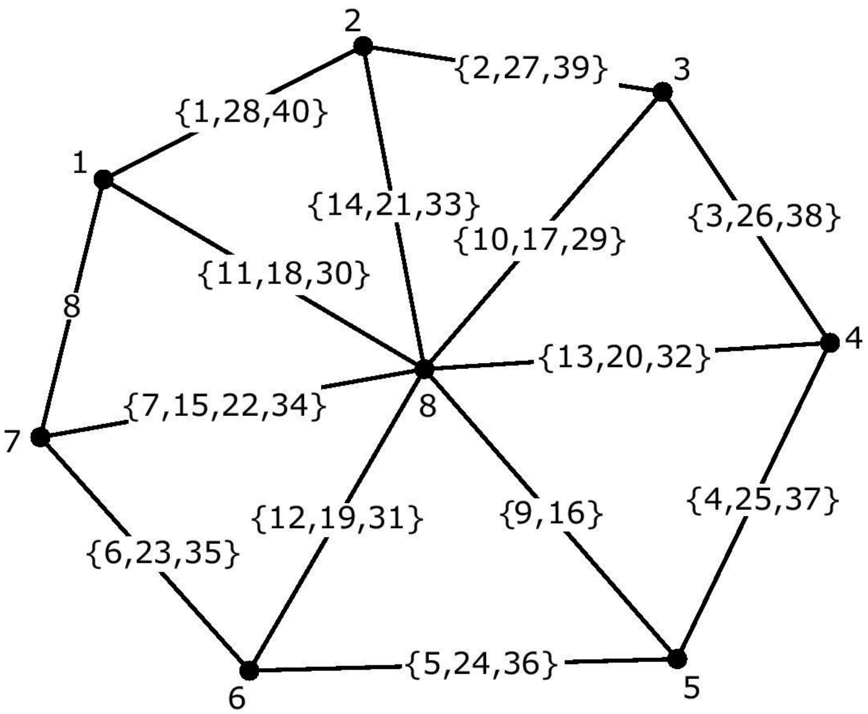

Fig. 19) k=5 fault-tolerant gossip graph

Let us count the number of edges of G in case when k is odd. In H_1 there are $(n-1)+1$ edges and the sum of edges of all H_2 and H_3 graphs is $k(n-2)+2$. So, for odd k our result is $(n-2)+2+(n-2)k+2 = (k+1)(n-2)+4$. Consider any two distinct vertices v_i and v_j in communication network G. It is an easy exercise for the reader to verify that there exist $k+1$ pairwise edge disjoint ascending paths from v_i to v_j.

At the end let's consider cases when $k=1$ and $k=2$. In these cases the construction of a graph becomes more simple. We will present it here without considering details. Just note, that $\tau(n,1) \leq 2n-3+\left\lfloor\frac{n}{2}\right\rfloor$ and $\tau(n,2) \leq 2n-3+n$. So, as we can see, it confirms the assumption of Hasunuma and Nagamochi $\tau(n,k) \leq nk/2+O(n)$.

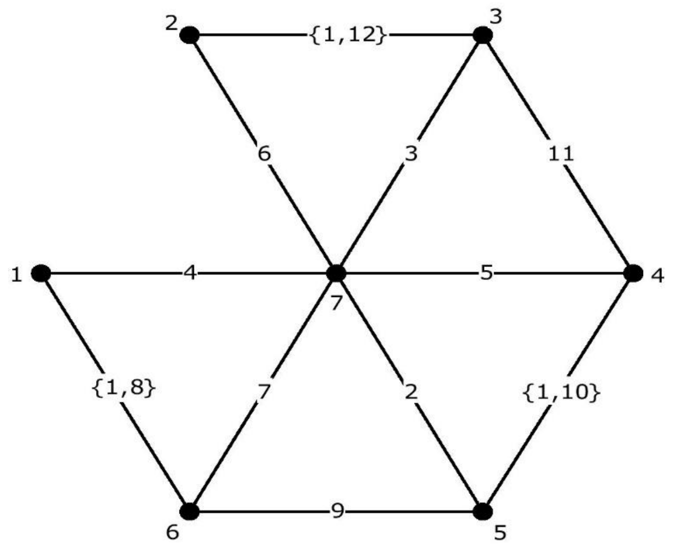

Fig. 20) k=1 fault-tolerant gossip graph

2.7 Fault-tolerant gossip graphs based on the Knodel graphs

In this section we will present a new way of fault-tolerant construction, which is based on the Knodel graphs as an underlying communication topology. Since the definition of Knodel graphs was already presented (see section 1.4), here we will bring several new definitions regarding the various concepts in Knodel graphs and fault-tolerant gossiping.

Definition. The interval between two vertices $(\propto,\beta)$ and (γ,δ) in Knodel graph $W_{\lfloor \log_2 n \rfloor, n}$ is defined to be

$$R\big((\propto,\beta);(\gamma,\delta)\big)=\begin{cases}\delta-\beta, & if\ \delta\geq\beta\ and\ \propto=1,\\ \dfrac{n}{2}-|\delta-\beta|, & if\ \delta<\beta\ and\ \propto=1,\\ |\delta-\beta|, & if\ \delta\leq\beta\ and\ \propto=2,\\ \dfrac{n}{2}-(\delta-\beta), & if\ \delta>\beta\ and\ \propto=2.\end{cases}$$

Given two arbitrary vertices $(\propto,\beta)$ and (γ,δ) in the Knodel graph $W_{\lfloor \log_2 n \rfloor, n}$, the ascending path from the vertex (γ,δ) to the vertex $(\propto,\beta)$ exists only if

$$R\big((\propto,\beta);(\gamma,\delta)\big) \le 2^{\lfloor \log_2 n \rfloor - 1} - 1.$$

The explicit expression for the ascending path from the vertex (γ,δ) to $(\propto,\beta)$ is described via 3 sequences $\{a_i\}$, $\{b_i\}$ and $\{f_i\}$ defined reclusively by the following way:

$$a_1 = \begin{cases} 1, & if \ \propto = 2, \\ 2, & if \ \propto = 1, \end{cases} \qquad a_i = \begin{cases} 1, & if \ a_{i-1} = 2, \\ 2, & if \ a_{i-1} = 1, \end{cases}$$

$$f_1 = \begin{cases} 2^{\lceil \log_2(\delta-\beta+1) \rceil} - 1, & if\ \delta \ge \beta \ and \ \propto = 1, \\ 2^{\left\lceil \log_2(\frac{n}{2}-|\delta-\beta|+1) \right\rceil} - 1, & if\ \delta < \beta \ and \ \propto = 1, \\ 2^{\lceil \log_2(|\delta-\beta|+1) \rceil} - 1, & if\ \delta \le \beta \ and \ \propto = 2, \\ 2^{\left\lceil \log_2(\frac{n}{2}-(\delta-\beta)+1) \right\rceil} - 1, & if\ \delta > \beta \ and \ \propto = 2. \end{cases}$$

$$b_1 = \begin{cases} (\beta + f_1) mod\ n/2, & if \ \propto = 1, \\ \left(\frac{n}{2} + \beta - f_1\right) mod\ n/2, & if \ \propto = 2. \end{cases}$$

$$f_i = \begin{cases} 2^{\lceil \log_2(|b_{i-1}-\delta|+1) \rceil} - 1, & if\ b_{i-1} < \delta \ and \ a_{i-1} = 1, \\ 2^{\left\lceil \log_2(\frac{n}{2}-(b_{i-1}-\delta)+1) \right\rceil} - 1, & if\ b_{i-1} \ge \delta \ and \ a_{i-1} = 1, \\ 2^{\lceil \log_2(b_{i-1}-\delta+1) \rceil} - 1, & if\ b_{i-1} \ge \delta \ and \ a_{i-1} = 2, \\ 2^{\left\lceil \log_2(\frac{n}{2}-|b_{i-1}-\delta|+1) \right\rceil} - 1, & if\ b_{i-1} < \delta \ and \ a_{i-1} = 2. \end{cases}$$

$$b_i = \begin{cases} \left(\beta + \sum_{j=1}^{i} (-1)^{j-1} f_j\right) mod\ n/2, & if \ \propto = 1, \\ \left(\frac{n}{2} + \beta - \sum_{j=1}^{i} (-1)^{j-1} f_j\right) mod\ n/2, & if \ \propto = 2. \end{cases}$$

The sequence $\{a_i\}$, $\{b_i\}$ and $\{f_i\}$ stop on the minimal index $i = L$, for which $a_L = \gamma$ and $b_L = \delta$. Then $((a_L,b_L),(a_{L-1},b_{L-1}),\dots,(a_1,b_1),(\alpha,\beta))$ is an ascending path from the vertex (γ,δ) to the vertex $(\propto,\beta)$. For example, the ascending path from the vertex $(2,2)$ to $(1,0)$ is

$$(2,2) \xrightarrow{1} (1,2) \xrightarrow{2} (2,3) \xrightarrow{3} (1,0),$$

where the numbers on arrows are the labels of the corresponding edges.

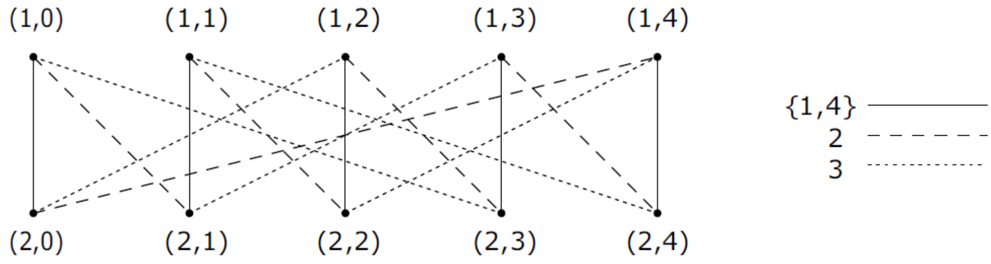

Fig. 21) The edge sum of two graphs: $W_{3,10} + W_{1,1}$.

Each of the vertical (solid) lines in the above graph consists of two edges with labels 1 and 4. The dashed and dotted edges are labeled correspondingly 2 and 3. This graph is a gossip graph with $n = 10$ nodes (0-fault-tolerant gossip graph based on Knodel graphs).

The following lemma is about gossiping properties of the Knodel graph. Although this topic was investigated very widely, we will formulate the lemma and proof it once more, since we use this concept throughout our research.

Lemma. The graph $W_{\lfloor log_2 n \rfloor, n} + W_{1,n}$ is a gossip graph (n is even). Moreover, if n is a power of 2, then $W_{log_2 n, n}$ itself is a gossip graph.

Proof. Consider the graph $G = W_{\lfloor \log_2 n \rfloor, n} + W_{1,n}$. Let us fix an arbitrary vertex $(1, \beta) \in V(G)$, $0 \leq \beta \leq n/2 - 1$. The set of vertices from which there are ascending paths in $W_{\lfloor \log_2 n \rfloor, n}$ to the vertex $(1, \beta)$ is

$$V_{(1,\beta)} = \left\{\left(\gamma, (\beta + \delta)\ mod \tfrac{n}{2}\right) : \gamma = 1,2;\ \delta = 0,1, \dots, 2^{\lfloor \log_2 n \rfloor - 1} - 1\right\}.$$

Similarly, the set of vertices from which there are ascending paths in $W_{\lfloor \log_2 n \rfloor, n}$ to the vertex $(1, \beta) \in V(G)$, $0 \leq \beta \leq n/2 - 1$ is

$$V_{(2,\beta)} = \left\{\left(\gamma, (\beta - \delta + n/2)\ mod \tfrac{n}{2}\right) : \gamma = 1,2;\ \delta = 0,1, \dots, 2^{\lfloor \log_2 n \rfloor - 1} - 1\right\}.$$

If n is a power of 2 ($n = 2^m$), for we have $V_{(1,\beta)} = V_{(2,\beta)} = V(W_{\log_2 n, n})$. Therefore, $W_{\log_2 n, n}$ is a gossip graph.

For the case, when $n \neq 2^m$, the addition of the graph $W_{1,n}$ to the graph $W_{\lfloor \log_2 n \rfloor, n}$ connects the vertices $(1, \beta)$ and $(2, \beta)$ with a new edge with a label $\lfloor \log_2 n \rfloor + 1$. This label is bigger than the labels in $W_{\lfloor \log_2 n \rfloor, n}$ (by construction), therefore this edge (call) exchanges the full information of the nodes $(1, \beta)$ and $(2, \beta)$ collected during the calls in $W_{\lfloor \log_2 n \rfloor, n}$. Therefore, after all calls the pieces of information known by nodes $(1, \beta)$ and $(2, \beta)$ coincide and are nothing but the following union of sets:

$$\widetilde{V_\beta} = V_{(1,\beta)} \cup V_{(2,\beta)}.$$

Note that, here (when $\Delta = \lfloor \log_2 n \rfloor$) $\widetilde{V_\beta} = V(G)$ for any β. Therefore, $W_{\lfloor \log_2 n \rfloor, n} + W_{1,n}$ is a gossip graph.

Consider the problem to find the minimal number of calls $\tau(n, k)$ in a k-fault-tolerant gossip scheme with n nodes. For the special case, when n is a power of 2 ($n = 2^m$) Hasunama and Nagamochi constructed k-fault-tolerant gossip graphs based on hypercube and found $\tau(n, k) \leq \frac{n}{2} \log_2 n + \frac{nk}{2}$. Here we will generalize this result for general n nodes. The following lemma is an important step towards achieving this goal. Note, in the end we will come up with only hypothesis (which was verified experimentally), but we will try to formulate several lemmas that we believe are important for the final proof of this hypothesis.

Hypothesis. Consider two arbitrary vertices $(\propto, \beta)$ and (γ, δ) in the Knodel graph $W_{\lfloor \log_2 n \rfloor, n}$ with $\propto, \gamma = 1,2;\ \beta, \delta = 0, 1, 2, \dots, \frac{n}{2} - 1$. There are $p = \lfloor \log_2 n \rfloor$ edge-disjoint folded ascending paths from the vertex (γ, δ) to the vertex $(\propto, \beta)$ and the sum of their folded numbers is at most $q = \lfloor \log_2 n \rfloor$.

Here are our observations in this regards that, we believe, can spread a bit of clarity on the bases of above hypothesis.

Since the graph $W_{\lfloor \log_2 n \rfloor, n}$ is symmetric, then without loss of generality it is enough to consider the vertex $(\propto, \beta) = (1, 0)$. Consider the following pairwise non-intersecting sets of vertices

$$V(1) = \{(2,0)\},$$
$$V(\Delta) = \{(i, 2^{\Delta-2} + j): i = 1,2; j = 0, 1, \dots, 2^{\Delta-2} - 1\}, \ \ \Delta = 2,3, \dots, \lfloor \log_2 n \rfloor.$$

From the above it follows that there exist an ascending path from every vertex in $V(\Delta)$, $1 \le \Delta \le \lfloor \log_2 n \rfloor$ to the vertex $(1, 0)$, which is described by the sequences $\{a_i\}$, $\{b_i\}$ and $\{f_i\}$. Moreover, all the endpoints of the edges of this path, except the last vertex of the last edge (vertex $(1,0)$), are in $V(\Delta)$.

In [12] it was shown that Knodel graph can be defined as a Cayley graph, hence is vertex transitive and has edge-connectivity equal to Δ. Hence, there are $\Delta = \lfloor \log_2 n \rfloor$ edge-disjoint paths between two arbitrary vertices of Knodel graph. In case of the colored Knodel graphs which we consider here, these paths can be either ascending or s-folded ascending paths.

Consider the case of $n = 2^m - 2$. It can be verified that there exists an ascending path or 1-folded ascending path from the vertex (γ, δ) to the one of the vertices in $V(\Delta)$ for any $1 \le \Delta \le \lfloor \log_2 n \rfloor$.

Therefore, there are $\lfloor \log_2 n \rfloor$ folded edge-disjoint ascending paths from (γ, δ) to $(1, 0)$ with folded numbers presumably being equal to 0, 1 or 2.

Depending on the given vertex (γ, δ) there are 2 possible cases:

Case 1: $R\big((1,0);\ (\gamma, \delta)\big) > 2^{\lfloor \log_2 n \rfloor}$. In this case, all paths are 1-folded. Each of the paths consists of two parts, which are themselves ascending paths. The first path starts from (γ, δ) and comes to one of the vertices of the sets $V(\Delta)$, $(1 \le \Delta \le \lfloor \log_2 n \rfloor)$. The second path goes from that vertex up to the vertex $(1;\ 0)$. Since there are $\lfloor \log_2 n \rfloor$ edge-disjoint folded paths between two vertices, thus the sum of folded numbers of all paths is $\lfloor \log_2 n \rfloor$.

Case 2: $R\big((1,0);\ (\gamma, \delta)\big) \le 2^{\lfloor \log_2 n \rfloor}$. In this case, there is an ascending path from (γ, δ) to $(1;\ 0)$. On the other hand, there is an ascending path started from the given vertex (γ, δ), whose single edge has a label $\lfloor\log_2 n\rfloor$ and finishes outside of any $V(\Delta)$, $(1 \le \Delta \le \lfloor\log_2 n\rfloor)$. Therefore, the folding number of a folded ascending path containing this path is 2. The remaining folded ascending paths from (γ, δ) to $(1, 0)$ are similar as in the Case 1 and are 1-folded. Therefore, the sum of folded numbers of all folded ascending paths from (γ, δ) to $(1, 0)$ is $\lfloor\log_2 n\rfloor$.

For the case of $n \neq 2^m - 2$ there exist vertices (γ, δ) with $R\big((1,0);\ (\gamma, \delta)\big) \le 2^{\lfloor\log_2 n\rfloor}$ such that paths from them to the vertex $(1, 0)$ will be as follows: one ascending path (folded number is 0) and the remaining $\lfloor\log_2 n\rfloor - 1$ 1-folded ascending paths. Hence, the sum of folded numbers of all paths will be $\lfloor\log_2 n\rfloor - 1$ for these vertices. All the adjacent vertices of this kind of vertices have $R\big((1,0);\ (\propto, \beta)\big) \le 2^{\lfloor\log_2 n\rfloor}$, which was not possible in case of $n = 2^m - 2$. The number of these vertices will be maximum when $n = 2^m + 2$ (since $\lfloor\log_2 n\rfloor = m$ and the graph is the "dense" in this case), but this is not important, since there always exist vertices with $R\big((1,0);\ (\gamma, \delta)\big) > 2^{\lfloor\log_2 n\rfloor}$, hence the sum of the folded numbers of the paths started from these vertices is $\lfloor\log_2 n\rfloor$. Since, the sum of the folded numbers for the graph is the maximum of all possible values for all the pairs of vertices, thus nothing will differ for $n \neq 2^m - 2$. Thus $q = \lfloor\log_2 n\rfloor$ for all the cases.

We believe, that the above said may contains some ambiguous statements and decisions, thus we formulate only the hypothesis regarding the fault-tolerant gossiping properties of the Knodel graph. For stricter claims we would need to construct actual folded ascending paths between any pairs of vertices (corresponding to the above said) and show that they are pairwise edge-disjoint.

Hypothesis. The minimal number of calls (edges) in a k-fault-tolerant gossip graphs has the following bound for any even n:

$$\tau(n, k) \le \frac{n}{2}\lceil\log_2 n\rceil + \frac{nk}{2}.$$

This bound follows by construction, considering the previous hypothesis. Well, to construct k-fault-tolerant gossip graph we take as base graph $G = W_{\lfloor\log_2 n\rfloor, n}$ and apply the theorem of fault-tolerance. The edge set $E(G)$ of G is divided into the following $l = \lfloor \log_2 n \rfloor$ subsets:

$$E(G) = F^{(0)} \cup F^{(1)} \cup \ldots \cup F^{(\lfloor \log_2 n \rfloor - 1)},$$

where

$$F^{(i)} = \{e: e \in G;\ \ t_G(e) = i + 1\};\ \ i = 0, 1, 2, \ldots, \lfloor \log_2 n \rfloor - 1.$$

The number of edges in $F^{(i)}$ is $|F^{(i)}| = n/2$ for any $i = 0, 1, 2, \ldots, \lfloor \log_2 n \rfloor - 1$. Each vertex in G has $\lfloor \log_2 n \rfloor$ incident edges with one edge in each of the sets $F^{(i)}$ for $i = 0, 1, 2, \ldots, \lfloor \log_2 n \rfloor - 1$. According to the previous hypothesis and the discussion immediately followed it, there are $p = \lfloor \log_2 n \rfloor$ edge-disjoint folded ascending paths between any two vertices of Knodel graph, and the sum of their folded numbers is at most $q = \lfloor \log_2 n \rfloor$. Therefore $r_i = 1$ for any $i = 0, 1, 2, \ldots, \lfloor \log_2 n \rfloor - 1$, and ω is any integer satisfying the inequality $\omega \geq k + q$. From this result, the value of $\partial(n, k)$, is

$$\partial(n, k) = \frac{n}{2}(\omega + 1) = \frac{n}{2}\lceil \log_2 n \rceil + \frac{nk}{2},$$

which concludes the above hypothesis for even n.

Additionally, let us note that the construction described in this section holds $\lceil \log_2 n \rceil + k$ upper bound on the minimum possible time of complete fault-tolerant gossiping. Considering the fact that this result was shown to be a lower bound as well (see [37] and [58]) we can claim (as a corollary from the above hypothesis) that the minimum time required to complete k-fault-tolerant gossiping is:

$$T(n, k) = \lceil \log_2 n \rceil + k.$$

2.8 Conclusion

In this chapter we presented the results of our research about various aspects of the gossiping process. First, we have introduced a method of the modification of gossip graphs, which we use extensively throughout our study to construct various new gossip topologies and obtain some theoretical results. We used this method to construct a new proof for the minimum number of calls in a complete gossip scheme being equal to 2n-4, where n is the number of vertices. Further, with the help of above mentioned method we have constructed NOHO gossip graphs, that can perform complete gossiping process in a minimum possible time ($\lceil \log_2 n \rceil$ for any even n). Another application of this method was useful when providing a new construction method for minimum gossip graphs, that have the number of calls equal to 2n-4 and can perform complete gossiping in the minimum possible time, given the above constrain.

Our further analysis were focused on the Knodel graphs and their gossiping properties. We have showed that the cyclic permutation of the weights of the edges of Knodel graph, in general, does not affect its gossiping properties. This is an important also for describing edge-disjoint folded ascending paths between any pairs of vertices in Knodel graphs, which can be crucial for verifying our hypothesis about the upper bound value of the minimum number of calls and minimum time of fault-tolerant gossiping.

The last 3 sections are focused on fault-tolerant gossiping and finding optimal structures for it (as base topologies). We suggest 3 different ways of construction of fault tolerant gossip graphs (1 in the form of hypothesis) that also improve the known upper bounds for some values of the number of vertices or the level of fault-tolerance (by construction).

Construction of the software tool (Graph Plotter) for the investigation of gossip problems

3.1 The organization of the software tool Graph Plotter

The whole chapter 3 is denoted to the description of a software package called Graph Plotter. Graph Plotter is a software package designed for the construction of gossip graphs, verification of the gossiping properties of the constructed graphs, such as verifying weather the graph is complete gossip graph and the level of fault-tolerance of the graph. The main purpose of the tool is to check for a given input graph whether it satisfies the given level of fault-tolerance. In addition, it is possible to check whether the input graph satisfies the NOHO (No One Hears Own) or a NODUP conditions. The NOHO graphs are graphs, which do not contain any node that listens to its own information, or equivalently, do not contain a cycle. The NODUP graphs are defined as the graphs, whose nodes listen to each piece of information exactly once. It means that there is exactly one increasing path between two arbitrary vertices.

In addition, the system provides a convenient means of working with graphs, such as, for example, auto numbering of vertices and edges of the graph, in order to show the paths between two vertices it is possible to paint vertices and edges with any color, if necessary, add a new edge with proper weight or sort weights of all edges with integer values, to color the paths between any two vertices of Knodel graph, color the graphs correspondingly to various Messy broadcast modes, etc. Apart from all this, the system makes it possible to change the position of vertices adjacent to the current edge, also to permute all edges which are adjacent to the current edge and have bigger (or smaller) weights.

The meaning of this action lies in the fact that if there has been a call between two arbitrary vertices at time t since then these two have exactly the same information, so nothing will be changed if from that moment onwards all the nodes that will connect to these two will change their calls direction from one vertex to another, if the considered graph is symmetric. This function might be useful if you need to disperse or collect edges into one channel, since in some modifications of Gossip problem the number of edges in one channel is limited. This feature can serve as a useful tool for the designers of such networks to ensure the correspondence of their networking topology with the constraints put on them (by environment, cost, etc.).

The system also has certain restrictions regarding the maximum number of vertices (999), which is connected to the measures of the vertices and the complexity (exponential) of gossip verification calculations.

It should be noted that the system does not have a sample or an analogue, at least in the field of universal access, and the main reason is that the system can be useful only in a narrow range of problems, so it can be used only for scientific research. Prior to this, exclusively a mathematical approach was used in solving problems of gossiping and the hardware/software model of calculations was not considered.

Graph Plotter was developed in Wolfram Mathematica environment and is easily extensible tool. Since Gossip test, NOHO and NODUP modules are written in C++ language and are separate processes in relation to the main process, new modules can be integrated with this software tool very easily.

It is expected to involve such technologies of parallel computing as MPI, OpenMP and CUDA, since it will let us to increase the performance of the tool, which is very important for the investigation of the graphs with relatively large number of vertices.

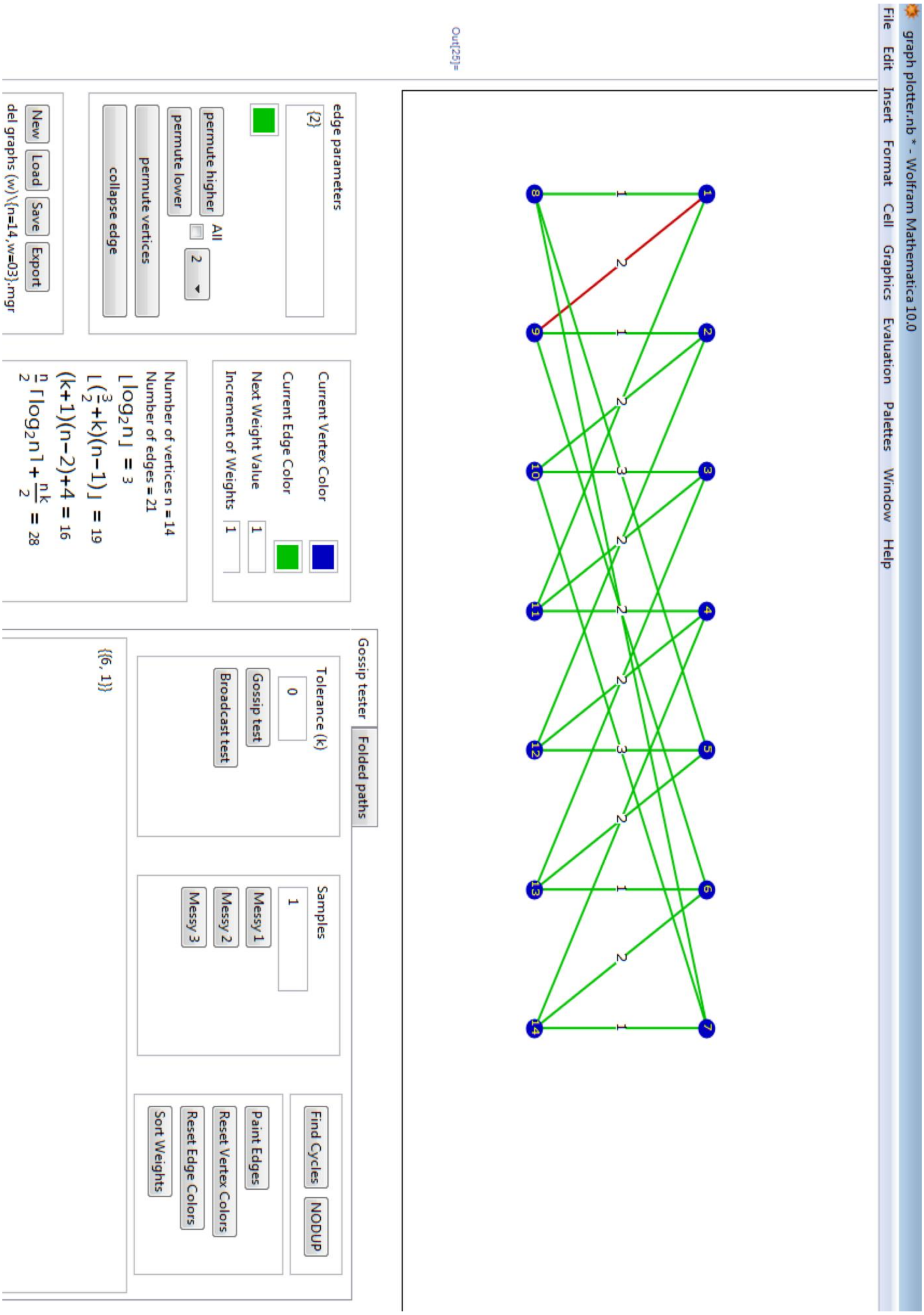

Fig. 22) Graph Plotter user interface

3.1.1 The choice of the software components

The software package consists of the main component that is responsible for the interactive UI and several functional flows and can be easily extended by various other components using IPC (inter-process communication). The main component of this tool was developed in Wolfram Mathematica environment and supports all the interactive UI and some of the functional flows.

Wolfram Mathematica (usually called Mathematica) is a modern software system that was designed for the researchers of all the areas of technical computing – including, but not limited to neural networks, machine learning, image processing, geometry, data science, visualizations. The software is used by the researchers of many technical, scientific, engineering, mathematical, and computing fields. It was proposed by Stephen Wolfram and is developed by Wolfram Research of Champaign, Illinois ([79]). Mathematica has its own programming language called the Wolfram Language.

The Wolfram Mathematica package includes features such as:

- Mathematical functions libraries
- Matrix manipulation possibilities
- Possibility to work with complex numbers, interval arithmetic, etc.
- Various numeric tools for discrete and continuous calculations
- Various tools for text mining, data mining, pattern matching, etc.
- Digital signal processing libraries
- Tools for graph visualization and analysis
- Tools for automated theorem proving
- Tools for combinatory problems
- Tools for connection with dynamic-link libraries (DLL), C++, SQL, Java, .Net, etc.

This was only small subset of its possibilities.

Wolfram Mathematica consists of two parts, the kernel part and the front end. The kernel is responsible for the interpretation of Wolfram Language and all the calculations. The GUI allows the creation and editing of program code with syntax highlighting and support for standard word processing.

The communication between Wolfram kernel and front-end, as well as between Wolfram kernel and other external applications is taken care with the help of protocol called Wolfram Symbolic Transfer Protocol (WSTP). For each of the supported programming languages it provides means for linking developer kit for linking the applications written in corresponding langue to the kernel through WSTP.

The second component of the system is a bunch of various tools, that were designed for performing various processing on top of gossip graphs and returning some results about its gossiping properties. These tools were developed in C++ and are integrated with the main component using WSTP and corresponding library provided for C language. The integration part was implemented as method calls from Mathematica kernel to methods contained in corresponding .dll libraries. The choice of C++ is based on the facts that it is easy to integrate it with OpenMP for further extension on multiprocessor calculations (for large number of vertices) and, also, it gives us convenient environment for developing multithreaded applications and improving performance.

3.1.2 The working modes of the tool

In the current section we are going to present a working modes of the software tool. As already was noted, this tool was designed mainly to serve researchers of this topic with the goal of helping them particularly with verification of their hypothesis.

The idea is to construct a model of a given network structure and verify their gossiping properties, such us the level of fault-tolerance, weather they satisfy NOHO or NODUP conditions, how is it possible to perform Messy broadcasting using this graph. In what follows let us go through each of this points in a bit detailed way.

- For the simple gossip check and fault-tolerance check we use an input box to give a level of fault tolerance. The result of the execution of this module is either OK, which means that the constructed graph satisfies given level of fault-tolerance or it shows the pairs of vertices that do not satisfy the required conditions. During the execution of gossip test module it performs the followings
    - constructs the incidents matrix of the graph, with each element being a vector containing all the weights of the edges connecting given two vertices
    - for each pair of vertices application tries to find k+1 edge-disjoint ascending paths going from the one vertex to the other
- For the NOHO (Find Cycles) check the module reuses the above described functionality and has one more addition: for each of the path between given 2 vertices it checks the first and last vertices of the path and their equality. If they are the same it interrupts its execution and returns that path, which then is being displayed by the system.
- For the NODUP check the module reuses simple gossip verification functionality as well. It counts all the pairs of vertices that have more than 1 ascending paths between them. In the end it returns all such pairs and this data is being displayed as output. In case there are no duplicates the output will be the string “OK”.
- For the Messy broadcast models we have tools for generating messy broadcast scheme on top of the given graph. The aim of these tools is to randomly generate coloring of the edges of constructed graphs, that will correspond to a given Messy model (M_1, M_2, M_3). By generating a lot of such samples we can try to find an upper bound on the minimum possible Messy broadcasting time for a given number of vertices.

- For the “Folded Paths” mode the application tries to generate all the folded paths between given 2 vertices, that correspond to a given fold number. This is very useful tool for the fault-tolerant gossip research. The number of edge-disjoint folded paths and the maximum among the sum of the fold numbers of these edge-disjoint paths between any two vertices are important to decide the correspondence of given network topology as a base for fault-tolerant construction.

3.2 The implementation details of the tools

In the current section (and its corresponding subsections) we are going to present some insights about the implementation details of the various tools that we have developed throughout this research. The main goal of this section and its 4 subsections is to give algorithmic description of the gossip verification, NOHO, NODUP, Messy Broadcast tools and, in some cases, also describe the mentioned models more precisely.

3.2.1 NOHO gossiping test implementation

NOHO stands for No Own Hears Own. This is an additional restriction put on the gossiping process to make sure that no own from the participants can hear his own information from the other participant. In the terms of graphs, this require that there is no any ascending path in the gossip graph the first and last vertices of which are the same. This condition comes to ensure that there is no much duplicate information flows in the gossiping network.

We have developed a tool in the Graph Plotter that allowed us to check whether the constructed graph satisfies NOHO gossiping requirement or not. The algorithm behind that requirement is to check that there is no any cycle in the graph ($O(n^2)$ complexity).

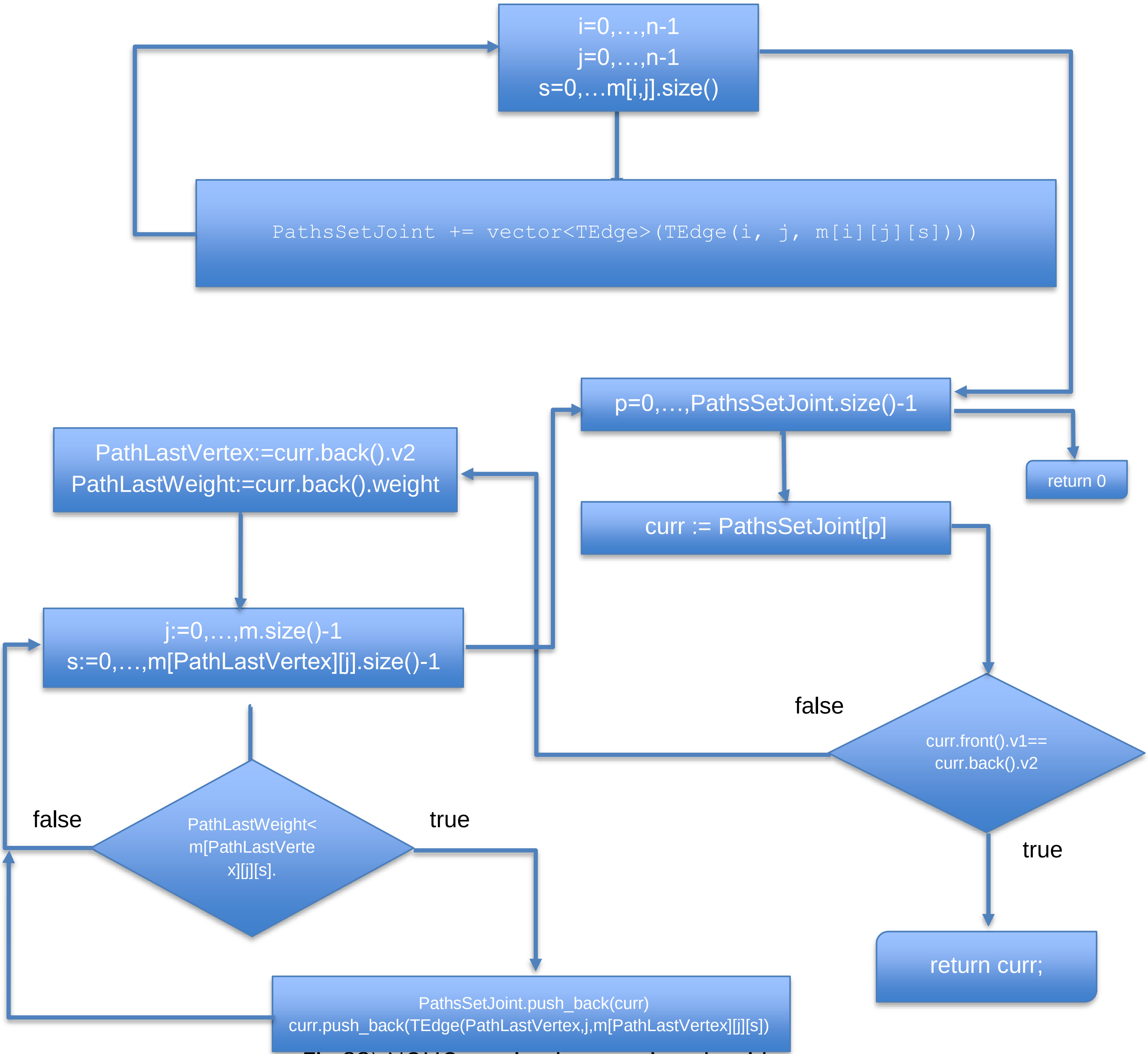

Fig 23) NOHO test implementation algorithm

In the scheme above it is shown the algorithm for determination of duplicate paths. The variable m is 3 dimensional vector containing the list of all the edges with their weights connecting any two vertices of the graph. The algorithm first tries to collect all the possible paths between given two vertices in the set PathsSetJoint variable. During this process it checks for the existence of cycles and interrupts its execution in case finds one.

3.2.2 NODUP gossiping test implementation

This section, in some sense, is the continuation of the previous one, because the restriction put on gossip schemes by NODUP (No Duplicates) term is that there should not be any kind of information heard by any participate more than once. In terms of gossip graph this restriction means that the number of ascending paths between any pair of vertices should be equal to exactly one.

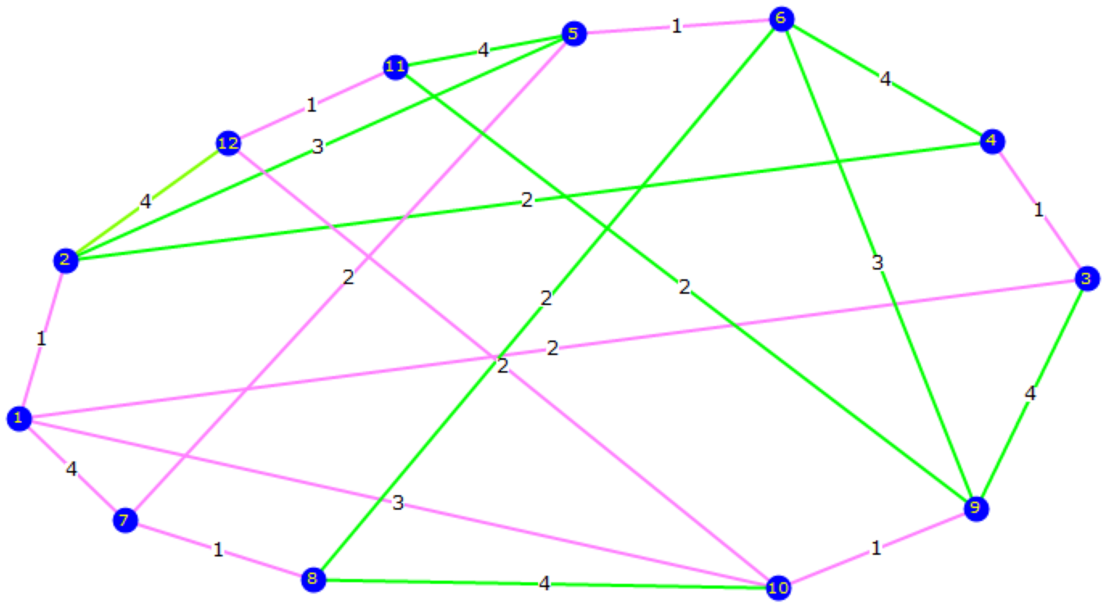

Fig. 24) NODUP graph (n=12)

The NODUP requirement comes in sight when there are some restrictions on the throughput of the channel that transmits the information between participates or the network was designed to perform some aggregation operation (in the form of gossiping) and it is not acceptable to count any part of the total aggregate more than once.

The NODUP verification algorithm is pretty much similar to the one of NOHO (see previous subsection). First, it tries to construct all the ascending paths between all the pairs of vertices and then checks for duplicated paths. All the duplicate paths are being collected and returned to Mathematica.

3.2.3 Fault Tolerant gossiping test implementation

The fault-tolerant gossiping verification tool was designed to give an answer to the question whether the constructed graph satisfies to a given level of fault-tolerance. This tool has played a crucial role in our research of fault-tolerant gossiping topologies. Especially, it was used when researching fault-tolerant gossip graphs based on the Knodel graph. With the help of this tool, we have experimentally verified that these graphs can serve as an excellent base topologies for fault-tolerant gossiping and yield an upper bound for the number of calls of the k-fault-tolerant gossip graph, that improves the previously known results for even n (in general).

As already was said the construction graph is passed from Wolfram to corresponding module developed in C++ in the form of 3 dimensional matrix. The main idea behind the implementation is the following:

1. Construct the ascending paths between any two vertices of the graph by traversing the given matrix using multiple nested cycles
2. After the above step, we perform grouping of the constructed paths into multiple groups and associate them with their start and end points (vertices)
3. In the end we check for the equality of the elements (paths) of the above generated sets. In case we count k+1 different elements in that set we return success.

Since the above describe algorithm is iterative and there is no direct dependency between its iterations, we can perform the iterations in a different threads.

3.2.4 Messy broadcasting implementation

Broadcasting differs from gossiping, since here the originator of the process is only one node and the "mission" of this node is to broadcast its information to every other node in the network. As already was discussed Messy broadcast models becoming actual when there is no common knowledge about the network structure, no any coordinating point that will tell each of the participates when to make each call and to which neighbor. In this situation every node has only some local knowledge about the topology of the system and the state of broadcasting process. With regards to the last (the level of ambiguity of each participate), there are 3 models defined in literature that are called Messy broadcasting models:

- M_1 – in this model in any unit of time every participate knowns the states of its neighbors. This model require that in any unit of communication time each of the informed vertices sends its information to own of its uninformed neighbors
- M_2 – in this model the node has no knowledge about the state of its neighbors that had no any contact with him before. This means that in any given time unit each of the nodes send his information to one of its neighbors with whom he did not have any contact (send/receive of information) before
- M_3 – in this model the requirement is that in any unit of time every informed participant sends its information to one of those neighbors that did not received the information from the itself before.

The Messy broadcast models are especially actual when there are some limitations of memory resources in the network. There might be cases when the node can't just remember from whom it received the information before, thus is forced to broadcast with M_3 model.

Note, that in all 3 models in any unit of time a node can receive information from different neighbors simultaneously, but can transmit to only one of its neighbors.

In Graph Plotter we have developed a tool, that allows us to construct Messy broadcast coloring of the constructed graph edges (if possible) that corresponds to corresponding model. The idea behinds this tool is to generate minimum time calling sequence that will allow Messy broadcasting or at least to find tighter upper bound on the minimum time Messy broadcasting. Currently, it is only known an upper bound for the minimum Messy broadcasting time for the models M_2 $(\tau(n) \leq 1.89 \log_2 n)$ and M_3 $(\tau(n) \leq 2.5 \log_2 n)$ for sufficiently large n ([80]).

Below we present one example coloring performed by this tool on the provided known topology for Messy broadcasting. The edges of the same weight are colored in the same color and the broadcast originator is the vertex 1.

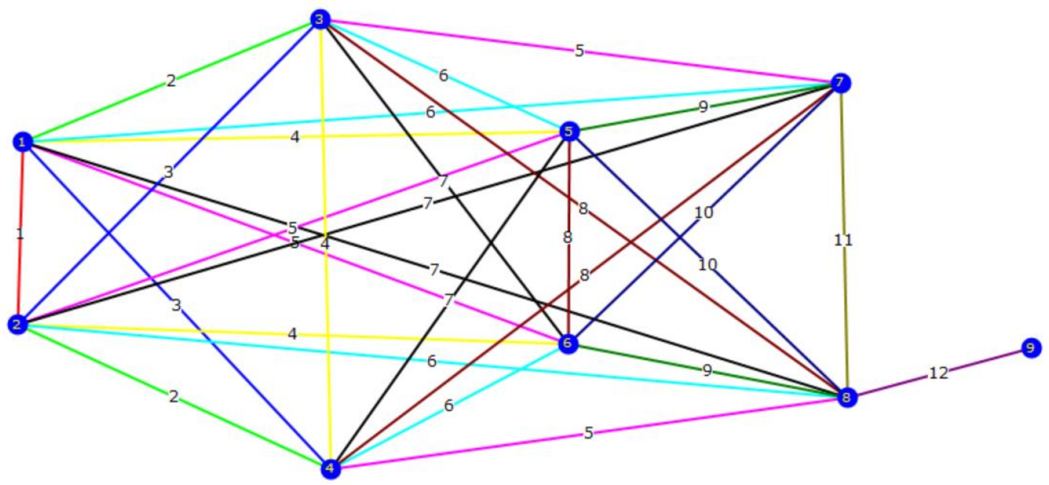

Fig. 25) Messy broadcast model M_2

From the above model it follows that the time required to complete broadcasting is 12. This is not the only possible solution that can be generated by the tool.

3.3 The description of the “Folded paths” mode

This functionality was designed for generating all the ascending folded paths between two given vertices of the graph. This is very useful tool, since it allows us to obtain a folded number of the total graph (the maximum among all the sums of all the folded numbers of edge-disjoint paths between any two vertices). This metrics is very important in the research of fault-tolerant gossiping, since it serves as one of the main metrics that determines the level of propriety of the suggested graph as a base for fault-tolerant gossiping.

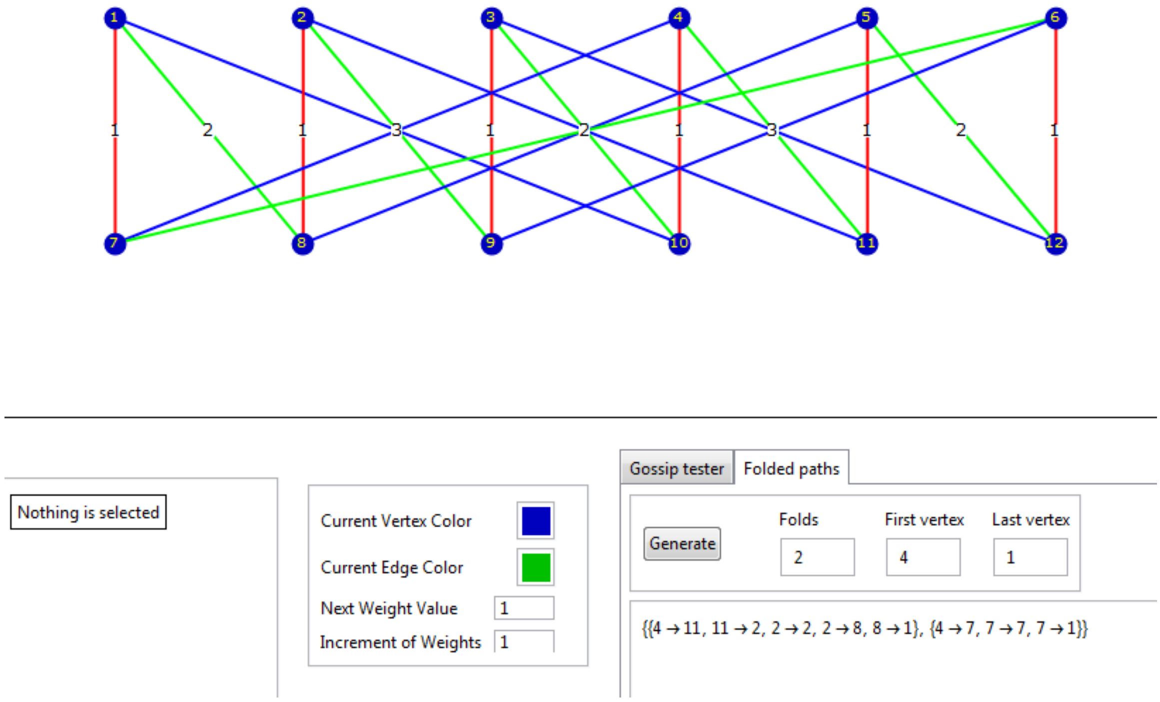

Fig. 26) “Folded Paths” from the vertex 4 to 1 (consists of 2 ascending components)

As we can see from the above figure, this tools accepts the numbers of first and last vertices of the seeking path, as well as the required number of folds. The number 2 (Folds) in the picture above means that folded ascending path between given two vertices should consists of 2 components, each of which is monotonically ascending. As we can see the output is a list of such paths, which also contain the exact folded points (2 for first path and 7 for the second).

3.4 Conclusion

In this chapter we presented the software package, called Graph Plotter and its various components. The package was designed for the researchers of gossiping/broadcasting processes and the main purpose of this tool is to model these processes, verify various hypothesis regarding the gossiping properties of constructed models and perform various modifications with the structure of constructed models. This modifications can be served for the adjustment of constructed networks to the requirements and restrictions that we have in the environment where these network should operate. These modifications allow us to change the structure of certain components of the constructed gossip graphs without modifying the main gossiping properties of the graph.

Graph Plotter was enhanced with more and more functionalities throughout our research. It helped us to obtain and verify our main theoretical results, such us the suggested NOHO graphs, Minimum Gossip Graphs, as well as to verify the fault-tolerance properties of the Knodel graphs. As already was mentioned in the first section of this chapter, the tool was designed to be easily extensible with the new functionalities.

The tool is available for downloading and using for any purposes from drive clouds storage ([81]). Note that, the prerequisite for this package is Wolfram Mathematica 10 or higher.

Conclusion

The main results of the work are the following:

- A new method of construction of minimum gossip graphs for some subsets of the number of vertices was obtained ({[5-6], [9-12], [17-24], [33-48], [65-96], [129-192], [257-384], [513-768], ...}). The proposed construction methods allows the tradeoff between the size of the "base" of the graph and the number of so-called "batch" in- and out-calls.
- A new method of construction of NOHO gossip schemas with minimum possible gossiping time. The proposed construction method was based on the application of the method of local interchange on the Knodel graphs and yields a NOHO gossip graph with gossiping time equal to $\lceil \log_2 n \rceil$, which is the minimum possible time. This addresses the open problem proposed by many authors and proves that NOHO restriction does not affect the minimum gossiping time of the graph (see Problem 4 of section 1.3 of the current study).
- A proof for the Knodel graphs preserving their gossiping properties even after a cyclic permutation of the weights of its edges. This was also an open problem and was proved only in case when the number of vertices is in the form of $n = 2^m - 2$.
- The new fault-tolerant gossip schemas based on the graph combination method and Wheel graphs.
- Hypothesis (verified experimentally) on the upper bound value for the minimum number of calls and minimum possible gossiping time value in fault-tolerant gossip schemes that are based on the construction using Knodel graphs.
- A software package for modeling gossip schemes, obtaining new topologies (more optimal for the given requirements) with the help of various tools, as well as to

experimentally verify their gossiping properties, the level of fault-tolerance, NOHO and NODUP requirements and to construct the folded ascending paths between any pair of vertices.

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