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Exact expressions for a class of observables in supersymmetric gauge theories and in 2d CFT

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Contents

0.1 Introduction

All papers [1–3] that form the basis of this thesis are located on a crossroad between the fields of supersymmetric field theories, gauge field theories and instanton calculus. Supersymmetry is a space-time symmetry discovered (rediscovered) in the 1970's independently by Gervais and B. Sakita (in 1971) [4], Golfand and Likhtman (also in 1971) [5], and Volkov and Akulov (1972) [6]. Its existence in nature is neither proved nor disproved nevertheless it plays a major role in theoretical physics. One of the reasons of its popularity is the Coleman-Mandula theorem [7] which elevates supersymmetry to the status of the single possible extension of the Poincare group, assuming some natural constraints. The existence of the conformal symmetry as an extension of the Poincare group is a loophole, where all particles are massless.

Gauge symmetry was first present in Maxwell's famous work on electrodynamics "A Dynamical Theory of the Electromagnetic Field" in 1864-65 [8], the more modern formulation was popularized by Pauli in 1941 [9]. Nowadays gauge fields are used to describe three of four fundamental forces, one would hardly need any other reasons to study them, furthermore they are present in broad areas of pure mathematics and theoretical physics such as differential geometries and Gravity.

Instantons are a more specific area of research and occur in various situations and contexts like in the calculation of tunneling effects of vacuum states in quantum field theories or in calculations of path integrals in the semiclassical limit [10]. The rest of our introduction is dedicated to the structure of this thesis.

The thesis is divided in to four chapters, the first chapter illustrates some of the background knowledge needed to understand the other chapters. We had to cherry pick the material out of the vast amount of necessary prerequisites, therefore we had to sacrifice consistency for the sake of having a brief summary. For a more broad understanding we urge the reader to have a look at the references. The other three chapters are dedicated to the works on which the thesis is based.

Chapter 1 starts with a short introduction of conformal symmetry(1.1). Hear the importance of the symmetry are highlighted, continued by the geometric and formal definitions. Followed by the derivation of the the algebra generators with space-time dimensions bigger then two. The generators are explained and paired with the corresponding group members. Afterwards the Witt and Virasoro algebras are introduced. Then a quick outline of the differences between different space time dimensions is given, followed by the necessary references for this part.

Because our main interest is in the application of instantons in gauge theories at the beginning of section 1.2 a short overview of path integrals and their connection with Feynman diagrams and non-perturbative effects is given. Further the definition of instanton is mentioned, with the deliberate choice of Euclidean metrics. The process of changing to Euclidean metric is also explained. In section 1.2.1 the system of a double well potential is discussed. The main goal here is to illustrate a situation where instantons arise in a well-known system. Then, it is argued that the shift in vacuum states is described by instantons, also by explicit calculations this effect is clarified. In the end the tunneling amplitude is written which operates as expected.

In section 1.2.2 an introduction of instantons in Yang-Mills theories is given, by presenting the action, equations of motion and the Bianchi identities, which makes possible to properly define instantons and and anti-instantons. Also a discus of the benefits of Wick rotation and its practicality and the differences between Euclidean space-time over Minkowski space-time in this setting is given, which in itself is a reacquiring theme in supersymmetry. Then the instanton number and the Chern character are defined, it is also indicated that instanton solutions are an essential part in approximations of path integrals and non-perturbative effects.

The section 1.2.3 is devoted to the Clifford algebra and its representations. The definition of Clifford algebra in Euclidean and Minkowski spaces are given continued by representations of the algebra for two dimensions, four dimensions (which are the Dirac gamma matrices) and in six dimensions (these are the famous t' Hooft symbols [11]), the representations are given in both Minkowski and Euclidean spaces. At the end a scheme for construction of arbitrary dimensional representations in Euclidean space is illustrated.

In 1.2.4 the connection between Young diagrams to partitions is given. Euler's famous equation is also mentioned, with a hint on how to proof it. This section is a tribute to actual calculations done in [1, 3].

Next, in section 1.2.5, the ADHM construction [12] is introduced, which is a method for constructing a self-dual field strength. Also, the moduli space for instantons with instanton number k is defined, and its dimension is indicated. By a straight check the correctness of ADHM is confirmed. At the end of this section the BPST [13] instanton is introduced by showing that it is a special case of the ADHM construction.

In section 1.3 the Lorentz algebra and its representations are discussed. The algebra of Lorentz transformations is given, the more familiar space rotations and boosts are also defined. The definitions of representations and equivalent representations are given. Also the notion of irreducible representations is highlighted. Then the direct sum and direct product, as methods to construct higher dimensional representations, are reviewed. As an example the $\mathbf{4} \otimes \mathbf{4}'$ representation and its reduction to a direct sum of irreducible representations is illustrated. In an simplistic fashion the notions of Hodge dual, tensor representations and spinor representation are discussed. At the end, the construction of irreducible representation via the $SU(2) \otimes SU(2)$ covering group is shown.

1.3.1 is devoted to Majorana spinors. This review is meaningfully divided into two, first the simple connection between the Dirac equation and Majorana spinors is described. The second part is devoted to the formally correct illustration of Majorana spinors. A basis for 4×4 matrices is constructed out of the gamma matrices. The γ_5 matrix and with it the Weyl spinors are defined. Then by the construction of some auxiliary operators the Majorana spinors are defined. A proof of the contradiction of the Weyl and Majorana conditions is derived.

Supersymmetry has a central role in theoretical physics. One way to see its importance and give a introduction to it is to look at the Coleman-Mandula theorem. In section 1.4.1 the Coleman-Mandula theorem is given and its implications are explored by a simple thought experiment. Then in a toy theory of two scalar fields it is argued that additional generators of internal symmetries must be Lorentz scalars. Then by adding a fermion field with interaction it is argued that the only extension of Poincar algebra is a spin one half conserved current, which are the generators of supersymmetry.

Next in 1.4.2 the superspace is introduced as the natural upgrade of Minkowski space with Grassmann coordinates. Necessary differential and integral relations are given.

In the following section (1.4.3) the superfield is introduced as a field on superspace. By expanding the superfield in Grassmann coordinates a connection is established between superfields and usual field on Minkowski space. The distinction of fermionic and bosonic superfields is established. The notions superderivatives and supercharges is also reviewed with their corresponding anticommutative and commutation rules. Then the chiral and vector superfields are introduced, the gauge superfield is illustrated as a natural sub case of the vector superfield. By gauge fixing the Wess-Zumino field is detached.

Chapter 2. Linear quiver $\mathcal{N} = 1$ 5D gauge theory in Ω background is considered. It is shown that under certain restrictions on the VEV's of the adjoint scalar field corresponding to the first node, only the array of Young diagrams, such that the first diagram has a single column only the others are empty, contribute to the partition function. Furthermore it is proved that this partition function in a simple way is related to the expectation values of Baxter's Q operator (at specific discrete values of the spectral parameter) in the gauge theory with the special node removed. Using known expression of the partition function in the $U(1)$ quiver, Baxter's T-Q difference equations are established and explicit expressions for the VEV of the Q operator in terms of generalized q-deformed Appel's functions is fond. Finally the corresponding expressions for the 4D limit are derived.

The chapter is organized as follows.

In section 2.2 a short review of 5d linear quiver gauge theory: the Nekrasov partition function and important observables Q, y are introduced.

In section 2.3 an extended quiver with specific parameters at the extra nod is introduced and its relation to the Q-observable is analyzed.

Section 2.4 specializes to the case of $U(1)^r$ theory. Difference equations Q-observable are derived. Explicit expressions for the Q observable in terms of generalized Appel and hypergeometric functions are found.

In section 2.5 through dimensional reduction, corresponding difference equations and their solutions for the 4d theory are found.

In sections 2.6, 2.7, 2.8 some technical details, used in the main text, are presented.

Chapter 3. In this short notes using AGT correspondence we express simplest fully degenerate primary fields of Toda field theory in terms an analogue of Baxter's Q-operator naturally emerging in $\mathcal{N}=2$ gauge theory side. This quantity can be considered as a generating function of simple trace chiral operators constructed from the scalars of the $\mathcal{N}=2$ vector multiplets. In the special case of Liouville theory, exploring the second order differential equation satisfied by conformal blocks including a degenerate at the second level primary field (BPZ equation) we derive a mixed difference-differential relation for Q-operator. Thus we generalize the T-Q difference equation known in Nekrasov-Shatashvili limit of the Ω-background to the generic case.

In Section 3.2 we show that an appropriate choice of parameters [14] in A_{r+1} linear quiver theory with $U(n)$ gauge groups is equivalent to insertion of the analoge of Baxters Q-operator into the partition function of a theory with one gauge node less A_r theory with generic parameters. In the 2d CFT side such special choice corresponds to insertion of a degenerated primary field in the conformal block [14].

In Section 3.3 restricting to the case of Liouville theory, starting from the second order differential equation satisfied by the multi-points conformal blocks including a degenerate field $V_{-b/2}$ [15] we derive the analogues equation satisfied by the gauge theory partition function with Q operator insertion. Then we show that this equation leads to a mixed linear difference-differential equation for Q operators which is a direct generalization of the $T-Q$ equation from NS limit to the case of generic Ω-Background. Finally we summarize our results and discuss a couple of further directions which we think are worth pursuing.

Chapter 4. We specify Gaiotto's proposal for the RG domain wall between some coset CFT models to the case of two minimal N=1 SCFT models SM_p and SM_{p-2} related by the RG flow initiated by the top component of the Neveu-Schwarz superfield $\Phi_{1,3}$. We explicitly calculate the mixing coefficients for several classes of fields and compare the results with the already known in literature results obtained through perturbative analysis. Our results exactly match with both leading and next to leading order perturbative calculations.

The chapter is organized as follows:

Section 4.2 is a brief review of the 2d $N = 1$ superconformal filed theories.

Section 4.3 is devoted to the description of the coset construction of $N = 1$ SCFT. Of course everything here is well known; our purpose here is to fix notations and list the relevant formulae in a form, most convenient for the further calculations.

In Section 4.4 we formulate Gaiotto's general proposal for a class of coset CFT models.

Section 4.5 is the main part of our paper. We explicitly calculate the mixing coefficients for the several classes of local fields in the case of the supersymmetric RG flow discussed above using RG domain wall proposal. Then we compare this with the perturbation theory results available in the literature finding a complete agreement.

Chapter 1

Preliminary ideas and concepts

1.1 Introduction to conformal symmetry

Before we start to introduce the symmetry itself we want discus one of its main applications, the Conformal Field theory (CFT). CFT is a field theory as the name suggests with an additional symmetry the conformal symmetry. CFT's are QFT's but because of the additional symmetry the approach can be somewhat different. In a standard QFT the goal is to calculate all correlation functions at least to some precision, which usually involves the Lagrangian or partition function. In contrast if our theory has a bigger (bigger then the standard Poincar group) symmetry it can be used to get some information about the correlation functions without actually solving or even knowing the full Lagrangian. The extreme of this situation arises in 2 dimensions where the conformal group is infinite dimensional. In this short overview we define some of the core definitions and relations in conformal symmetry.

Conformal transformations are transformations that conserve the angle between two intersecting lines at the point of intersection. The more abstract definition states that if we have a map φ from a metric space M_1 to a metric space M_2 then the map conserves the metric up to a function

$$g'_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu}(x)\,. \tag{1.1}$$

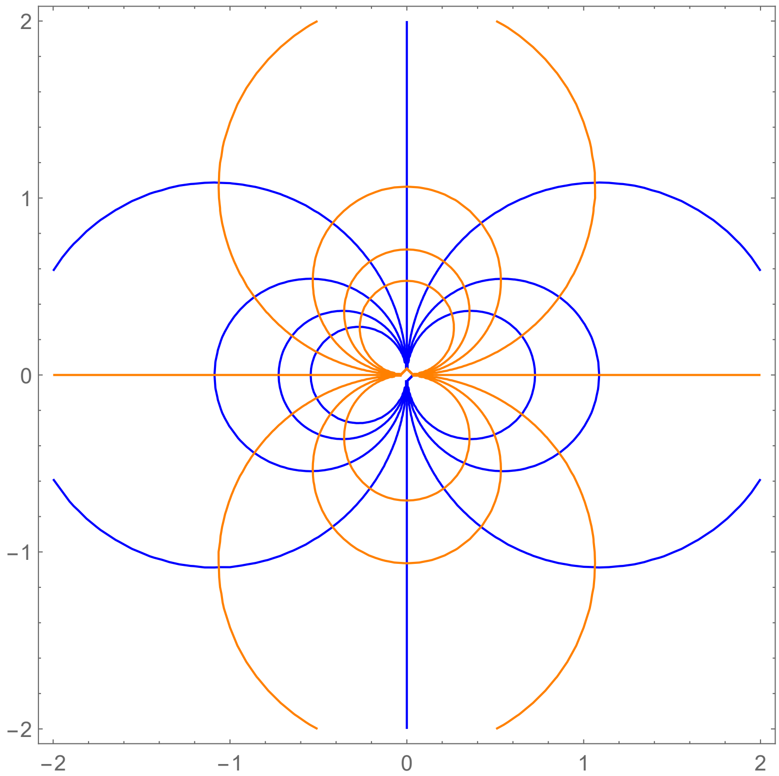

Figure 1.1: The conformal map of the complex mapping $z \rightarrow \frac{1}{z}$, where the blue contours are the real part, the orange contours the imaginary. Notice that at the intersections the angle is $\frac{\pi}{2}$.

We use Einsteins convention [16] by assuming a sum over all repeating indexes. From now on, for convenience, we take $M_1 = M_2 = M$ and, furthermore, that M is a flat Minkowski space with signature $(-, ..., -, +, ..., +)$, we denote this metric by η. The condition of conformality has this simplified form:

$$\eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) \eta_{\mu\nu} \,. \tag{1.2}$$

If $\Lambda(x)$ is 1 than we get the condition for Poincar transformations, which implies that the Poincare transformations are a sub-case of the more general conformal transformations.

To study a symmetry it is nearly always a good idea to study its Lie algebra: the infinitesimal expansion near the identity. These objects have an additional structure, besides the product operation that they inherit from the group, they also have a summation operation. Nevertheless they are sufficiently general(at least for our case), by which we insist that there is a "reverse expansion" from the Lie algebra to the group.

Infinitesimal coordinate transformations have this general form:

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho} + O(\epsilon^2)\,. \tag{1.3}$$

To select only the conformal transformations out of all transformations we demand eq 1.2 to be true.

$$\begin{aligned}
\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} &= \eta_{\rho\sigma}\Big(\delta^{\rho}_{\mu} + \frac{\partial \epsilon^{\rho}}{\partial x^{\mu}} + O(\epsilon^2)\Big)\Big(\delta^{\sigma}_{\nu} + \frac{\partial \epsilon^{\sigma}}{\partial x^{\nu}} + O(\epsilon^2)\Big) \\
&= \rho_{\mu\nu} + \eta_{\mu\sigma}\frac{\partial \epsilon^{\sigma}}{\partial x^{\nu}} + \eta_{\rho\nu}\frac{\partial \epsilon^{\rho}}{\partial x^{\mu}} + O(\epsilon^2) \\
&= \eta_{\mu\nu} + \Big(\frac{\partial \epsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial \epsilon_{\nu}}{\partial x^{\mu}}\Big) + O(\epsilon^2)\,.
\end{aligned} \tag{1.4}$$

From here we can conclude

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = K(x)\eta_{\mu\nu}\,, \tag{1.5}$$

where $K(x)$ is derived from the expansion of $\Lambda(x)$, and can be calculated by taking the trace of eq 1.5

$$K(x) = \frac{2}{d}(\partial^{\mu}\epsilon_{\mu})\,, \tag{1.6}$$

where d is the dimension of our space. So we get the conformality condition on the parameter of coordinate transformations ϵ^{μ}.

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial^{\mu}\epsilon_{\mu})\eta_{\mu\nu}\,. \tag{1.7}$$

This (1.7) can be used to construct higher derivative conditions namely:

$$(\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu)(\partial\cdot\epsilon) = 0\,, \tag{1.8}$$

$$(d-1)\Box(\partial\cdot\epsilon) = 0\,, \tag{1.9}$$

$$\partial_\mu\partial_\nu\epsilon_\rho = \frac{1}{d}(\eta_{\rho\mu}\partial_\nu + \eta_{\rho\nu}\partial_\mu - \eta_{\mu\nu}\partial_\rho)(\partial\cdot\epsilon) = 0\,. \tag{1.10}$$

Note that 1.8 and 1.9 make it clear that $d = 1$ and $d = 2$ are special. So we first look at the case where $d \geq 3$. From equations 1.8 an 1.9 follows:

$$\partial_\mu\partial_\nu(\partial\cdot\epsilon) = 0\,. \tag{1.11}$$

which means $(\partial\cdot\epsilon)$ is at most linear in x, so by expanding ϵ in powers of x we get:

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\lambda}x^\nu x^\lambda\,, \tag{1.12}$$

where $a_\mu, b_{\mu\nu}$ and $c_{\mu\nu\lambda}$ are infinitesimal small constants. All this constants stand for various transformations and in general any conformal transformation is a sequence of them.

a_μ stands for space-time translations. $b_{\mu\nu}$ can be divided (like every matrix) into a sum of a symmetric and an antisymmetric parts, from equation 1.5 we see that the symmetric part is proportional to the metric.

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu}\,, \tag{1.13}$$

hear α represents dilatations and $m_{\mu\nu}$ represents rotations. Using eq 1.10 we can reduce the number of independent components of $c_{\mu\nu\lambda}$, by doing so we get:

$$c_{\mu\nu\lambda} = \eta_{\mu\lambda}b_\lambda + \eta_{\mu\nu}b_\lambda - \eta_{\nu\lambda}b_\mu, \qquad where \qquad b_\mu = \frac{1}{d}c^\rho{}_{\rho\mu}\,. \tag{1.14}$$

$c_{\mu\nu\lambda}$ stands for the special conformal transformations(SCT). By exponentiation the generators

Name	Transformation	Generator
translation	$x^\mu \to x^\mu + a^\mu$	$P_\mu = -i\partial_m u$
dilation	$x^\mu \to \alpha x^\mu$	$D = -ix^\mu \partial_\mu$
rotation	$x^\mu \to M^\mu_\nu x^\nu$	$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
SCT	$x^\mu \to \frac{x^\mu - (x\cdot x)b^\mu}{1-2(b\cdot x)+(b\cdot b)(x\cdot x)}$	$K_m u = -i(2x_\mu x^\nu \partial_\nu - (x\cdot x)\partial_\mu)$

Table 1.1: Conformal transformation and their generators for $d > 2$

of the infinitesimal transformations one can recover the finite (aka group) transformations(see table 1.1).

For two dimensional space-time its customary to use complex coordinates.

$$z = x^0 + ix^1, \qquad \bar{z} = x^0 - ix^1. \tag{1.15}$$

the bar denotes complex conjugation, then the infinitesimal transformations

$$z \to z + \epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n(-z^{n+1}),$$
$$\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n(-\bar{z}^{n+1}),$$

correspond to the algebra of the conformal group, denoted l and $\bar{l}$, the generators l_n and $\bar{l}_n$ form the de Witt algebra.

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n}, \\ [\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n}, \\ [l_m, \bar{l}_n] &= 0. \end{aligned} \tag{1.16}$$

This algebra has a central extension, with central charge c, the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} = \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{1.17}$$

the L's are the Virasoro generators. Both the de Witt and Virasoro algebras are infinite dimensional which makes them entirely different from the four dimensional algebra. This remarkable property has an important role in string theory and some integrable models. There are some excellent books and lecture notes on this topic, for a broader discussion or the continuation of it see [17–19] .

1.2 Introduction to Instantons

In quantum field theories correlators contain all the information about the observables of the system. The path integral approach enables us to calculate the correlations in QFT's. Historically this approach was created to calculate Feynman diagrams for perturbative phenomena, but now it is believed that the path integrals, even for small coupling constants, can describe non-perturbative effects. The path integrals are notoriously hard to calculate, to counter this hardship a number of approaches were developed. One of the more consistent approaches is to look at path integrals as limits of integrals over fields on latices.

$$\left\langle \prod_i O(x_i, t_i) \right\rangle = \int \mathcal{D}\phi \prod_i O(x_i, t_i) e^{\frac{i}{\hbar} S(\phi)} \,. \tag{1.18}$$

Where S is the action, in particular for gauge theories

$$S = -\frac{1}{4g^2} \int d^4 F^a_{\mu\nu} F^{a\mu\nu} \,. \tag{1.19}$$

We always assume the Einstein summation convention if not explicitly stated otherwise. In the classical limit $\hbar \to 0$ the integral 1.18 is dominated by the extreme $\delta S = 0$ point which can be a maximum, minimum or a saddle point. It is convenient to take the analytic continuation:

$$t_E = it \,, \tag{1.20}$$

we change to a Euclidean space time.

$$iS = -\frac{1}{4g^2}\int dt_E d^3x F^a_{ij}F^a_{ij} + ... = -S_E\,. \tag{1.21}$$

Here we have the advantage of having a positive defined action. The correlation functions are dominated by the minimum of $S_E(\phi)$ these can be vacuum minimum or solutions to the equation of motion. Instantons are solutions to the Euclidean equation of motion. If the theory has a small coupling constant g^2 we are enabled to calculate the usual vacuum bubble diagrams and perturbations around the instanton solutions. Besides, the role in calculating path integrals instantons have many other uses. One such situation arises when the tunneling between two vacuum states are calculated.

1.2.1 Instantons in Quantum Tunneling

Instantons can be used to calculate tunneling effects between two vacuum states. To see this lets look at a particle in a double well potential

$$V(x) = V_0\Big(1 - \frac{x^2}{x_0^2}\Big)^2\,. \tag{1.22}$$

V_0 is the height of the barrier between the two sectors, and in case if $V_0 >> 1$ the solutions of the Schrdinger equation gives rise to a spectrum similar to the harmonic oscillator with corrections

$$E_n = (n + \frac{1}{2} \pm \frac{1}{2}\Delta_n)\omega_0\,, \tag{1.23}$$

where

$$\Delta_n \sim e^{-\frac{2V_0\omega_0}{3}}, \qquad \omega_0^2 = \frac{8V_0}{x_0^2}\,. \tag{1.24}$$

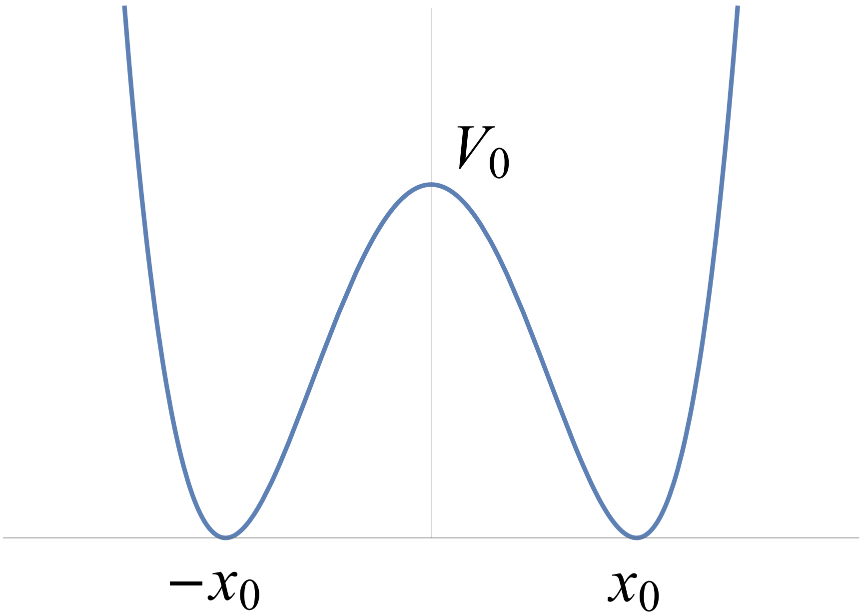

Figure 1.2: The double-well potential

This corrections are a result of the tunneling effect between the two sectors, and as expected the corrections get exponentially smaller when we rise the barrier. To see this let us compute the probability of a particle shifting from $x = -x_0$ to $x = x_0$ in δt time

$$\langle x_0|e^{-H\delta t}| - x_0\rangle = \int Dx(t)\mathrm{exp}\Big(- \int_0^{\delta t} dt_E\big[\frac{\dot{x}^2}{2} + V_E(x)\big]\Big) . \tag{1.25}$$

where

$$V_E = -V(x) . \tag{1.26}$$

The transition to Euclidean space-time has the effect of changing the usual Lagrangian like in equation 1.18 by the Hamiltonian with the Euclidean potential V_E. The equation of motion for the double well potential are

$$\ddot{x} = -V_E'(x) , \tag{1.27}$$

and its solution

$$\dot{x} = \sqrt{-2V_E(x)} = \frac{\sqrt{2V_0}}{x_0^2}(x^2 - x_0^2)\,, \tag{1.28}$$

$$x = x_0 \text{tanh}\ t_E \sqrt{\frac{2V_0}{x_0^2}}\,. \tag{1.29}$$

Inserting these into the Euclidean action one finds

$$S = \int_{-infty}^{\infty} dt_E [\frac{\dot{x}^2}{2} - V_E] = \int_{-infty}^{\infty} dt_E \dot{x}^2 = \frac{2V_0}{3}\sqrt{\frac{8x_0^2}{V_0}}\,. \tag{1.30}$$

The first approximation to the tunneling effect are instanton solutions

$$A \sim e^{-S_E} \sim \exp\Big(-\frac{2V_0}{3}\sqrt{\frac{8x_0^2}{V_0}}\Big)\,. \tag{1.31}$$

for the proper factor in front of 1.31 one needs to calculate the fluctuations around the instanton solutions.

1.2.2 Instantons in Gauge Theories

Instantons are solutions to the Euclidean Yang-Mills equation of motion. The Euclidean Yang-Mills action has this form

$$S_E = \frac{\text{Im}\tau}{8\pi}\int d^4x TrF_{ij}F_{ij} - i\frac{\text{Re}\tau}{8\pi}\int d^4x TrF_{ij}\tilde{F}_{ij}\,, \tag{1.32}$$

with

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]\,, \tag{1.33}$$

$$\tilde{F}_{ij} = \frac{1}{2}\epsilon_{ijkl}F_{kl}\,. \tag{1.34}$$

The field with the tilde is the dual field, it naturally obeys the Bianchi identity, which can be demonstrated by a simple insertion:

$$
\begin{aligned}
D_i \tilde{F}_{ij} &= \partial_i \tilde{F}_{ij} + [A_i, \tilde{F}_{ij}] \\
&= \frac{1}{2}\epsilon_{ijkl}(2\partial_i\partial_k A_l + \partial_i[A_k, A_l] + 2[A_i, \partial_k A_l] + [A_i, [A_k, A_l]]) \\
&= 0\,,
\end{aligned}
$$

where the first three terms in the last equation vanish because of the fully antisymmetric tensor or cancel themselves out, the last term vanishes because of the Jacobi identity.

The Yang-Mills equation of motion:

$$D_i F_{ij} = \partial_i F_{ij} + [A_i, F_{ij}] = 0\,, \tag{1.35}$$

has the same form as the Bianchi identity for the dual field. This suggests a solution for the Yang-Mills equation, where the field is proportional to the dual field

$$F = c\tilde{F}. \tag{1.36}$$

From $\tilde{\tilde{F}} = F$ we conclude that $c = \pm 1$. This condition is valid only for the Euclidean space time. In contrast for Minkowski space-time the corresponding condition $\tilde{\tilde{F}} = -F$ differs by a minus sign. This minus sign is a consequence of the determinant of the metric in the definition of Levi-Civita tensors, so for the Minkowski space-time we get $c = \pm i$. This definitions and identities can be written in the language of forms, where they have a simpler look:

$$F = \frac{1}{2}F_{ij}dx^i dx^j\,, \qquad *F = \frac{1}{2}\tilde{F}_{ij}dx^i dx^j\,, \tag{1.37}$$

with

$$F = DA = dA + A \wedge A\,. \tag{1.38}$$

The equation and motion

$$D * F = d * F + [A, *F] = 0 \tag{1.39}$$

The Bianchi identities for the dual field

$$DF = dF + [A, F] = 2dAA + 2AdA + AA^2 - A^2A = 0\,. \tag{1.40}$$

As mentioned for the Euclidean case in equation 1.36 we have $c = \pm 1$. An instanton with the plus sign is called Yang-Mills instanton and for the minus sign anti-instanton. We classify instantons by the topological integer k.

$$k = -\frac{1}{16\pi^2}\int d^4x \mathrm{tr}(F_{mn}\tilde{F}_{mn}) = -\frac{1}{8\pi^2}\int d^4x \mathrm{tr}(F \wedge F)\,. \tag{1.41}$$

k is also called the instanton number. The Chern character and Chern numbers are defined as:

$$ch(F) = \sum_i ch_i(F) = \exp\left(\frac{iF}{2\pi}\right)\,. \tag{1.42}$$

The instanton number is the second Chern number. To show that for a fixed instanton number instantons minimize the action in the space of gauge connections we start with this trivial inequality:

$$\int d^4x \mathrm{tr}(F \pm \tilde{F})^2 \geq 0\,, \tag{1.43}$$

then we open the parentheses and and use $\mathrm{tr}F^2 = \mathrm{tr}\tilde{F}^2$, which is a direct consequence of equation 1.36, we get:

$$\int d^4x \mathrm{tr}F^2 \geq \Big| \int d^4x \mathrm{tr}F\tilde{F} \Big| = 16\pi^2|k|\,, \tag{1.44}$$

for the plus sign in 1.43 the inequality 1.44 saturates for anti-instantons, the same is also true for the minus sign and instantons. After inserting this in the Yang-Mills action we conclude that for instantons:

$$-S_{\mathrm{inst}} = 2\pi i k\tau\,, \tag{1.45}$$

or for anti-instantons

$$-S_{\mathrm{inst}} = 2\pi i k\tau^*\,. \tag{1.46}$$

Instanton corrections of the correlators take this form:

$$\langle \mathcal{O} \rangle = \int DAe^{-S}\mathcal{O} = \sum c_k d^k + \sum_{k=1}^{\infty} h_k e^{-\frac{8\pi^2 k}{g^2}}\,. \tag{1.47}$$

Correlators in Yang-Mills theories are computed by path integrals which can be approximated by the contribution of the saddle point solutions and the fluctuations around them. The Euclidean saddle point solutions are known as instantons.

1.2.3 Representations of Clifford algebra and construction in higher dimensions

Because of the heavy use of spinors and their properties we will introduce the Clifford algebra. Also we'll construct the representations in higher dimensions for both Minkowski and Euclidean spaces. For the Minkowski space as before we choose the metric as $\eta_{\mu\nu} = \mathrm{diag}(-,+,...,+)$. The general d dimensional Clifford algebra in Minkowski space is defined by generators Γ_μ where μ

goes from 0 to $d-1$. The generators are also defined in Euclidean space, we denote them as Γ_m, where m goes from 1 to d. For the sake of clarity we will use Greek letters for Minkowski space indexes and Latin letters for Euclidean space indexes. The generators obey the following constraints.

For Minkowski space:

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}\,. \tag{1.48}$$

For the Euclidean space

$$\{\Gamma_m, \Gamma_n\} = 2\delta_{mn}\,, \tag{1.49}$$

where $\delta_{mn} = 1$ if $m = n$ otherwise $\delta_{mn} = 0$. We will directly introduce the Clifford algebra for a number of even dimensions and review a scheme to construct them for arbitrary even dimensions. We always choose a representation where the additional generator is given as:

$$\Gamma_{d+1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$d = 2$ for Minkovski space

$$\Gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$d = 2$ for Euclidean space

$$\Gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In $d = 4$ we have the famous gamma matrices, for the Euclidean case

$$\gamma_n = \begin{pmatrix} 0 & -i\sigma_n \\ i\sigma_n & 0 \end{pmatrix} ,$$

where $\sigma_n = (i\vec{\tau}, 1)$ and $\vec{\tau}$ are the usual Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The gamma matrices for Minkowski space

$$\gamma_\nu = \begin{pmatrix} 0 & \sigma_\nu \\ -\sigma_\nu & 0 \end{pmatrix} ,$$

where $\sigma_\nu = (-1, \vec{\tau})$. For further uses we also define the self-dual σ_{mn} and anti-self-dual $\bar{\sigma}_{mn}$ quantities in Euclidean space. The self-duality and anti self duality will play a central role in building the solution for self-dual and anti-self-dual Yang-Mills equations of motion.

$$\sigma_{nm} = \frac{1}{4}(\sigma_n \bar{\sigma}_m - \sigma_m \bar{\sigma}_n) , \tag{1.50}$$

$$\bar{\sigma}_{nm} = \frac{1}{4}(\bar{\sigma}_n \sigma_m - \bar{\sigma}_m \sigma_n) . \tag{1.51}$$

The bar notation denotes the Hermitian adjoin (Hermitian conjugate). Now we will discuss the six dimensional case. For the Euclidean case we have

$$\Gamma_m = \begin{pmatrix} 0 & \Sigma_m \\ \bar{\Sigma}_m & 0 \end{pmatrix} ,$$

with

$$\Sigma_m = (\eta^3, i\bar{\eta}^3, \eta^2, i\bar{\eta}^2, \eta, i\bar{\eta})\,, \tag{1.52}$$

$$\bar{\Sigma}_m = (-\eta^3, i\bar{\eta}^3, -\eta^2, i\bar{\eta}^2, -\eta, i\bar{\eta})\,, \tag{1.53}$$

where the η-s are three 4x4 matrices known as t' Hooft symbols.

$$\bar{\eta}^m_{AB} = \eta^m_{AB} = \epsilon_{mAB}\,, \tag{1.54}$$

$$\bar{\eta}^m_{4A} = \eta^m_{A4} = \delta_{mA}\,, \tag{1.55}$$

$$\eta^m_{AB} = -\eta^m_{BA}, \qquad \bar{\eta}^m_{AB} = -\bar{\eta}^m_{BA}\,. \tag{1.56}$$

indexes m, A and B run from 1 to 3. In Minkowski space the relation between Γ and Σ stay the same but with diferent Σ's.

$$\Sigma_\mu = (i\eta^3, i\bar{\eta}^3, \eta^2, i\bar{\eta}^2, \eta, i\bar{\eta})\,, \tag{1.57}$$

$$\bar{\Sigma}_\mu = (-i\eta^3, i\bar{\eta}^3, -\eta^2, i\bar{\eta}^2, -\eta, i\bar{\eta})\,. \tag{1.58}$$

Now suppose we have two representations in Euclidean space, one with dimension d_1 and an another with dimension d_2. The generators are written as $\Gamma^{(d_1)}_n$ and $\Gamma^{(d_2)}_m$ respectively, n runs from 1 to d_1 and m runs from 1 to d_2. Then a representation with dimension $D = d_1 + d_2$, also in Euclidean space, can be constructed out of the formal representations.

$$\Gamma_k = \{\Gamma^{(d_1)}_n \otimes 1, \Gamma^{(d_1)}_{d_1+1} \otimes \Gamma^{(d_2)}_m\}\,, \tag{1.59}$$

here k goes from 1 to D.

$$
\begin{gathered}
4 \\
3+1 \\
2+2 \\
2+1+1 \\
1+1+1+1.
\end{gathered}
\tag{1.60}
$$

Figure 1.3: The partitions of the number four.

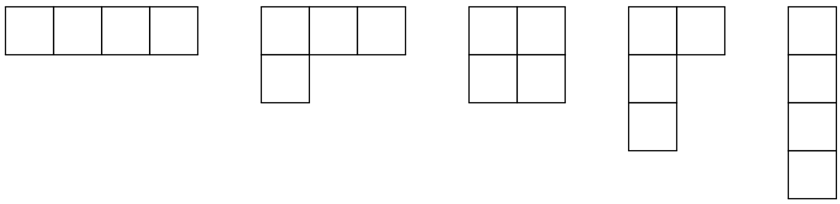

Figure 1.4: All Young diagrams with four boxes.

1.2.4 Young diagrams and partition of numbers

The partition of a natural number n is the process of writing n as a sum of positive integers, where the order of summands is neglected. So lets look at the example of the number 4. There are a total of five distinct partitions:

In contrast Young diagrams are diagrams of ordered rows of boxes where the number of boxes in a row newer decreases. There is a one to one correspondence between Young diagrams and the partitions of natural numbers. To see this look at the figure 1.3 and 1.4 the diagrams and partitions are corresponding. We denote the number of partitions by $p(n)$, for our example $p(4) = 5$. Here are the partition numbers for 0 to 9

$$1,\ 1,\ 2,\ 3,\ 5,\ 7,\ 11,\ 15,\ 22,\ 30. \tag{1.61}$$

The partition number has a famous generating function discovered by Euler.

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \left(\frac{1}{1-x^n}\right). \tag{1.62}$$

The later in fact is a special case of the q-Pochhammer symbol which we will discuss later. The fact that equation 1.62 is correct can be seen when one notices that the r.h.s has the form of a product of sums over infinite geometric series. The connection between Young diagrams and partitions is heavily used in calculations in(cite my last 2 pps)

1.2.5 ADHM construction

In mathematical physics, the ADHM construction is the construction of all instantons in YM theories by Atiyah, Drinfeld, Hitchin and Manin in their paper "Construction of Instantons" [12]. In this section we will introduce the ADHM construction for $\mathbb{R}^4$. The ADHM constriction is a method to construct solutions for eq. 1.36. We consider only the instanton aka $c = 1$ case. First we will look at an ansatz and then proof that it really constitutes self-dual instantons. The first observation is that if

$$F_{mn} \sim \sigma_{mn} \,, \tag{1.63}$$

then the field strength is an instanton. We start with an ansatz matrix Δ of dimensions $(N + 2k) \times 2k$ of a peculiar form

$$\Delta(x) = \mathbf{a} + \mathbf{x_n}\mathbf{b^n} = \begin{pmatrix} \omega_{u,i\dot{\alpha}} \\ \\ a_{i\alpha,j\dot{\alpha}} \end{pmatrix} + \mathbf{x_n} \begin{pmatrix} 0 \\ \\ \sigma^n_{\alpha\dot{\alpha}\delta_{i,j}} \end{pmatrix} . \tag{1.64}$$

The indexes can be a little confusing but to make it bearable we will hold to this notations: i, j go from 1 to k, u, c go from 1 to N, μ, λ go from 1 to $N + 2k$, $\alpha, \beta, \dot{\alpha}, \dot{\beta}$ are spinor indexes and are 1 or 2. For example a quantity with indexes $u, i\alpha$ is a $(N \times 2k)$ matrix. The moduli space can be divided into sectors of different instanton numbers k, and are denoted as $\mathfrak{M}_k$. To avoid cumbersome factors we think of the moduli space as already factorized by local gauge transformations. The a and b matrices are parametrising the moduli space of instanton solutions. We also need the Hermitian conjugate of Δ:

$$\Delta_{\lambda i\dot{\alpha}}(x) = a_{\lambda i\dot{\alpha}} + b^{\alpha}_{i\lambda} x_{\alpha\dot{\alpha}}, \qquad \bar{\Delta}^{\lambda\dot{\alpha}}_{i}(x) = \bar{a}^{\lambda\dot{\alpha}}_{i} + \bar{b}^{\lambda}_{i\alpha}\bar{x}^{\alpha\dot{\alpha}}. \tag{1.65}$$

Hear we used the quaternion form of the coordinate x, as one can see this definition is broader then in eq. 1.64 .The reason is that the a and b matrices are not uniquely fixed by ADHM. By rotating Δ and U.

$$\Delta \rightarrow \Lambda\Delta\Gamma^{-1}, \qquad U \rightarrow \Lambda U\,, \tag{1.66}$$

we conserve the ADHM constraints, where $\Lambda \in U(N+2k)$, $\quad \Gamma \in Gl(k,\mathbb{C})$. We'll make use of the already "fixed" matrices defined in eq. 1.64. One of the requirements of ADHM is that $\Delta_{\dot{\alpha}}(x) : \mathbb{C}^{k} \rightarrow \mathbb{C}^{N+k}$ is injective and $\bar{\Delta}^{\dot{\alpha}}$ is surjective. Furthermore we will see that the parameter k is the instanton charge introduced in section 1.2.2. We also need the normalized kernel of Δ, denoted as U. U is a $(N+2k)\times N$ dimensional complex valued matrix, which as expected is also coordinate dependent.

$$\bar{\Delta}U = 0 = \bar{U}\Delta, \qquad U\bar{U} = \bar{U}U = \mathbb{1}_{N\times N}\,. \tag{1.67}$$

The U matrices play an important role in constructing the ansatz gauge connections.

$$A_m = \bar{U}\partial_m U\,, \tag{1.68}$$

for the $k=0$ instantons we recovers the pure gauge. Because for a pure gauge the field strength vanishes it naturally obeys the self-duality constraint. The ADHM constructions works for arbitrary instanton number. The ADHM construction also requires that:

$$\bar{\Delta}^{\dot{\alpha}\lambda}_{i}\Delta_{\lambda j\dot{\beta}} = f^{-1}_{ij}\delta^{\dot{\alpha}}_{\dot{\beta}}\,, \tag{1.69}$$

the non-degeneracy condition was employed to guaranty the existence of f^{-1}. Here f is a arbitrary $k \times k$ Hermitian matrix dependent on space coordinates. The Δ's are dependent on the a and b, by inserting the definitions we get this eq. in its components.

$$\bar{\omega}\tau^c\omega - i\bar{\eta}^v_{mn}[a_m, a_n] = 0\,. \tag{1.70}$$

The η's are t'Hooft symbols which we defined earlier, and a_m is connected to a a by $a_{\dot{\alpha}\alpha} = a_m\sigma^m_{\dot{\alpha}\alpha}$. We repress some indexes now and then to make the equations more readable, but often only the spinor indexes are important to be followed. For consistency we want a completeness relation

$$U_{\lambda u}\bar{U}^{\mu u} = \delta^\mu_\lambda - \Delta_{\lambda ij}f^{ij}\bar{\Delta}^{\dot{\alpha}\mu}_j\,, \tag{1.71}$$

this relation enables us to convert U sums with Δ sums a trick used extensively in ADHM calculus. By calculating the dimensions of the moduli space , which involves counting.the number of freedom in ADHM and subtracting the number of symmetries, one gets

$$\dim_{\mathbb{R}}\mathfrak{M}_k = 4kN\,. \tag{1.72}$$

Now after the illustration of the ADHM construct we are able to proof its preposition. To proof that the field strength is self dual we start with the definition of F_{mn} and insert the ansatz gauge field

$$\begin{aligned}
F_{mn} &= \partial_m A_n - \partial_n A_m - [A_m, A_n] \\
&= \partial_{[m}(\bar{U}\partial_{n]}U) + (\bar{U}\partial_{[m}U)(\bar{U}\partial_{n]}U) = \partial_{[m}\bar{U}(1-\bar{U}U)\partial_{n]}U = \partial_{[m}\bar{U}(\Delta f\bar{\Delta})\partial_{n]}U \\
&= \bar{U}\partial_{[m}\Delta f\partial_{n]}\bar{\Delta}U = \bar{U}b\sigma_{[m}f\bar{\sigma}_{n]}\bar{b}U = \bar{U}b\sigma_{[m}f\sigma_{mn}\bar{b}U\,,
\end{aligned} \tag{1.73}$$

here we used eq. 1.67 and eq. 1.71. The matrices σ_{mn} are defined in section 1.2.3. So, we see that F_{mn} is proportional to σ_{mn} which insures the self duality, this concludes the proof, before we conclude this chapter we present some explicit forms and remarks. As we can see we didn't use the special form of b, if we use it we'll get:

$$F_{mn} = 4\bar{U} \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{mn} \otimes f_{[k\times k]} \end{pmatrix} U \sim \sigma_{mn} \,. \tag{1.74}$$

To wind up this chapter lets look at a special case the BPST instanton [13]. BPST stands for Belavin, Polyakov, Schwarz and Tyupkin who found this solution. The BPST is the $N = 2,\ k = 1,$ and $a_{\alpha\dot{\alpha}} = 0$ case. By simple insertions we find:

$$A_m = \frac{2x_n \sigma_{mn}}{x^2+\rho^2}, \qquad F_{mn} = 4\bar{U} \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{mn}}{\rho^2+r^2} \end{pmatrix} U = \frac{4\rho^2 \sigma_{mn}}{(\rho^2+r^2)^2} \,, \tag{1.75}$$

$$\Delta = \begin{pmatrix} \rho \mathbb{1}_{[2\times 2]} \\ x + 2\times 2 \end{pmatrix} , \qquad \bar{U} = \frac{1}{(\rho^2+r^2)^{\frac{1}{2}}} \big(- x_{[2\times 2]} \rho \mathbb{1}_{[2\times 2]} \big) \,,$$

$$\bar{\Delta}\Delta = (\rho^2+r^2)\mathbb{1}_{[2\times 2]} \quad \Rightarrow \quad f = \frac{1}{\rho^2+r^2} \,, \qquad r^2 = x^m x_m \,. \tag{1.76}$$

Before we continue to the next section we'll give a short list of references for the various subjects discussed here. For instantons and instantons in gauge theories look [10,20]. For Clifford algebra see [21].

1.3 Lorentz Algebra and its Representations

In this section well introduce the Lorentz algebra. As already mentioned supersymmetry is an extension of the Poincar group. In it self the Poincar algebra is the Lorentz algebra with the addition of space-time translations. Lorentz algebra generators in four dimensional($d = 4$) Minkowski space [21]:

$$i[J^{\mu\nu}, J^{\lambda\rho}] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho} \,, \tag{1.77}$$

where μ, ν go from zero to three, $J^{\mu\nu}$ is antisymmetric and contains the generators of space rotations(J) and boosts (P).

$$J^i = \frac{1}{2}\epsilon_{ijk}J^{jk}\,, \tag{1.78}$$

$$P^i = J^{i0}\,, \tag{1.79}$$

here i, j and k go from one to three. η is the Minkowski metric with signature $(-, +++)$ and ϵ is the the Levi-Civita symbol for $d = 3$ Euclidean space. Because we want our laws of nature to behave in a predictable manner under the Lorentz transformations, we classify our field under the finite representations of the Lorentz group. First of all a representation is a set of matrices $M(g)$, where g is the group element, which obey the following constraints:

$$M(e) = I, \qquad M(g_1 g_2) = M(g_1)M(g_2), \tag{1.80}$$

where e is the identity in the group and I is a unit matrix. For Lie groups the Lie algebra(Λ) can be expressed as an infinitesimal deviation from the identity:

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \eta^{\mu\rho}\omega_{\rho\nu}\,, \tag{1.81}$$

ω is a infinitesimal and antisymmetric parameter. The fields are transforming under the n dimensional representation by this formula

$$\delta\phi^i = \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^i_j\,\phi^j \tag{1.82}$$

where i, j go from one to n and $\mathcal{J}^{\mu\nu}$ is a n dimensional representation of Lorentz algebra. This formula can be shortened if we think of the various fields as n-tuples and assume matrix multiplication:

$$\delta\phi = \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\phi\,. \tag{1.83}$$

The corresponding finite transformation for Lie algebras coincides with the exponentiation of the small transformation:

$$\phi \to D(\omega)\phi = \exp(\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu})\phi. \tag{1.84}$$

For compact groups any dimensional representation has a equivalent unitary representation so their generators are Hermitian [22]. But the same is not true for non-compact groups. The Lorentz group is in fact non compact , the rotation generators are Hermitian but the boosts are anti-hermitian:

$$(\mathcal{J}^{ij})^\dagger = \mathcal{J}^{ij}\,, \qquad (\mathcal{J}^{0j})^\dagger = -\mathcal{J}^{0j}. \tag{1.85}$$

From a finite dimensional representation one can construct other representations by sandwiching them in by a unitary matrix and its inverse. A simple check can verify that the new representation $U^{-1}\mathcal{J}^{\mu\nu}U$ is obeying the necessary constants of 1.80. Any two representations that are connected by the mentioned procedure are called equivalent. The name "equivalent" is chosen correctly because it constitutes a equivalence relation [23]. If the representation matrices have a invariant subspace smaller than the dimension of the space where the representation acts, then a suitable equivalent representation can be chosen that has a block diagonal form. This kind of representation are know as reducible representations. If there not reducible they are irreducible. Irreducible representations play a central role in group theory and are used to decompose reducible representations and to construct new. For clarity we will write the dimension of the particular representation of the group in boldface, two different representations of the same dimension are differentiated by a prime e.g. $\mathbf{m}$ and $\mathbf{m}'$. There are two usual ways to construct higher dimensional representations one to take the direct sum and the other the direct product.

1. direct sum.

In the direct sum we concatenate two fields into a field with higher dimension, the new representation has a block diagonal form.

$$\phi^{\mathbf{m}\oplus\mathbf{n}} = \begin{pmatrix} \phi^{\mathbf{m}} \\ \\ \phi^{\mathbf{n}} \end{pmatrix}, \qquad \mathcal{J}_{\mu\nu}^{\mathbf{m}\oplus\mathbf{n}} = \begin{pmatrix} \mathcal{J}_{\mu\nu}^{\mathbf{m}} & 0 \\ \\ 0 & \mathcal{J}_{\mu\nu}^{\mathbf{n}} \end{pmatrix}. \tag{1.86}$$

This is one reason of why we only need to occupy ourselves with irreducible representations.

2. direct product

Another way of constructing representations with higher dimensions is to take the direct product also known as the tensor product. It is defined by the following:

$$\phi_{ij}^{\mathbf{m}\otimes\mathbf{n}} = \phi_i^{\mathbf{m}} \phi_j^{\mathbf{n}}, \qquad (\mathcal{J}_{\mu\nu}^{\mathbf{m}\otimes\mathbf{n}})_{kl}^{ij} = (\mathcal{J}_{\mu\nu}^{\mathbf{m}})_k^i \delta_l^j + (\mathcal{J}_{\mu\nu}^{\mathbf{n}})_l^j \delta_k^j. \tag{1.87}$$

where i, k are indexes of the $\mathbf{m}$ dimensional representation and j, l of the $\mathbf{n}$ dimensional representation. This occurs when adding angular momentum in quantum mechanics.

$$\mathbf{m} \otimes \mathbf{n} = (\mathbf{m} - \mathbf{n} + \mathbf{1}) \oplus \cdots \oplus (\mathbf{m} + \mathbf{n} - \mathbf{1}). \tag{1.88}$$

Now we want to address the question of how to construct tensor and spinor fields. To construct a rank $\mathbf{n}$ tensor field we take the tensor product of n vector fields. These fields do not correspond to irreducible representations. But they can be decomposed to irreducible representations by looking at tensors with fixed symmetric and antisymmetric indexes. This procedure in fact is similar to finding irreducible representations for the symmetric group. As for the symmetric group the irreducible representations can be constructed by looking at Young tableau with n boxes. For an other place where Young tableau emerge see section 1.2.4. For example lets look at the product representation of $\mathbf{1} \otimes \mathbf{1}'$ (for the Young tableau see table 1.5). By dividing the direct product into two different parts.

$$\mathbf{1} \otimes \mathbf{1}' = (\mathbf{1} \otimes_S \mathbf{1}') \oplus (\mathbf{1} \otimes_A \mathbf{1}'), \tag{1.89}$$

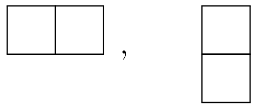

Figure 1.5: Young tableau with two boxes.

where $\otimes_A$ and $\otimes_S$ are defined by

$$(\phi^{\mathbf{1}\otimes_S\mathbf{1}'})_{ij} = (\phi^{\mathbf{1}})_i(\phi^{\mathbf{1}'})_j + (\phi^{\mathbf{1}})_j(\phi^{\mathbf{1}'})_i\,, \tag{1.90}$$

$$(\phi^{\mathbf{1}\otimes_A\mathbf{1}'})_{ij} = (\phi^{\mathbf{1}})_i(\phi^{\mathbf{1}'})_j - (\phi^{\mathbf{1}})_j(\phi^{\mathbf{1}'})_i\,, \tag{1.91}$$

both are not yet irreducible representations and accordingly correspond to the Young tableau the symmetric representation has $\frac{n(n+1)}{2}$ terms for $\mathbf{1}$ we have 10 terms. To have a irreducible representation we also need to separate the trace.

$$\phi^T_{\{\mu\nu\}} = \phi_{\{\mu\nu\}} - \frac{1}{4}\eta_{\mu\nu}(\eta^{\lambda\rho}\phi_{\{\lambda\rho\}})\,, \tag{1.92}$$

the {} brackets stand for symmetrization and the [] for antisymmetrization. So in short we have $\mathbf{4}\otimes_S\mathbf{4} = \mathbf{9}\oplus\mathbf{1}$. The antisymmetric part is also reducible. To see this we need to introduce the Hodge dual tensor. The Hodge dual tensor is a the tensor contracted with the invariant Levi-Civita symbol. The Levi-Civita symbol in four dimensional Minkowski space is a fully antisymmetric four tensor. defined by the equality

$$\epsilon^{0123} = 1 = -\epsilon_{0123}\,. \tag{1.93}$$

The Hodge dual tensor for the antisymmetric two tensor is defined as:

$$\phi^*_{[\mu\nu]} = \frac{i}{2}\epsilon_{\mu\nu\lambda\rho}\phi^{[\lambda\rho]}\,. \tag{1.94}$$

The Hodge dual tensor has the property of $(\phi^*)^* = \phi$. For this case the dual tensor is a antisymmetric two tensor. In fact here the dual tensor is proportional to the antisymmetric tensor.

$$\phi^*_{[\mu\nu]} = \pm\phi_{[\mu\nu]} \, , \tag{1.95}$$

here if considered a generic constant the equation would contradict the $\phi^{**} = \phi$ constraint. The two options are known as self dual and anti self dual representations for +1 and -1 correspondingly. They are three dimensional and are denoted as $\mathbf{3}^+$ and $\mathbf{3}^-$. In a simpler notation we got $\mathbf{4} \otimes_A \mathbf{4} = \mathbf{3}^+ \oplus \mathbf{3}^-$. So we showed all the irreducible representations of the tensor field. So we got 9 terms from the traceless symmetric, 1 from the invariant trace, 3 from the self dual and 3 from the anti self dual representations in total we have $4 \times 4 = 16$ terms which was expected and correct. We want to note that this is only a illustration not a proof. The tensor representations are not the only one, there are also the spinor representations. The spinor representations are associated with the covering group $SU(2) \times SU(2)$. The spinor representation has a advantage of being a construction block for tensor representations and in general if we take the product of even number of spin representations we'll get tensor representations so in the following part of this section the already discussed tensor representations will be seen as products of spinor representations. We start with the four dimensional representation of the Clifford algebra (for more details look up section 1.2.3)

$$\{\gamma^\mu \gamma^\nu\} = 2\eta^{\mu\nu} \, . \tag{1.96}$$

From tease we can construct the generators of the Lorentz algebra :

$$\mathcal{J}^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu] \, , \tag{1.97}$$

this is easily checked by substituting 1.97 into 1.77 . There is a simple way to construct spin

Name	Field	dimension	(m,n)
scalar	ϕ	$\mathbf{1}$	$(0,0)$
left-handed spinor	ψ_L	$\mathbf{2}_L$	$(\frac{1}{2},0)$
right-handed spinor	ψ_R	$\mathbf{2}_R$	$(0,\frac{1}{2})$
vector	ϕ^μ	$\mathbf{4}$	$(\frac{1}{2},\frac{1}{2})$
self dual antisymmetric	$\phi^+_{[\mu\nu]}$	$\mathbf{3}^+$	$(1,0)$
anti self dual antisymmetric	$\phi^-_{[\mu\nu]}$	$\mathbf{3}^-$	$(0,1)$
traceless symmetric	$\phi_{\{\mu\nu\}}$	$\mathbf{9}$	$(1,1)$

Table 1.2: Some of the irreducible representations of Lorentz algebra

representations. One needs to look at the operators:

$$L_i = \frac{1}{2}(J_i + iP_i)\,, \qquad R_i = \frac{1}{2}(J_i - iP_i) \tag{1.98}$$

J_i and P_i are the rotation and boost operators defined in the beginning of this chapter. The commutation relations of L and R

$$[L_i, L_j] = i\epsilon_{ijk}L_k\,, \quad [R_i, R_j] = i\epsilon_{ijk}R_k\,, \qquad [R_i, L_j] = 0\,, \tag{1.99}$$

reveal that the Lorentz group has a covering group of $SU(2)\times SU(2)$. The representations of the two $SU(2)$'s are distinguished by the subscripts R and L. Now we can present the irreducible representation as combination of irreducible representations of $SU(2)$, denoted (n,m)(see table 1.2).

1.3.1 Majorana spinors

Majorana spinors are different then the more familiar Dirac spinors in a number of ways and mixing them can create unexpected complications. So, because we need Majorana spinors for describing supersymmetry we'll give a short introduction of their properties. The "spinor" part in Majorana spinors means that they are functions of energy and momentum and when multiplied by e^{ipx} or e^{-ipx} become solutions to the Dirac equation [24], therefore they are a special case of the Dirac spinor. The Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0\,, \tag{1.100}$$

which can be understood as Schrdinger's equation

$$i\frac{\partial \Psi}{\partial t} = H\Psi\,, \tag{1.101}$$

with the Hamiltonian:

$$H = \gamma_0(\gamma^i p_i + m)\,, \tag{1.102}$$

the γ symbols are defined in section 1.2.3. The Dirac equation can be formulated as a Euler-Lagrange equation of a system with Lagrangian:

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi\,, \tag{1.103}$$

bar notation refers to Hermitian conjugate times γ_0. Now we want to look at the real solutions of the Dirac equations. To have real solutions we need the equation also to be real so the task is to find gamma matrices that are imaginary. One such possible solution is this:

$$\tilde{\gamma}^0 = \begin{bmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{bmatrix}, \qquad \tilde{\gamma}^1 = \begin{bmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{bmatrix}$$

$$\tilde{\gamma}^2 = \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}, \qquad \tilde{\gamma}^3 = \begin{bmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{bmatrix}.$$

the σ's are Pauli matrices. This observation has been done by Majorana. Because σ^2 is imaginary and σ^1 , σ^3 are real as proposed we have

$$\tilde{\gamma}^{*\mu} = -\tilde{\gamma}^{\mu}. \tag{1.104}$$

The reality conditions of the equation should make it possible to find real solutions for equation 1.100. The now found representation of the gamma matrices not unique, as we know gamma matrices in general are not unique and can be redefined by a unitary matrix, the same situation is also true for us with an additional constraint that the redefinition should not violate the reality condition. So, we have

$$\tilde{\gamma}^{\mu} \to U\tilde{\gamma}^{\mu}U^{\dagger}\,, \tag{1.105}$$

with the correspondent transformation for the field

$$\Psi \to U\Psi\,, \tag{1.106}$$

here U is a unitary matrix, and a straightforward check shows that equation 1.100 still hold true. The reality condition:

$$\tilde{\Psi} = \tilde{\Psi}^{*}\,, \tag{1.107}$$

implies that

$$U^{\dagger}\Psi = (U^{\dagger}\Psi)^{*} \tag{1.108}$$

or

$$\Psi = UU^{T}\Psi^{*} \tag{1.109}$$

This part illustrated the connection between Dirac spinors and Majorana spinors. Now we discus the connection between the Weyl spinor and Majorana spinor. First we have to return to the general setting of gamma matrices, without fixing their form. One can construct all the possible combinations of the gamma matrices,

$$\gamma^\mu\,,\ \gamma^\mu\gamma^\nu\,,\ \gamma^\mu\gamma^\nu\gamma^\lambda\,,\ \gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\,,\dots\,. \tag{1.110}$$

Fortunately the independent terms in this list are finite. First we want to note that any ordered list of indexes can be decomposed into lists with fixed symmetrization and antisymmetrization. The Clifford algebra (1.96) lowers the number of gamma matrices with symmetric indexes so we are left with a subset of possibly independent combinations.

$$\gamma^\mu\,,\ \gamma^{[\mu}\gamma^{\nu]}\,,\ \gamma^{[\mu}\gamma^\nu\gamma^{\lambda]}\,,\ \gamma^{[\mu}\gamma^\nu\gamma^\lambda\gamma^{\rho]}\,,\dots\,. \tag{1.111}$$

But we know that there is only one four tensor with fully antisymmetrized indexes the Levi-Civita symbol. So we have:

$$\gamma^{[\mu}\gamma^\nu\gamma^\lambda\gamma^{\rho]} \sim \epsilon^{\mu\nu\lambda\rho}\,. \tag{1.112}$$

The relative constant is a 4! because we define symmetrization and antisymmetrization without the $\frac{1}{k!}$, and an i for convenience. This enables us to discard all entries in 1.111 that have five or more indexes. For we clarity use the dual vector of the third entry in 1.111.

$$\gamma^{[\mu}\gamma^\nu\gamma^\lambda\gamma^{\rho]} = i4!\epsilon^{\mu\nu\lambda\rho}\gamma_5\,, \qquad \gamma^{[\mu}\gamma^\nu\gamma^{\lambda]} = i3!\epsilon^{\mu\nu\lambda\rho}\gamma_\rho\gamma_5\,. \tag{1.113}$$

γ_5 gets fixed by equations 1.113:

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{1.114}$$

the set of independent combinations of gamma matrices $M = \{1, \gamma^\mu, \gamma^{[\mu}\gamma^{\nu]}, \gamma^\mu\gamma_5, \gamma_5\}$ are a basis for 4×4 matrices. There are 16 matrices in M. γ_5 has some nice properties:

$$\{\gamma_5, \gamma_\mu\} = 0\,, \quad (\gamma_5)^2 = 1\,, \quad \gamma_5 = \gamma_5^\dagger\,, \quad [\mathcal{J}_{\mu\nu}, \gamma_5] = 0\,. \tag{1.115}$$

the second property restricts the eigenvectors of γ_5 to ± 1, from the last property we can conclude that by diagonalizing γ_5 the Lorentz operators obtain a block diagonal form of two 2×2 matrices. The corresponding spinors also get divided into two kinds, spinors that have $+1$ as eigenvalue and spinors that have -1 as eigenvalue of γ_5, denote them as left(L) and right(R) correspondingly.

$$\psi_L = \gamma_5 \psi_L\,, \quad \psi_R = -\gamma_5 \psi_R\,. \tag{1.116}$$

The corresponding projection operators are known as chirality operators.

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5)\,. \tag{1.117}$$

In the basis where γ_5 is diagonal the projection operators $P_\pm$ take the down half or the top half of a 4 spinor respectively, tease 2 spinors are known as Weyl spinors. Note that the R spinors correspond to the *minus* sign in the projection.

Now lets observe the fact that $(\pm\gamma^\mu)^T$ and $\pm\gamma^{\mu\dagger}$ obey the Clifford algebra 1.96, T stands for the transpose of a matrix. The Clifford algebra only has one four dimensional representation therefore these representation are connected by a similarity transformation.

$$\beta\gamma^\mu\beta^{-1} = -\gamma^{\mu\dagger}\,, \tag{1.118}$$

$$\mathcal{C}\gamma^\mu\mathcal{C}^{-1} = -\gamma^{\mu T}\,, \tag{1.119}$$

β is made from γ_0 by multiplying with i. $\mathcal{C}$ is know as the charge conjugate matrix. With the help of these we can construct the complex conjugate matrix and as before define Majorana spinors. Majorana spinors are Dirac spinors who obey this reality condition

$$\psi^* = \beta\mathcal{C}\psi\,, \tag{1.120}$$

like with the Weyl spinors the Dirac spinor can be broken up into two Majorana spinors

$$\psi_+ = \frac{1}{2}(\psi + \beta\mathcal{C}\psi^*)\,, \quad \psi_- = -i\frac{1}{2}(\psi - \beta\mathcal{C}\psi^*)\,. \tag{1.121}$$

Majorana spinors and Weyl spinors cant be combined. If we try to impose both conditions we get the trivial zero as solutions. Here we use left Weyl and Majorana conditions (eqs. 1.120 and 1.116).

$$\psi^* = \beta\mathcal{C}\psi = \beta\mathcal{C}\gamma_5\psi = -\gamma_5\beta\mathcal{C}\psi = -\gamma_5\psi^* = -(\gamma_5\psi)^* = -\psi^*\,. \tag{1.122}$$

1.4 Supersymmetry

In this section we try to introduce the reader with supersymmetry. In the first subsection we combine the introduction to supersymmetry with the question " why supersymmetry?".

1.4.1 Why supersymmetry?

The answer to this question lies in the Coleman-Mandula theorem. The theorem states that every quantum theory in $d > 2$ that has this three natural properties [7]

1. The spectra is restricted from below.
2. The S matrix has non-trivial scattering for two body scattering.
3. The amplitude of an elastic scattering is an analytical function of the scattering angle.

Then all internal symmetry generators are Lorentz scalars. A basic understanding can be gained from this thought experiment: if there are conservation laws other then the conservation of the energy-momentum and the angular momentum in an elastic two body scattering then the scattering angle is non-vanishing only in finite angles, and from condition 3 we conclude that the amplitude is zero everywhere. To describe this situation in terms of symmetries and their generators lets look at a simple system of two real scalar fields. The Lagrangian:

$$\mathcal{L} = -\frac{1}{2}(\partial\psi_1)^2 - \frac{1}{2}(\partial\psi_2)^2 \,. \tag{1.123}$$

From the Euler-Lagrange equations we get the equation of motions:

$$\Box\phi_1 = 0\,, \qquad \Box\phi_2 = 0\,. \tag{1.124}$$

where the d'Alembert operator is defined by $\Box = \partial^\mu\partial_\mu$. From these equations one can construct an infinite amount of conserving currents:

$$J_\mu = (\partial_\mu\phi_1)\phi_2 + \phi_1\partial_\mu\phi_2\,, \tag{1.125}$$

$$J_{\mu\rho} = (\partial_\mu\partial_\rho\phi_1)\phi_2 + \partial_\rho\phi_1\partial_\mu\phi_2\,, \tag{1.126}$$

$$J_{\mu\nu\rho} = (\partial_\mu\partial_\rho\phi_1)\partial_\nu\phi_2 + \partial_\rho\phi_1\partial_\mu\partial_\nu\phi_2\,, \tag{1.127}$$

$$\cdots\cdots\cdots$$

The fact that all currents are conserving can be shown directly by calculating the derivative with the index μ. The theorem does not apply here because we have no interactions, and therefore have a trivial two particle scattering contradicting point 2 of the theorem. If we add an interaction of the form $V(\phi_1^2 + \phi_2^2)$ only J_μ remains a conserved current. In this setting the Coleman-Mandula theorem states that there cant be any form of Lorentz invariant interaction that conserves currents of higher rank then the charge four-current , even there can't be any rank two conserving current for any kind of Lorentz invariant interaction. This situation changes when we add a fermion.

$$\mathcal{L} = -\frac{1}{2}(\partial\psi_1)^2 - \frac{1}{2}(\partial\psi_2)^2 - \frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi \tag{1.128}$$

This is the Lagrangian of two real scalars and a Majorana spinor. With equations of motion:

$$\Box\phi_1 = 0\,, \qquad \Box\phi_2 = 0\,. \qquad \gamma^\mu\partial_\mu\psi = 0 \tag{1.129}$$

as before we can tailor an infinite amount of conserved currents

$$S_{\mu\alpha} = \partial_\rho(\psi_1 - i\psi_2)(\gamma^\rho\gamma_\mu\psi)_\alpha\,, \tag{1.130}$$

$$S_{\mu\alpha\beta} = \partial_\rho(\psi_1 - i\psi_2)(\gamma^\rho\gamma_\mu\partial_\beta\psi)_\alpha\,, \tag{1.131}$$

$$S_{\mu\alpha\beta\tau} = \partial_\rho(\psi_1 - i\psi_2)(\gamma^\rho\gamma_\mu\partial_\beta\partial_\tau\psi)_\alpha\,, \tag{1.132}$$

$$\cdots\cdots\cdots$$

To check, one needs to use the equations of motion, the Clifford algebra for gamma matrices and the commutativity of partial derivatives. They give rise to conserved charges:

$$Q_\alpha = \int dx^3 S_{0\alpha}\,, \quad Q_{\alpha\beta} = \int dx^3 S_{0\alpha\beta}\,, \ldots \tag{1.133}$$

An analogue situation arises wen we add interaction term to the Lagrangian.

$$\mathcal{L} = \mathcal{L}_{free} - V \tag{1.134}$$

where an example of V is:

$$V = g\bar{\psi}(\phi_1 + i\gamma_5\phi_2)\psi + \frac{1}{2}g^2(\psi_1^2 + \psi_2^2)^2 \tag{1.135}$$

The Coleman-Mandula theorem states that after adding an Lorentz invariant interaction only $S_{\mu\alpha}$ will stay a conserved current even if ones try to change the form of the other currents. Note tat if there was a conserved current $S_{\mu\alpha\beta}$ one could construct a spin three symmetry generator out of it. But a spin half current $S_{\mu\alpha}$ gives rise to a spin one generator:

$$\{Q, \bar{Q}\} = 2\gamma^{\mu} P_{\mu}\,, \tag{1.136}$$

$$[Q, P] = 0 \tag{1.137}$$

This is known as the $N = 1$ $d = 4$ superalgebra. Because this type of algebra was the only possible construction to build it retains the position of the natural extension of Poincar algebra.

1.4.2 Superspace

The superspace is an extension of the familiar Minkowski space. Space time coordinates x^{μ} are supplemented with the coordinates $\theta_{1,2}$, which in essence are complex Grassmann numbers. Its also customary to think of them as Weyl spinors, the reason is that Grassmann numbers anticommute, and we need two of them so we could just take Weyl spinors instead. To work with Grassmann numbers we need to know how to commute with space-time coordinates, how to differentiate and how to integrate them. The first question is just a matter of definitions

$$[x^{\mu}, \theta_{1,2}] = [x^{\mu}, \bar{\theta}_{1,2}] = 0\,. \tag{1.138}$$

additionally the Grassmann numbers anticommute like spinors, so, two different Grassmann numbers anticommute and the square of a Grassmann number is zero. The derivative is all but usual.

$$\frac{\partial}{\partial\theta}\theta\bar{\theta} = \bar{\theta}\,, \quad \frac{\partial}{\partial\theta}\bar{\theta}\theta = -\bar{\theta}\,, \quad \frac{\partial}{\partial\theta_1}\theta_1\theta_2 = \theta_2\,, \quad \frac{\partial}{\partial\theta_1}\theta_2\theta_1 = -\theta_2\,. \tag{1.139}$$

To define integration we need a scalar measure. There are three to choose from:

$$d^2\theta = d\theta_1 d\theta_2\,, \quad d^2\bar{\theta} = d\bar{\theta}_1 d\bar{\theta}_2\,, \quad d^4\theta = d^2\theta d^2\bar{\theta}\,. \tag{1.140}$$

The integration rules

$$\int d^2\theta\,\theta_2\theta_1 = \int d^2\bar{\theta}\,\bar{\theta}_2\bar{\theta}_1 = \int d^4\theta\,\bar{\theta}_2\bar{\theta}_1\theta_2\theta_1 = 1\,. \tag{1.141}$$

These formula can be compressed into this simple anticommutativity expressions:

$$\left\{\frac{\partial}{\partial\theta_\alpha}\,,\theta_\beta\right\} = \int d\theta_\alpha\,\theta_\beta = \delta_{\alpha,\beta}\,. \tag{1.142}$$

This shows us that differentiation and integration for Grassmann numbers are roughly the same. The superderivatives are defines as:

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta_\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \tag{1.143}$$

$$\bar{\mathcal{D}}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\sigma^\mu_{\alpha\dot{\alpha}}\theta^\alpha\partial_\mu \tag{1.144}$$

1.4.3 Superfields

Now we're able to define superfields. Superfields are fields in superspace. It is sometimes convenient to look at the expansion of the field in their Grassmann coordinates. The expansion yields a finite number of terms because the square of a Grassmann number is always zero.

$$\begin{aligned} S(x,\theta,\bar{\theta}) = {} & \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\xi}(x) + \bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x) + \theta\theta f(x) + \bar{\theta}\bar{\theta}g^*(x) \\ & + i\theta\theta\bar{\theta}\bar{\lambda} + i\bar{\theta}\bar{\theta}\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta}D(x)\,, \end{aligned} \tag{1.145}$$

we suppress the indexes of θ and $\bar{\theta}$ when we think it wont cause confusion or increase readability. We classify , as always, the fields as bosonic or fermionic depending on their commutation or anticommutation relations. Bosonic fields commute with Grassmann numbers fermionic fields anticommute.

$$[S_b, \theta] = [S_b, \bar{\theta}] = 0\,, \tag{1.146}$$

$$\{S_f, \theta\} = \{S_f, \bar{\theta}\} = 0 \tag{1.147}$$

To restrict the general field to a fixed statistics we restrict the component in the expansion (1.145). For a bosonic superfield we expect to have even numbers of Grassmann coordinates, therefore $\psi(x)\,, A_\mu(x)\,, f(x)\,, g(x)\,,$ and $D(x)$ are bosonic field in Minkowski space the others are fermions. For a fermionic superfield the roles are switched $\psi(x)\,, \bar{\xi}(x)\,, \bar{\lambda}(x)$ and $\rho(x)$ are bosonic the others fermionic. The superfields are classified under the algebra of all isometries of the superspace. We saw that the fields can be categorized under the categories of formions and bosons this is a consequence of the Z_2 graded algebra. We denote the grading of the algebra A as $\pi(A)$. The distribution law for superderivatives are dependent of the grade of the fields:

$$\mathcal{D}(AB) = (\mathcal{D}A)B + (-)^{\pi(A)\pi(B)}A(\mathcal{D}B)\,, \tag{1.148}$$

$$\bar{\mathcal{D}}(AB) = (\bar{\mathcal{D}}A)B + (-)^{\pi(A)\pi(B)}A(\bar{\mathcal{D}}B)\,. \tag{1.149}$$

This mechanism is in essence the same that we have encountered in eq. 1.139.

Now well discus the supersymmetry generators and their algebra. The supersymmetry generator act linearly on the superfields this stand in full analogue, with the Poincar algebra. For Poincar translations we have:

$$\delta_\tau \phi(x) = i\tau^\mu P_\mu \phi(x)\,, \tag{1.150}$$

where $P_\mu = i\partial_\mu$. For the superalgebra

$$\delta_\chi S = (\chi Q + \bar{\chi}\bar{Q})S\,, \tag{1.151}$$

where Q and $\bar{Q}$ are the supercharges, which are also operators:

$$Q_\alpha = \frac{\partial}{\partial\theta_\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \,, \tag{1.152}$$

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + i\sigma^\mu_{\alpha\dot{\alpha}}\theta^\alpha\partial_\mu \,. \tag{1.153}$$

They obey the following anticommutation relations:

$$\{Q,\bar{Q}\} = 2\sigma^\mu P_\mu \,, \quad \{\mathcal{D},\bar{\mathcal{D}}\} = -2\sigma^\mu P_\mu \,, \tag{1.154}$$

all the other anticommutation rules which mix $\mathcal{D}\,,\bar{\mathcal{D}}$ and $Q,\bar{Q}$ are zero.

The superfields that we constructed belong to a representation of the superalgebra but the representation is not an irreducible representation as with $\phi^{\mu\nu}$ in section 1.3 we will construct the irreducible representations out of these although without proof.

1. The chiral superfield. One can get a chiral superfield by imposing this property:

$$\bar{\mathcal{D}}S_c = 0 \,, \tag{1.155}$$

or for the anti-chiral superfield $\mathcal{D}S_c^\dagger$=0. Note that these restrictions don't change the supercharge. These equations can be solved by trivially expanding the superderivative and the superfields then writing a set of equations, that will restrict the form of the general superfield. An equivalent but fare more short solution is to notice that for:

$$x_\pm = x \pm i\theta\sigma\bar{\sigma} \,, \tag{1.156}$$

the following equations hold true:

$$\mathcal{D}x_- = 0 \,, \qquad \bar{\mathcal{D}}x_+ = 0 \,. \tag{1.157}$$

Therefore fields constructed out of them are also obeying these equations.

$$S_c(x,\theta,\bar{\theta}) = \phi(x_+) + \theta\psi(x_+) + \theta\theta F(x_+)\,, \tag{1.158}$$

$$S_c^\dagger(x,\theta,\bar{\theta}) = \phi^*(x_-) + \bar{\theta}\bar{\psi}(x_-) + \bar{\theta}\bar{\theta}F^*(x_-)\,. \tag{1.159}$$

here ϕ is a scalar, ψ is a left spinor and F is a auxiliary field with no dynamics.

2. the vector superfield. A vector superfield is a superfield that obeys this condition:

$$S_c = S_c^\dagger\,, \tag{1.160}$$

this restriction identifies (in 1.145) ψ with ξ, f with g and $\bar{\lambda}$ with ρ, so we can write:

$$\begin{aligned} S_v(x,\theta,\bar{\theta}) = \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\psi}(x) + \bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x) + \theta\theta f(x) + \bar{\theta}\bar{\theta}f^*(x) \\ +i\theta\theta\bar{\theta}\bar{\lambda} + i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)\,, \end{aligned} \tag{1.161}$$

if the vector superfield is a gauge field with a gauge transformation

$$S_v \to S_v + i\Lambda - i\Lambda^\dagger\,, \tag{1.162}$$

where Λ is a chiral superfield. Note that the gauge transformation conserves the condition 1.160. It is customary and convenient to write the gauge superfield in this way:

$$\begin{aligned} S_{gauge}(x,\theta,\bar{\theta}) = \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\psi}(x) + \bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x) + \theta\theta f(x) + \bar{\theta}\bar{\theta}f^*(x) \\ +i\theta\theta\bar{\theta}(\bar{\lambda} + \frac{1}{2}\bar{\sigma}\partial\psi(x)) - i\bar{\theta}\bar{\theta}\theta(\lambda(x) + \frac{1}{2}\sigma\partial\bar{\psi(x)}) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D(x) + \frac{1}{2}\partial\partial v(x))\,, \end{aligned} \tag{1.163}$$

the gauge transformations of the fields v, ψ and f are pure algebraic:

$$v \to v + i\phi - i\phi^* \,, \tag{1.164}$$

$$\psi \to \psi + i\sqrt{2}\xi \,, \tag{1.165}$$

$$f \to f + iF \,, \tag{1.166}$$

the A_μ transforms in a familiar way:

$$A_\mu \to A_\mu + \partial_\mu(\phi + \phi^*) \,. \tag{1.167}$$

Doing calculations in gauge theories it is frequently advantageous to use a gauge, one such gauge is the Wess-Zumino gauge where the fields v, ψ and f vanish. In this gauge the gauge superfield has this convenient form:

$$S_{W-S}(x, \theta, \bar{\theta}) = \bar{\theta}\bar{\sigma}^\mu \theta A_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \,. \tag{1.168}$$

For a more comprehensive review of the topic see [25–27]

Chapter 2

VEV of Q-operator in linear quiver 5d $U(1)$ gauge theories

2.1 Introduction

The 4d $\mathcal{N} = 2$ gauge theories have natural uplift to the 5 dimensions. Embedding $\mathcal{N} = 2$ gauge theory in Ω-background was instrumental in all developments related to the instanton counting with the help of equivariant localization technics. In fact the geometric meaning of Ω-background is more transparent in 5d theory compactified on a circle. One simply considers a 5d geometry fibered over a circle of circumference L so that the complex coordinates (z_1, z_2) of the (four real dimensional) fiber get rotated along the circle as: $z_1 \to \exp(iL\epsilon_1)$, $z_2 \to \exp(iL\epsilon_2)$ accompanied with suitable **R**-symmetry and gauge rotations [28, 29]. $\epsilon_{1,2}$ are the *Omega*-background parameters. In 5d setting we'll use the notation $T_{1,2} = \exp(-\beta\epsilon_{1,2})$, where $\beta = iL$ and for technical reasons it will be assumed that β has a tiny real positive part. The initial 4d theory is recovered by sending $R \to 0$. Furthermore, sending both Ω-background parameters $\epsilon_{1,2}$ to 0, one gets the standard Seiberg-Witten theory [30, 31]. It is interesting that even the case of $U(1)$ gauge group, in contrast to the case without Ω-background, the theory is non-trivial. A characteristic feature of this case, is that the instanton sums become tractable, and for Nekrasov partition function one obtains closed formulae. In this paper it is shown that not only the partition function, but also a more refined quantity, namely the expectation value of the Q-observable can be computed in closed form. It was shown in [32] that the analog of Baxters Q operator in purely gauge theory context naturally emerges in Necrasov-Shatashvili limit ($\epsilon_2 = 0$) [33] as an entire function whose zeros are given in terms of an array of "critical" Young diagrams, namely those, that determine the most important instanton configuration contributing to the partition function. This observable encodes perfectly not only information about partition function (which is simply related to the total sum of column lengths of Young diagrams) but also the entire chiral ring [34] constructed from $\langle \Phi^J \rangle$, $J = 0, 1, 2, \ldots$ (Φ is the scalar of vector multiplet) which can be expressed in terms of power sum symmetric functions of the column lengths. This is why it is not surprising that the logarithmic differential of (shifted) ratio $y(x) \sim Q(x)/Q(x + \epsilon_1)$ is the direct analog of Seiberg-Witten differential: $x d \log(y(x)) \sim \omega_{SW}$. Subsequently Q and y observables have been extended for theories with various matter and gauge contents [32, 35–37]. In particular [37] interprets the equations satisfied by $y(x)$ as deformed character relations and also considers the 5d setup, while in [38] a relation between the $T - Q$ difference equation and the AGT dual [39, 40] (quasi-classical) 2d Toda conformal blocks with a fully degenerate insertion is found. The next step, namely extension to the case of generic Ω-background has been achieved in [41] and [42], where Dyson-Shwinger type equations (called qq-character relations) for y-observable are derived. For recent developments see also the series of papers by Nekrasov [41, 43–46].

In [1] the already mentioned link between Q observable and Toda conformal blocks with a degenerate field insertion remains valid for the case of generic Ω-background and, in AGT dual 2d CFT side, fully quantum conformal blocks as well. The case of the gauge group $SU(2)$ corresponding to the Liouville theory was analyzed in much details and starting from second order the BPZ differential equation [15] a difference-differential equation, generalizing conventional Baxters $T - Q$ relation [?] was derived. In present paper simpler $U(1)$ case in 5d setting is analyzed. The corresponding $T - Q$ difference equations as well as their solutions in closed form are found. The solution is expressed in terms of generalized Appel's function.

2.2 5d linear quiver theory

2.2.1 The partition function

The (instanton part of) partition function of the 5d, A_{r+1} linear quiver theory with gauge group $U(n)$ is given by (see Fig.2.1 where the setup and the notations are briefly described)

$$\mathcal{Z} = \sum_{(\vec{Y}_1,\ldots,\vec{Y}_r)} Z_{\mathbf{Y}} q_1^{|\vec{Y}_1|} \ldots q_r^{|\vec{Y}_r|} \tag{2.1}$$

The sum in (2.1) is over all possible r-tuples of arrays of n Young diagrams. $|\vec{Y}_k|$ is the total number of boxes in the k-th array of n Young diagrams and $Z_{\mathbf{Y}}$ is defined as:

$$\begin{gathered} Z_{\mathbf{Y}} = Z_{\vec{Y}_1,\ldots,\vec{Y}_r}(\vec{a}_0, \vec{a}_1, \ldots, \vec{a}_{r+1}) = \\ \prod_{u,v=1}^{n} \frac{Z_{bf}(\emptyset, a_{0,u}|Y_{1,v}, a_{1,v}) Z_{bf}(Y_{1,u}, a_{1,u}|Y_{2,v}, a_{2,v}) \ldots Z_{bf}(Y_{r,u}, a_{r,u}|\emptyset, a_{r+1,v})}{Z_{bf}(Y_{1,u}, a_{1,u}|Y_{1,v}, a_{1,v}) \ldots Z_{bf}(Y_{r,u}, a_{r,u}|Y_{r,v}, a_{r,v})} \end{gathered} \tag{2.2}$$

For a pair of Young diagrams λ, μ the bifundamental contribution is given by [47, 48]

$$Z_{bf}(\lambda, a|\mu, b) = \prod_{s\in\lambda} \left(1 - \frac{a}{b} T_1^{-L_\mu(s)} T_2^{1+A_\lambda(s)}\right) \prod_{s\in\mu} \left(1 - \frac{a}{b} T_1^{1+L_\lambda(s)} T_2^{-A_\mu(s)}\right) \tag{2.3}$$

A_λ and L_λ, known as the arm and leg lengths respectively, are defined as: if s is a box with coordinates (i, j) and λ_i (λ'_j) is the length of i-th (j-th) column (row), then:

$$L_\lambda(s) = \lambda'_j - i \ , \quad A_\lambda(s) = \lambda_i - j \tag{2.4}$$

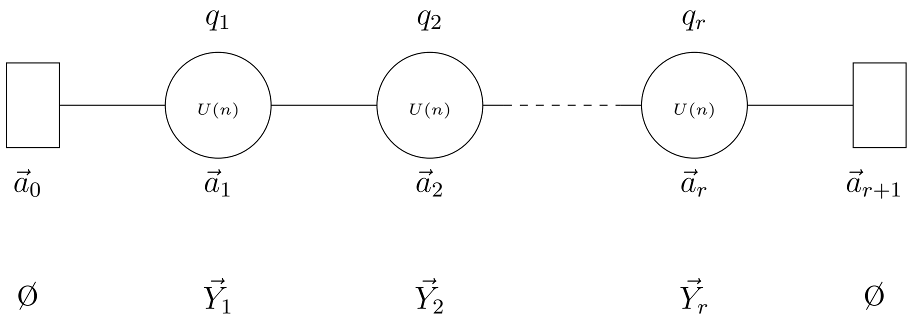

Figure 2.1: The linear quiver U(n) gauge theory: r circles stand for gauge multiplets; two squares represent n anti-fundamental (on the left edge) and n fundamental (the right edge) matter multiplets while the line segments connecting adjacent circles represent the bi-fundamentals. $q_1, \dots, q_r$ are the exponentiated gauge couplings, the n-dimensional vectors $\vec{a}_0, \dots, \vec{a}_{r+1}$ encode respective (exponentiated) masses/VEV's and $\vec{Y}_0, \dots, \vec{Y}_{r+1}$ are n-tuples of young diagrams specifying fixed (ideal) instanton configurations.

2.2.2 Important observables

The important observable of main interest in this paper, the Q-observable, is defined as

$$\mathbf{Q}(x,\lambda) = \prod_{(i,j)\in\lambda} \frac{x - T_1^i T_2^{j-1}}{x - T_1^{i-1} T_2^{j-1}} \tag{2.5}$$

Of course an analogous observable with the roles of T_1 and T_2 exchanged can be introduced as well. In 4d case $\beta \to 0$ and in Nekrasov-Shatashvili limit $\epsilon_1 \to 0$ this observable satisfies Baxter's T-Q equation [32]: a difference equation introduced by Baxter in context of lattice integrable models [?]. Generalization for the case of generic Ω background (in both 4d and 5d cases) is due to [41].

An important role is played also by the observable

$$y(x,\lambda) = \frac{\mathbf{Q}(x,\lambda)}{\mathbf{Q}(x/T_2,\lambda)} \equiv \prod_{(i,j)\in\lambda} \frac{(x - T_1^i T_2^{j-1})(x - T_1^{i-1} T_2^{j})}{(x - T_1^{i-1} T_2^{j-1})(x - T_1^i T_2^{j})} \tag{2.6}$$

In 4d Nekrasov-Shatashvili limit the logarithmic derivative of this observable generates all expectation values $\langle \phi^J \rangle$ of the vector multiplet scalar. Besides, its expectation value satisfies the (quantized analog of) Seiberg-Witten curve equation [32]. In generic Ω-background the corresponding equations (the so called qq-character equations) were introduced and investigated in [41] (see also [42]).

2.3 The special quiver and its relation to the Q observable

The expectation value of the Q-operator associated to the first node, by definition is

$$Q(x) = Z^{-1} \sum_{(\vec{Y}_1,\ldots,\vec{Y}_r)} \prod_{u=1}^{n} \mathbf{Q}\left(\frac{x}{a_{1,u}}, Y_{1,u}\right) Z_{\mathbf{Y}} q_1^{|\vec{Y}_1|} \ldots q_r^{|\vec{Y}_r|} \tag{2.7}$$

It was noticed in [1] that such insertion of the operator Q is equivalent to adding an extra node with specific expectation values. Here this statement will be proved in more general 5d setting. Note that a detailed proof in [1] was absent, so that also this gap automatically will be filled.

Let's look at a quiver with $r+1$ nodes with expectation values at the additional node (denoted as $\tilde{0}$) specified as (see Fig.2.2):

$$a_{\tilde{0},u} = \frac{a_{0,u}}{T_1^{\delta_{1,u}}}. \tag{2.8}$$

Due to the specific choice of $\vec{a}_{\tilde{0}}$, in order to give a nonzero contribution, the array of n diagrams associated with the special node $\tilde{0}$ has to be severely restricted. Namely, the diagram $Y_{\tilde{0},1}$ should consist of a single column and the remaining $n-1$ diagrams $Y_{\tilde{0},2}$, $\ldots$, $Y_{\tilde{0},n-1}$ must be empty. The proof of this statement is given in the Appendix 2.7.

There is a close relation between the Nekrasov partition function associated to above described specific length $r+1$ quiver and the expectation value of a particular Q operator in a generic quiver with r nodes. This relation is a consequence of the identity

$$Z_{\vec{Y}_{\tilde{0}},\vec{Y}_1,\ldots,\vec{Y}_r}\left(\vec{a}_0, \vec{a}_{\tilde{0}}, \vec{a}_1, \ldots, \vec{a}_{r+1}\right) q_{\tilde{0}}^l \left(T_1 q_1\right)^{|\vec{Y}_1|} q_2^{|\vec{Y}_2|} \ldots q_r^{|\vec{Y}_r|} =$$
$$\prod_{u=1}^{n} \left(\mathbf{Q}\left(\frac{a_{0,1}}{a_{1,u}} T_2^l, Y_{1,u}\right) \frac{\left(\frac{a_{0,1}}{a_{1,u}} T_2; T_2\right)_l}{\left(\frac{a_{0,1}}{a_{0,u}} T_2; T_2\right)_l} \right) Z_{\vec{Y}_1,\ldots,\vec{Y}_r}\left(\vec{a}_0, \vec{a}_1, \ldots, \vec{a}_{r+1}\right) q_{\tilde{0}}^l q_1^{|\vec{Y}_1|} \ldots q_r^{|\vec{Y}_r|}, \tag{2.9}$$

where $Y_{\tilde{0},u}$ for $u = 1$ is a one column diagram with length l and the rest are empty diagrams. The q-analog of Pochhammer's symbol is defined as:

$$(a;q)_l = (1-a)(1-aq)\cdots\left(1-aq^{l-1}\right). \qquad (2.10)$$

Inserting the definition (2.2) of $Z_{\vec{Y}}$ and canceling out the common factors of q and Z_{bf}, we see that (2.9) is equivalent to

$$\prod_{u,v=1}^{n}\left(\frac{Z_{bf}(\varnothing, a_{0,u}|Y_{\tilde{0},v}, a_{\tilde{0},v})Z_{bf}(Y_{\tilde{0},u}, a_{\tilde{0},u}|Y_{1,v}, a_{1,v})}{Z_{bf}(Y_{\tilde{0},u}, a_{\tilde{0},u}|Y_{\tilde{0},v}, a_{\tilde{0},v})}\right)T_1^{|\vec{Y}_1|} =$$

$$\prod_{u=1}^{n}\left(\mathbf{Q}\left(\frac{a_{0,1}}{a_{1,u}}T_2^l, Y_{1,u}\right)\frac{\left(\frac{a_{0,u}}{a_{1,u}}T_2, T_2\right)_l}{\left(\frac{a_{0,1}}{a_{0,u}}T_2; T_2\right)_l}\right)\prod_{u,v=1}^{n} Z_{bf}\left(\varnothing, a_{0,u}|Y_{1,v}, a_{1,v}\right) \qquad (2.11)$$

The last equality is proven in Appendix 4.6.

Clearly, the eq. (2.9) shows that the VEV (2.7) at specific values $x = x_l$

$$x_l = a_{0,1}T_2^l\,, \qquad l = 0,1,2,\ldots \qquad (2.12)$$

is related to the partition function of the special quiver with the fixed instanton number $|\vec{Y}_{\tilde{0}}| = l$ at the node $\tilde{0}$

$$\begin{aligned} Q(x_l) &= Z^{-1}\prod_{u=1}^{n}\frac{\left(\frac{a_{0,1}}{a_{0,u}}T_2; T_2\right)_l}{\left(\frac{a_{0,u}}{a_{1,u}}T_2; T_2\right)_l} \\ &\times \sum_{(\vec{Y}_1,\ldots,\vec{Y}_r)} Z_{\vec{Y}_{\tilde{0}},\vec{Y}_1,\ldots,\vec{Y}_r}\left(\vec{a}_0, \vec{a}_{\tilde{0}}, \vec{a}_1, \ldots, \vec{a}_{r+1}\right)\left(T_1q_1\right)^{|\vec{Y}_1|}q_2^{|\vec{Y}_2|}\ldots q_r^{|\vec{Y}_r|} \end{aligned} \qquad (2.13)$$

2.4 Difference equation for Q and its solution

From now on we'll restrict ourselves to the simplest case of the quiver of $U(1)$'s. 5d Nekrasov partition function of such linear quiver can be found using refined topological vertex method

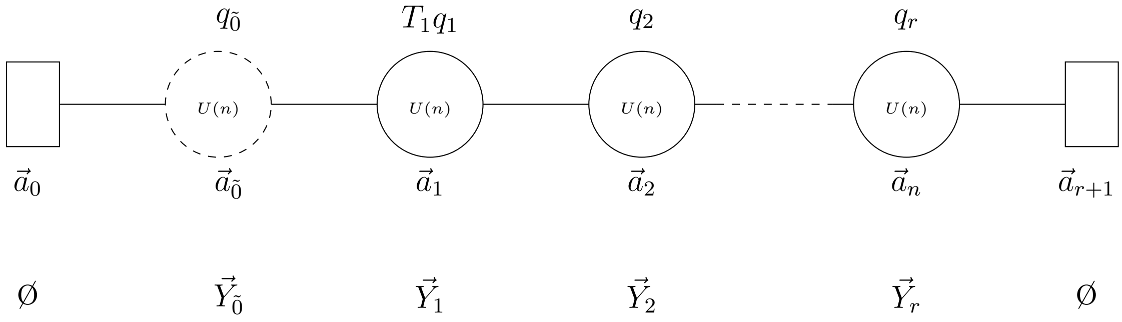

Figure 2.2: The quiver diagram with an extra node, labeled by $\tilde{0}$, added. Note that the gauge coupling at the node 1 is chosen to be $T_1\, q_1$.

[49], [50], [51–53] or through a direct instanton calculation (see e.g. [54] and references therein). The result can be represented as the infinite product

$$\mathcal{Z} = \prod_{l,s=0}^{\infty} \prod_{i=1}^{r} \prod_{j=i}^{r} \frac{\left(1 - \frac{a_{i-1} p_i}{a_j p_j}\, T_1^l T_2^s\right) \left(1 - \frac{a_i p_i}{a_{j+1} p_j}\, T_1^{l+1} T_2^{s+1}\right)}{\left(1 - \frac{a_i p_i}{a_j p_j}\, T_1^l T_2^s\right) \left(1 - \frac{a_{i-1} p_i}{a_{j+1} p_j}\, T_1^{l+1} T_2^{s+1}\right)} \tag{2.14}$$

where

$$p_i = a_1 \prod_{l=1}^{i} q_l \tag{2.15}$$

Applying the formula (2.14) for the special quiver discussed in Section 2.3, and for brevity denoting the partition function of the special quiver simply as $\mathcal{Z}(q_{\tilde{0}})$, up to factors independent of $q_{\tilde{0}}$ we get

$$\mathcal{Z}(q_{\tilde{0}}) \simeq \prod_{s=0}^{\infty} \prod_{i=0}^{r} \frac{1 - \frac{a_0 p_i}{a_1 a_{i+1}}\, q_{\tilde{0}} T_1^{1-\delta_{i,0}} T_2^{1+s}}{1 - \frac{a_0 p_i}{a_1 a_i}\, q_{\tilde{0}} T_2^s} \tag{2.16}$$

Note now that in ratio $Z(q_{\tilde{0}})/Z(T_2^{-1} q_{\tilde{0}})$ nearly all factors cancel out and one is lead to the relation

$$\mathcal{Z}(q_{\tilde{0}}) \prod_{i=0}^{r} \left(1 - \frac{a_0 p_i}{a_1 a_{i+1}}\, q_{\tilde{0}} T_1^{1-\delta_{i,0}}\right) = \mathcal{Z}(T_2^{-1} q_{\tilde{0}}) \prod_{i=0}^{r} \left(1 - \frac{a_0 p_i}{a_1 a_i}\, q_{\tilde{0}} T_2^{-1}\right) \tag{2.17}$$

Expanding this equality in powers of $q_{\tilde{0}}$ and taking into account (2.13), we'll get a linear relation (with rational in x_l coefficients) among $r+2$ quantities $Q(x_l)$, $Q(x_l/T_2)$,..., $Q(x_l/T_2^{r+1})$.

First let consider the simplest case $r = 1$. An easy computation allows us to establish the equality

$$Q(x) - \left(1 + q_1 \frac{a_1 T_1 x - a_0 a_2}{a_2(x - a_1)}\right) Q\left(\frac{x}{T_2}\right) + q_1 \frac{a_1(x - a_0 T_2)(T_1 x - a_2)}{a_2(x - a_1)(x - a_1 T_2)} Q\left(\frac{x}{T_2^2}\right) = 0 \qquad (2.18)$$

which is valid for infinitely many values $x = x_l$, $l = 0, 1, 2, \ldots$ (see eq. (2.12)).

An essential observation is in order here. Since $Q(x)$ and hence the entire LHS of the eq. (2.18) restricted up to an arbitrary instanton order is a rational function of x, the equality must be valid also for generic values of x. It is not difficult to check that the q-hypergeometric function (see Appendix C for definition)

$$Q(x) = \frac{(q_1; T_2)_\infty}{\left(\frac{q_1 a_1 T_1 T_2}{a_2}; T_2\right)_\infty} \, {}_2\phi_1\left(\begin{matrix} \frac{a_0}{x}, \frac{a_1 T_1 T_2}{a_2} \\ \frac{a_1}{x} \end{matrix}; T_2, q_1\right) \qquad (2.19)$$

is a solution of (2.18). The (x-independent) normalization coefficient in (2.19) is fixed from the asymptotic condition

$$\lim_{x \to \infty} Q(x) = 1 \qquad (2.20)$$

In fact it is possible to argue that (2.19) is the only solution of (2.18) with correct asymptotic and rationality properties discussed above. Using the special $n = 1$ case of the identity (2.59) the eq. (2.19) can be rewritten also as (this equality is referred as Heine's first transformation [55])

$$Q(x) = \frac{\left(\frac{a_0}{x}; T_2\right)_\infty}{\left(\frac{a_1}{x}; T_2\right)_\infty} \, {}_2\phi_1\left(\begin{matrix} \frac{a_1}{a_0}, \; q_1, \\ \frac{q_1 a_1 T_1 T_2}{a_2} \end{matrix}; T_2, \frac{a_0}{x}\right). \qquad (2.21)$$

The general case with an arbitrary r though more cumbersome, could be analyzed in the same way. The resulting difference equation reads:

$$\sum_{s=0}^{r+1} (-)^s C_s Q\left(T_2^{-s} x\right) = 0 \qquad (2.22)$$

where C_s

$$C_s = \frac{xT_1^{s-1}(O_+^{(s-1)} + T_1 O_+^{(s)}) - a_1 O^{(s-1)} - a_0 O^{(s)}}{x - a_0} \prod_{n=0}^{s-1} \frac{x - a_0 T_2^n}{x - a_1 T_2^n} \tag{2.23}$$

and $O^{(i)}, O_+^{(i)}$ are the coefficients of the expansions:

$$\prod_{i=1}^{r} \left(t + \frac{p_i}{a_i} \right) = \sum_{s=-\infty}^{\infty} O^{(s)} t^{r-s} \tag{2.24}$$

$$\prod_{i=1}^{r} \left(t + \frac{p_i}{a_{i+1}} \right) = \sum_{s=-\infty}^{\infty} O_+^{(s)} t^{r-s} \tag{2.25}$$

or, explicitly

$$O^{(s)} = \sum_{1 \le c_1 < \ldots < c_s \le r} \frac{p_{c_1} \ldots p_{c_s}}{a_{c_1} \ldots a_{c_s}} \tag{2.26}$$

$$O_+^{(s)} = \sum_{1 \le c_1 < \ldots < c_s \le r} \frac{p_{c_1} \ldots p_{c_s}}{a_{c_1+1} \ldots a_{c_s+1}} , \tag{2.27}$$

supplemented with conditions $O^{(-1)} = O_+^{(-1)} = 0$ and $O^{(0)} = O_+^{(0)} = 1$.

It is possible to find a closed expression for Q by expanding the RHS of eq. (2.16) in powers of $q_{\tilde{0}}$ and, with the help of eq. (2.13), relating the coefficients to $Q(x_l)$. After few manipulations, with a key role played by the identity

$$\frac{(ax;q)_\infty}{(a;q)_\infty} = \sum_{l=0}^{\infty} \frac{(a;q)_l}{(q;q)_l} x^l , \tag{2.28}$$

we finally get the expression

$$Q(x) = C \sum_{m_1, \ldots, m_r \ge 0} \frac{\left(\frac{a_0}{x}; T_2\right)_{m_1+m_2+\ldots+m_r} \left(\frac{a_1 T_1 T_2}{a_2}; T_2\right)_{m_1} \ldots \left(\frac{a_r T_1 T_2}{a_{r+1}}; T_2\right)_{m_k}}{\left(\frac{a_1}{x}; T_2\right)_{m_1+m_2+\ldots+m_r} (T_2;T_2)_{m_1} \ldots (T_2;T_2)_{m_r}} \left(\frac{p_1}{a_1}\right)^{m_1} \ldots \left(\frac{p_r}{a_r}\right)^{m_r} , \tag{2.29}$$

which is a generalization of the q-Appell's $\Phi^{(1)}$ series. As earlier, the normalization constant C can be fixed from the condition (2.20). Indeed in large x limit the RHS of eq. (2.29) breaks down to r independent sums which are easy to evaluate using eq. (2.28). The end result is:

$$C = \prod_{i=1}^{r} \frac{\left(\frac{p_i}{a_i}; T_2\right)_\infty}{\left(\frac{p_i T_1 T_2}{a_{i+1}}; T_2\right)_\infty} \tag{2.30}$$

It is remarkable that the multiple sum (2.29) can be expressed in terms of basic hypergeometric series ${}_{r+1}\phi_r$ (see Appendix 2.8) so that we finally get

$$Q(x) = \frac{\left(\frac{a_0}{x}; T_2\right)_\infty}{\left(\frac{a_1}{x}; T_2\right)_\infty} \; {}_{n+1}\phi_n \left(\begin{matrix} \frac{a_1}{a_0}, \frac{p_1}{a_1}, \frac{p_2}{a_2}, \;\ldots\;, \frac{p_r}{a_r} \\ \frac{p_1 T_1 T_2}{a_2}, \frac{p_2 T_1 T_3}{a_3}, \ldots, \frac{p_r T_1 T_2}{a_{r+1}} \end{matrix} ; T_2, \frac{a_0}{x} \right). \tag{2.31}$$

2.5 Reduction to 4 dimensions

In this section we reduce our results to the case of four dimensions. We substitute:

$$a_{i,u} \to e^{-\beta a_{i,u}}, \qquad T_1 \to e^{-\beta \epsilon_1}, \qquad T_2 \to e^{-\beta \epsilon_2}. \tag{2.32}$$

where a, ϵ_1 and ϵ_2 are the parameters of our 4d quiver theory. The reduction corresponds to the small β limit. Let me briefly list how the various quantities and relations get modified.

1. The link between expectation value of the Q operator and the partition function:

$$\begin{aligned} &Z_{\vec{Y}_0,\vec{Y}_1,\ldots,\vec{Y}_r}(\vec{a}_0, \vec{a}_{\tilde{0}}, \vec{a}_1, \ldots, \vec{a}_{r+1}) q_0^l q_1^{|\vec{Y}_1|} q_2^{|\vec{Y}_2|} \ldots q_r^{|\vec{Y}_r|} = \\ &\prod_{u=1}^{n} \left(\mathbf{Q}(a_{0,1} - a_{1,u} + l\epsilon_2, Y_1) \frac{\left(\frac{a_{0,u} - a_{1,u} + \epsilon_2}{\epsilon_2}\right)_l}{\left(\frac{a_{0,1} - a_{0,u} + \epsilon_2}{\epsilon_2}\right)_l} \right) Z_{\vec{Y}_1,\ldots,\vec{Y}_r}(\vec{a}_0, \vec{a}_1, \ldots, \vec{a}_{r+1}) q_0^l q_1^{|\vec{Y}_1|} \ldots q_r^{|\vec{Y}_r|} \end{aligned} \tag{2.33}$$

where $(a)_l$ is the standard Pochhammer's symbol, $Z_{\mathbf{Y}}$ is still given by eq. (2.2) but now Z_{bf} and Q are given by

$$Z_{bf}(\lambda, a|\mu, b) =$$
$$\prod_{s\in\lambda}(a-b-L_\mu(s)\epsilon_1+(1+A_\lambda(s)\epsilon_2)\prod_{s\in\mu}(a-b+(1+L_\lambda(s))\epsilon_1-A_\mu(s)\epsilon_2)$$
$$\mathbf{Q}(x,\lambda) = \prod_{x\in\lambda}\frac{x-i\epsilon_1-(j-1)\epsilon_2}{x-(i-1)\epsilon_1-(j-1)\epsilon_2} \tag{2.34}$$

2. The difference equation and its solution:

$$\sum_{s=0}^{r+1}(-)^s C_s Q(x-a_1-s\epsilon_2) = 0 \tag{2.35}$$

where C_s is defined as:

$$C_s =$$
$$\frac{(a_1-x-(s-1)\epsilon_1)V^{(s-1)}+W^{(s-1)}+(a_0-x-s\epsilon_1)V^{(s)}+W^{(s)}}{a_0-x}\prod_{n=0}^{s-1}\frac{a_0+n\epsilon_2-x}{a_1-m\epsilon_2-x} \tag{2.36}$$

where

$$V^{(i)} = \sum_{1\le c_1<...<c_i\le r} p_{c_1}...p_{c_i} \tag{2.37}$$
$$W^{(i)} = \sum_{1\le c_1<...<c_i\le r} p_{c_1}...p_{c_i}(a_{c_1+1}-a_{c_1}+...+a_{c_i+1}-a_{c_i})\,. \tag{2.38}$$

Here p_i's are redefined as:

$$p_i = \prod_{j=1}^{i} q_j \tag{2.39}$$

and, by definition, we set $V^{(-1)} = W^{(-1)} = W^{(0)} = 0$, $V^{(0)} = 1$.

The solution reads:

$$\begin{aligned} Q(x) &= C \sum_{m_1,\ldots,m_r \geq 0} \frac{(\frac{a_0-x}{\epsilon_2})_{m_1+m_2+\ldots+m_r}(\frac{a_1-a_2+\epsilon_1+\epsilon_2}{\epsilon_2})_{m_1}\cdots(\frac{a_r-a_{r+1}+\epsilon_1+\epsilon_2}{\epsilon_2})_{m_k}}{(\frac{a_1-x}{\epsilon_2})_{m_1+m_2+\ldots+m_r} m_1!\ldots m_r!} p_1^{m_1}\ldots p_k^{m_r} \\ &= C F_1^{(r)}\big(\frac{a_0-x}{\epsilon_2}, \frac{a_1-a_2+\epsilon_1+\epsilon_2}{\epsilon_2}, \ldots, \frac{a_r-a_{r+1}+\epsilon_1+\epsilon_2}{\epsilon_2}; \frac{a_1-x}{\epsilon_2}; p_1, .., p_r\big)\,, \end{aligned} \tag{2.40}$$

where $F_1^{(r)}$ is a generalization of Appel's function (see Appendix 2.8). C is x-independent and can be fixed from normalization:

$$C = \prod_{i=1}^{r} (1-p_i)^{\frac{a_i-a_{i+1}+\epsilon_1+\epsilon_2}{\epsilon_2}} \tag{2.41}$$

In the special case $r = 1$ we get

$$Q(x) = (1-q)^{\frac{a_1-a_2+\epsilon_1+\epsilon_2}{\epsilon_2}} \, {}_2F_1\left(\frac{a_0-x}{\epsilon_2}, \frac{a_1-a_2+\epsilon_1+\epsilon_2}{\epsilon_2}; \frac{a_1-x}{\epsilon_2}; q\right) \tag{2.42}$$

It is not difficult to check that after decoupling of both hypermultiplets by sending their masses to infinity, in Nekrasov-Shatashvili limit we recover the result presented in [32].

2.6 Proof of the equality (2.11)

Here we present the derivation of (2.11). We first derive two auxiliary identities.

Denote a Young diagram λ with column lengths $\lambda_1 \geq \lambda_2 \geq \cdots$ as $\{\lambda_1, \lambda_2, \ldots\}$. The corresponding row lengths we'll indicate as $\lambda'_1 \geq \lambda'_2 \geq \cdots$. In particular λ'_1 would be the number of columns. We want to show that

$$Z_{bf}(\{l\}, a|\lambda, b) = \mathbf{Q}\left(\frac{a}{b} T_1 T_2^l, \lambda\right) T_1^{-|\lambda|} Z_{bf}(\varnothing, aT_1|\lambda, b) \left(\frac{a}{b} T_1 T_2; T_2\right)_l \tag{2.43}$$

To prove (2.43) we divide and multiply the LHS by $Z_{bf}(\emptyset, aT_1|\lambda, b)$, then insert the definitions of Z_{bf} and the values of arm and leg lengths. We get

$$Z_{bf}(\{l\}, a|\lambda, b) = \\ Z_{bf}(\emptyset, aT_1|\lambda, b) \prod_{j=1}^{l} \left(1 - \frac{a}{b} T_1^{1-\lambda'_j} T_2^{1+l-j}\right) \prod_{j=1}^{\lambda_1} \prod_{i=1}^{\lambda'_j} \frac{1 - \frac{a}{b} T_1^{1+\theta(l-j)-i} T_2^{-\lambda_i+j}}{1 - \frac{a}{b} T_1^{2-i} T^{-\lambda_i+j}}, \tag{2.44}$$

where $\theta(x)$ is the Heaviside step function

$$\theta(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \tag{2.45}$$

Now we divide the problem into two separate cases when $\lambda_1 \leq l$ or $\lambda_1 > l$.

1. $\lambda_1 \leq l$.

In this case $\theta(l - j) = 1$ and the double product in (2.44) cancels out. The remaining single product by a simple manipulation can be rewritten as

$$Z_{bf}(\emptyset, aT_1|\lambda, b) \prod_{j=\lambda_1+1}^{l} \left(1 - \frac{a}{b} T_1^1 T_2^{1+l-j}\right) \prod_{j=1}^{\lambda_1} \prod_{i=1}^{\lambda'_j} \frac{1 - \frac{a}{b} T_1^{1-i} T_2^{1+l-j}}{1 - \frac{a}{b} T_1^{2-i} T_2^{1+l-j}} \prod_{j=1}^{\lambda_1} \left(1 - \frac{a}{b} T^1 T_2^{1+l-j}\right). \tag{2.46}$$

Notice that the middle double product is nothing but $\mathbf{Q}(\frac{a}{b}\, T_1 T_2^l, \lambda) T_1^{-|\lambda|}$ which concludes the first case.

2. $\lambda_1 > l$

We split λ into two parts: λ^{top} consisting of boxes with vertical coordinates $j > l$, and the part λ^{down} of lower lying boxes with $j \leq l$. Now the part of the double product in (2.44) corresponding to the boxes of λ^{top} survives. For the single product part we do the same manipulation as in previous case. As a result we get

$$Z_{bf}(\emptyset, aT_1|Y, b) \prod_{j=1}^{l} \prod_{i=1}^{\lambda'_j} \frac{1 - \frac{a}{b} T_1^{1-i} T_2^{1+l-j}}{1 - \frac{a}{b} T_1^{2-i} T_2^{1+l-j}} \prod_{j=1}^{l} (1 - \frac{a}{b} T_1 T_2^{1+l-j}) \prod_{(i,j) \in \lambda^{top}} \frac{1 - \frac{a}{b} T_1^{1-i} T_2^{-\lambda_i+j}}{1 - \frac{a}{b} T_1^{2-i} T^{-\lambda_i+j}}. \tag{2.47}$$

It is easy to see that the product over λ^{top} can be rewritten as

$$\prod_{(i,j)\in\lambda^{top}} \frac{1-\frac{a}{b}T_1^{1-i}T_2^{-\lambda_i+j}}{1-\frac{a}{b}T_1^{2-i}T^{-\lambda_i+j}} = \prod_{(i,j)\in\lambda^{top}} \frac{1-\frac{a}{b}T_1^{1-i}T_2^{1+l-j}}{1-\frac{a}{b}T_1^{2-i}T_2^{1+l-j}}. \tag{2.48}$$

Thus the first double product in (2.47) (which is a product over the boxes of λ^{down}) naturally combines with that of over λ^{top} to give a product over entire λ. As a result, instead of (2.47) we may as well write

$$Z_{bf}(\emptyset, aT_1|\lambda, b)\left(\frac{a}{b}T_1T_2;T_2\right)_l \prod_{(i,j)\in\lambda} \frac{1-\frac{a}{b}T_1^{1-i}T_2^{1+l-j}}{1-\frac{a}{b}T_1^{2-i}T_2^{1+l-j}}. \tag{2.49}$$

As before, the product over λ gives $\mathbf{Q}(\frac{a}{b}\,T_2^l,\lambda)T_1^{-|\lambda|}$, which concludes the proof of eq. (2.43).

We'll need also the simple identity

$$Z_{bf}(\emptyset, a|\lambda, b) = \prod_{(i,j)\in\lambda}\left(1-\frac{a}{b}\,T_1^{1-i}T_2^{1-j}\right) \tag{2.50}$$

Now the only thing that remains to be done is to make use of (2.43) and (2.50):

$$\begin{aligned}
&\prod_{u,v=1}^{n} \frac{Z_{bf}(\emptyset, a_{0,u}|Y_{\tilde{0},v}, a_{\tilde{0},v})Z_{bf}(Y_{\tilde{0},u}, a_{\tilde{0},u}|Y_{1,v}, a_{1,v})}{Z_{bf}(Y_{\tilde{0},u}, a_{\tilde{0},u}|Y_{\tilde{0},v}, a_{\tilde{0},v})} = \\
&\prod_{u,v=2}^{n} Z_{bf}(\emptyset, a_{0,u}|Y_{1,v}, a_{1,v}) \prod_{u=2}^{n} \frac{Z_{bf}(\emptyset, a_{0,u}|Y_{1,1}, a_{1,1})Z_{bf}(Y_{0,1}, \frac{a_{0,1}}{T_1}|Y_{1,u}, a_{1,u})}{Z_{bf}(Y_{0,1}, \frac{a_{0,1}}{T_1}|\emptyset, a_{0,u})} \\
&\qquad\times \frac{Z_{bf}(\emptyset, a_{0,1}|Y_{0,1}, \frac{a_{0,1}}{T_1})Z_{bf}(Y_{0,1}, \frac{a_{0,1}}{T_1}|Y_{1,1}, a_{1,1})}{Z_{bf}(Y_{0,1}, \frac{a_{0,1}}{T_1}|Y_{0,1}, \frac{a_{0,1}}{T_1})} \\
&= T_1^{-|\vec{Y}_1|}\prod_{u=1}^{n}\left(\mathbf{Q}(\frac{a_{0,u}}{a_{1,u}}T_2^l, Y_{1,u})\frac{(\frac{a_{0,u}}{a_{1,u}}T_2;T_2)_l}{(\frac{a_{0,1}}{a_{0,u}}T_2;T_2)_l}\right)\prod_{u,v=1}^{n} Z_{bf}(\emptyset, a_{0,u}|Y_{1,v}, a_{1,v})
\end{aligned} \tag{2.51}$$

2.7 Restriction on Young diagrams at the special node

To prove that the diagram $Y_{\tilde{0},1}$ at the special node $\tilde{0}$ should have at most one column in order to have a nonzero contribution to the partition function, let us assume in contrary that $Y_{\tilde{0},1}$ has a non-empty second column with length $l \geq 1$. This means that the box with coordinates

$(i,j) = (2,l)$ belongs to this diagram. Any term of the instanton sum corresponding to such choice includes a factor

$$Z_{bf}(\varnothing, a_{0,1}|Y_{\tilde{0},1}, a_{0,1}T_1^{-1}) = \prod_{s\in Y_{\tilde{0},1}} \left(1 - \frac{a_{0,1}}{a_{0,1}T_1^{-1}} T_1^{1+L_{\varnothing}(s)} T_2^{-A_{Y_{\tilde{0},1}}(s)}\right) \tag{2.52}$$

The arm and leg lengths of the box $(2,l)$ are easy to calculate: $L_{\varnothing}(2,l) = -2$ and $A_Y(2,l) = 0$ and the corresponding factor in eq. (2.52) vanishes.

In a similar way we can easily argue that all remaining $n-1$ diagrams $Y_{\tilde{0},i}$, $i = 2,\ldots,n$ must be empty. In fact, if any of this diagrams is non-empty (denote it as λ), then $Z_{\mathbf{Y}}$ will include a factor

$$Z_{bf}(\varnothing, a_{1,i}|\lambda, a_{1,i}) = \prod_{s\in\lambda}(1 - T_1^{1+L_{\varnothing}(s)} T_2^{-A_{\lambda}(s)}) \tag{2.53}$$

In this product the factor corresponding to the top box $(1,\lambda_1)$ of its first column becomes zero, since for this box $L_{\varnothing}(1,\lambda_1) = -1$ and $A_{\lambda}(1,\lambda_1) = 0$.

Thus we have proven that *at the special node* the first diagram has at most one column while the remaining diagrams are empty.

2.8 Generalized Appel and hypergeometric functions

Appels functions and their q-analogues generalize ordinary hypergeometric and q-hypergeometric functions for the case with more than one arguments. Here are the definitions:

- Appel's function F_1 and its generalization for the arbitrary number of variables:

$$F_1(a, b_1, b_2; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n} m! n!} x^m y^n \tag{2.54}$$

$$F_1^{(k)}(a, b_1, ..., b_k; c; x_1, .., x_k) = \sum_{m_1,...,m_k \geq 0} \frac{(a)_{m_1+...+m_r}(b_1)_{m_1}...(b_k)_{m_k}}{(c)_{m_1+...+m_r} m_1!...m_k!}(x_1)^{m_1}...(x_k)^{m_k} \tag{2.55}$$

- The corresponding q-analogs:

$$\Phi_1(a, b_1, b_2; c; q; x, y) = \sum_{m,n=0}^{\infty} \frac{(a;q)_{m+n}(b_1;q)_m(b_2;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n} x^m y^n \tag{2.56}$$

$$\Phi_1^{(n)}(a, b_1, b_2, ..., b_n; c; q; x_1, .., x_n) = \sum_{m_1,...,m_n=0}^{\infty} \frac{(a;q)_{m_1+...+m_r}(b_1;q)_{m_1}...(b_n;q)_{m_n}}{(c;q)_{m_1+...+m_r}(q;q)_{m_1}...(q;q)_{m_n}} x_1^{m_1}...x_n^{m_n}, \tag{2.57}$$

- The basic Hypergeometric function ${}_{n+1}\phi_n$:

$${}_{n+1}\phi_n \left(\begin{matrix} a_1, \ \ldots \ , \ a_{n+1} \\ b_1, \ldots, b_n \end{matrix} ; T_2, x \right) = \sum_{m=0}^{\infty} \frac{(a_1;q)_m \cdots (a_{n+1};q)_m}{(q;q)_m (b_1;q)_m \cdots (b_n;q)_m} x^m \tag{2.58}$$

There is a nice identity relating $\Phi_1^{(n)}$ with ${}_{n+1}\phi_n$ (see [56]):

$$\begin{gathered} \Phi_1^{(n)}(a, b_1, b_2, ..., b_n; c; q; x_1, .., x_n) = \\ \frac{(a, b_1 x_1, b_2 x_2, \ldots, b_n x_n; q)_\infty}{(c, x_1, x_2, \ldots, x_n; q)_\infty} \, {}_{n+1}\phi_n \left(\begin{matrix} c/a, x_1, x_2, \ldots, x_n \\ b_1 x_1, b_2 x_2, \ldots, b_n x_n; q, a \end{matrix} \right), \end{gathered} \tag{2.59}$$

where

$$(a_1, a_2, \ldots, a_k; q)_\infty = \prod_{l=1}^{k} (a_l; q)_\infty \, . \tag{2.60}$$

This identity allows us to rewrite the eq. (2.29) in terms of the function ${}_{n+1}\phi_n$ (see eq. (2.31)).

Chapter 3

VEV of Baxter's Q-operator in N=2 gauge theory and the BPZ differential equation

3.1 Introduction

Instanton [57] partition function of $\mathcal{N} = 2$ supersymmetric gauge theory in Ω-background admits exact investigation by localization methods [28, 29, 47, 48, 58]. In the limit when the background parameters ϵ_1, ϵ_2 vanish, the famous Seiberg-Witten solution [30, 31] is recovered. The case of non-trivial Ω-background has surprisingly rich area of applications. In particular when one of parameters is set to zero (Nekrasov-Shatashvili limit [33]), deep relations to quantum integrable system emerge (see e.g. [32, 35–37, 59–62] to quote a few from many important works). These are quantum versions of classical integrable systems, which played central role already in Seiberg-Witten theory on trivial background [63, 64]. The remaining non-zero Ω-background parameter just plays the role of Plank's constant. Many familiar concepts of exactly integrable models of statistical mechanics and quantum field theory such as Bethe-ansatz or Baxters $T-Q$ equations [?, 65] naturally emerge in this context [32]. In the case of generic Ω-background instanton partition function is directly related to the conformal blocks of a 2d

CFT (AGT correspondence) [39, 40, 66–68]. In this context the NS limit corresponds to the semi-classical limit of the related CFT [37, 38, 61, 69–72].

In [38] one of present authors (R.P.) has investigated the link between Deformed Seiberg-Witten curve equation and underlying Baxter's $T-Q$ equation in gauge theory side and the null-vector decoupling equation [15] of 2d CFT in quite general setting of linear quiver gauge theories with $U(n)$ gauge groups and 2d A_{n-1} Toda field theory multi-point conformal blocks in semi-classical limit (see also [14, 72–75] for earlier discussions on the role of degenerate fields in AGT correspondence).

In this short notes we'll extend some of the results of [38] to the case of generic Ω-background corresponding to the genuine quantum conformal blocks. For technical reasons we'll restrict ourselves to the case of $U(2)$ gauge groups corresponding to the Liouville theory leaving Toda field theory case for future work.

3.2 A special choice of parameters, leading to $\mathbf{Q}_{\vec{Y}}$ insertion

Consider the instanton partition function of the linear quiver theory A_{r+1} with gauge groups $U(n)$ with parameters specified as in Fig.3.1a. Note that the parameters of the first gauge factor (depicted as a dashed circle) are chosen to be $\tilde{a}_{0,u} = a_{0,u} - \epsilon_1 \delta_{1,u}$, where $a_{0,u}$ are the parameters of the "frozen" node corresponding to the n antifundamental hypermultiplets. It has been shown in [14] that under such choice of parameters all n-tuples of Young diagrams $Y_{\tilde{0},u}$ corresponding to the special node $\tilde{0}$ (the dashed circle) give no contribution in partition function unless the first diagram $Y_{\tilde{0},1}$ consists of a single column while the remaining $n-1$ diagrams are empty. Taking into account this huge simplification we'll be able to separate the contribution of the special node explicitly. According to the rules of construction of the

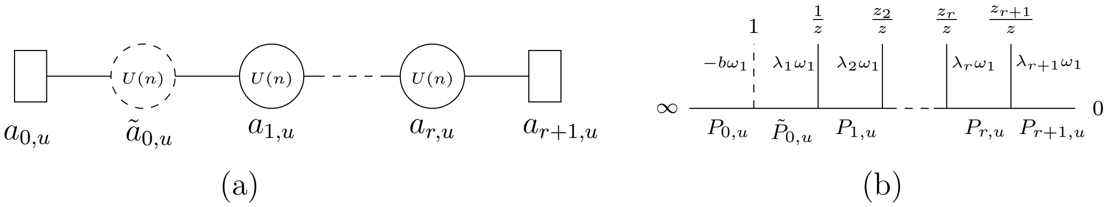

Figure 3.1: (a) The quiver diagram for the conformal linear quiver $U(n)$ gauge theory: r circles stand for gauge multiplets; two squares represent n anti-fundamental (on the left edge) and n fundamental (the right edge) hypermultiplets; the lines connecting adjacent circles are the bi-fundamentals. (b) The AGT dual conformal block of the Toda field theory.

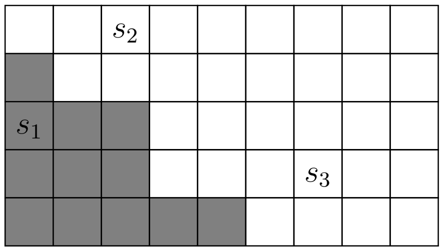

Figure 3.2: Arm and leg length with respect to a Young diagram $\lambda = \{4, 3, 3, 1, 1\}$ (the gray area): $A_\lambda(s_1) = 1$, $L_\lambda(s_1) = 2$, $A_\lambda(s_2) = -2$, $L_\lambda(s_2) = -3$, $A_\lambda(s_3) = -2$, $L_\lambda(s_3) = -4$.

partition function for this contribution we have

$$\prod_{u,v=1}^{n} \frac{Z_{bf}\left(a_{0,u}, \varnothing | \tilde{a}_{0,v}, Y_{\tilde{0},v}\right) Z_{bf}\left(\tilde{a}_{0,u}, Y_{\tilde{0},u} | a_{1,v}, Y_{1,v}\right)}{Z_{bf}\left(\tilde{a}_{0,u}, Y_{\tilde{0},u} | \tilde{a}_{0,v}, Y_{\tilde{0},v}\right)} \tag{3.1}$$

where for a pair of Young diagrams λ, μ the bifundamental contribution is given by

$$\begin{aligned} &Z_{bf}(a, \lambda | b, \mu) = \\ &\qquad \prod_{s \in \lambda} (a - b - \epsilon_1 L_\mu(s) + \epsilon_2(1 + A_\lambda(s))) \prod_{s \in \mu} (a - b + \epsilon_1(1 + L_\lambda(s)) - \epsilon_2 A_\mu(s)), \end{aligned} \tag{3.2}$$

the arm and leg lengths of a box s $A_\lambda(s)$ and $L_\lambda(s)$ towards a Young diagram λ are defined as

$$A_\lambda(s) = \lambda_i - j\,; \qquad L_\lambda(s) = \lambda'_i - j\,, \tag{3.3}$$

where (i, j) are coordinates of the box s with respect to the center of the corner box and λ_i (λ'_j) is the i-th column length (j-th row length) of λ as shown in Fig.3.2.

Using (3.2) It is not difficult to compute the factors Z_{bf} present in (3.1). In particular

$$Z_{bf}\left(a,\varnothing|b,\lambda\right)=\prod_{s\in\lambda}(a-b-\varphi(s)) \tag{3.4}$$

where

$$\varphi(s)=\epsilon_1(i_s-1)+\epsilon_2(j_s-1) \tag{3.5}$$

(e.g. in Fig.3.2 $\varphi(s_3)=6\epsilon_1+\epsilon_2$). To present the final result for the contribution (3.1) it is convenient to introduce the notation

$$\mathbf{Q}(v|\lambda)=\frac{(-\epsilon_2)^{\frac{v}{\epsilon_2}}}{\Gamma(-\frac{v}{\epsilon_2})}\prod_{s\in\lambda}\frac{v-\varphi(s)+\epsilon_1}{v-\varphi(s)} \tag{3.6}$$

The analogues quantity was instrumental in construction of Baxters T-Q relation in the context of Nekrasov-Shatashvili limit of $\mathcal{N}=2$ gauge theories [32]. Recently the importance of this quantity in the case generic Ω-background was emphasized in [41]. A careful examination shows that the contribution (3.1) can be conveniently represented as

$$\prod_{u=1}^{n}\frac{\mathbf{Q}\left(a_{0,1}-a_{1,u}+\epsilon_2 k|Y_{1,u}\right)}{\epsilon_2^k\left(\frac{a_{0,1}-a_{0,u}+\epsilon_2}{\epsilon_2}\right)_k\mathbf{Q}\left(a_{0,1}-a_{1,u}|Y_{1,u}\right)}\prod_{u,v=1}^{n}Z_{bf}\left(\tilde{a}_{0,u},\varnothing|a_{1,v},Y_{1,v}\right) \tag{3.7}$$

where

$$(x)_k=x(x+1)\cdots(x+k-1)=\frac{\Gamma(x+k)}{\Gamma(x)} \tag{3.8}$$

is the Pochammer's symbol. Using (3.4) we can see that the Young diagram dependent part of factor Q in the denominator can be absorbed in the double product. The net effect is a simple replacement of parameters $\tilde{a}_{u,0}$ by $a_{u,0}$ in arguments of the functions Z_{bf}:

$$\prod_{u=1}^{n} \frac{\Gamma\left(-\frac{a_{0,1}-a_{1,u}}{\epsilon_2}\right) \mathbf{Q}\left(a_{0,1}-a_{1,u}+\epsilon_2 k|Y_{1,u}\right)}{\epsilon_2^k \left(-\epsilon_2\right)^{\frac{a_{0,1}-a_{1,u}}{\epsilon_2}} \left(\frac{a_{0,1}-a_{0,u}+\epsilon_2}{\epsilon_2}\right)_k} \prod_{u,v=1}^{n} Z_{bf}\left(a_{0,u}, \varnothing | a_{1,v}, Y_{1,v}\right) \tag{3.9}$$

Thus we conclude that k-instanton sector of the dashed circle in A_{r+1} linear quiver theory can be treated as insertion of the operator

$$\mathbf{Q}_{\vec{Y_1}}(a_{0,1}+k\epsilon_2) = \prod_{u=1}^{n} \mathbf{Q}\left(a_{0,1}-a_{1,u}+\epsilon_2 k|Y_{1,u}\right) \tag{3.10}$$

in a generic A_r theory. It was already known [14], that the special choice of parameters $\tilde{a}_{0,u} = a_{0,u} - \epsilon_1 \delta_{u,1}$ corresponds to the insertion of the completely degenerate field $V_{-b\omega_1(z)}$ in AGT dual Toda CFT conformal block. Thus (3.10) gives an explicit realization of this field in terms of $\mathcal{N}=2$ gauge theory notions.

Until now we were discussing arbitrary gauge $U(n)$ gauge factors. In what follows, we'll restrict ourselves with the case $n=2$, corresponding to the Liouville theory in AGT dual side. The reason is that in Liouville theory conformal blocks including this degenerate field, satisfy second order differential equation *. In remaining part of the paper we'll translate this differential equation in gauge theory terms, finding a linear difference-differential equation, satisfied by the expectation values of the operators $\mathbf{Q}(v)$. Since the equation is valid for infinitely many discrete values of the spectral parameter $v = a_{0,1} + k\epsilon_2$, $k = 0, 1, 2, \ldots$, it can be argued that it is valid for generic values of v as well. The last statement we have checked also by explicit low order instanton computations.

*In generic Toda theory, the analogues null vector decoupling equation is not investigated in full details yet. Instead there is a recent progress in the case of quasi-classical limit [38].

3.3 Degenerate field decoupling equation in Liouville theory

Let us briefly remind that the Liouvill theory (see e.g. [76]) is characterized by the central charge c of Virasoro algebra parameterized as

$$c = 1 + 6Q^2 \qquad Q = b + \frac{1}{b} \tag{3.11}$$

where b is the Liouvill's dimensionless coupling constant related to the Ω-background parameters via

$$b = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \tag{3.12}$$

The conformal dimensions of primary fields are V_λ are given by

$$h(\lambda) = \lambda(Q - \lambda)\,. \tag{3.13}$$

The parameters α are usually referred as charges. One alternatively uses the Liouville momenta $P = Q/2 - \lambda$. In Fig.3.1b we found it convenient to specify the fields associated to the horizontal lines by their momenta, while those of vertical lines by charges. The relations between this parameters and the gauge theory VEV's are very simple[†]

$$p_\alpha = \frac{1}{\sqrt{\epsilon_1\epsilon_2}} \frac{a_{\alpha,1} - a_{\alpha,2}}{2}\,; \qquad \lambda_\alpha = \frac{1}{\sqrt{\epsilon_1\epsilon_2}} \left(\frac{a_{\alpha,1} + a_{\alpha,2}}{2} - \frac{a_{\alpha-1,1} + a_{\alpha-1,2}}{2} \right) \tag{3.14}$$

[†]The reader should be careful, there are various factors of 2 between specialized to $n = 2$ Toda notations compared to the standard Liouville theory conventions, adopted also in this paper.

for $\alpha = 2, 3, \ldots, r+1$. With the same logic we have

$$p_0 = \frac{1}{\sqrt{\epsilon_1\epsilon_2}}\frac{a_{0,1}-a_{0,2}}{2}\,; \qquad p_{\tilde{0}} = \frac{1}{\sqrt{\epsilon_1\epsilon_2}}\frac{a_{0,1}-\epsilon_1-a_{0,2}}{2}$$
$$\lambda_{\tilde{0}} = -\frac{\epsilon_1}{\sqrt{\epsilon_1\epsilon_2}} = -\frac{b}{2}\,; \qquad \lambda_1 = \frac{\epsilon_1}{\sqrt{\epsilon_1\epsilon_2}}\left(\frac{a_{1,1}+a_{1,2}}{2} - \frac{a_{0,1}-\epsilon_1+a_{0,2}}{2}\right) \tag{3.15}$$

Notice that the field $V_{\lambda_{\tilde{0}}} = V_{-b/2}$ is indeed a degenerate field satisfying second order differential equation due to the null vector decoupling condition (below L_m are the Virasoro generators)

$$(b^{-2}L_{-1}^2 + L_{-2})V_{-b/2} = 0 \tag{3.16}$$

The differential equation satisfied by our $r+4$-point conformal block

$$G(z|z_\alpha) = \langle p_0|V_{-b/2}(z)V_{\lambda_1}(1)V_{\lambda_2}(z_2)\cdots V_{\lambda_{r+1}}(z_{r+1})|p_{r+1}\rangle_{\{\tilde{p}_0,\ldots,p_r\}} \tag{3.17}$$

reads [15]

$$\left(b^{-2}\partial_z^2 - \frac{2z-1}{z(z-1)}\,\partial_z + \frac{\delta}{z(z-1)} + \sum_{\alpha=2}^{r+1}\frac{z_\alpha\,(z_\alpha-1)}{z(z-1)\,(z-z_\alpha)}\,\partial_{z_\alpha}\right.$$
$$\left. + \sum_{\alpha=1}^{r+2}\frac{h(\lambda_\alpha)}{(z-z_\alpha)^2}\right)G(z|z_\alpha) = 0 \tag{3.18}$$

where

$$\delta = h\,(Q/2-p_0) - h\,(-b/2) - \sum_{\alpha=1}^{r+2} h(\lambda_\alpha) \quad \text{and} \quad \lambda_{r+2} = Q/2 - p_{r+1}\,. \tag{3.19}$$

According to AGT correspondence the instanton part of the partition function of the $\mathcal{N}=2$ theory considered in previous section with $U(2)$ gauge group factors is related to the conformal block (3.17) as

$$G(z|z_\alpha) = Z_{inst}\; z^{h(Q/2-p_0)-h(-b/2)-b\sum_{\alpha=1}^{r+1}(Q-\lambda_\alpha)} \prod_{\alpha=1}^{r+1} (z-z_\alpha)^{b(Q-\lambda_\alpha)}$$
$$\times \prod_{1\le\alpha<\beta\le r+1} (z_\alpha - z_\beta)^{-2\lambda_\alpha(Q-\lambda_\beta)} \prod_{\alpha=2}^{r+1} z_\alpha^{p_\alpha^2-p_{\alpha-1}^2-h(\lambda_\alpha)+2\lambda_\alpha\sum_{\beta=\alpha+1}^{r+1}(Q-\lambda_\beta)} . \tag{3.20}$$

To complete the map (3.14), (3.14) between two sides let us mention also that the exponentiated gauge couplings (instanton counting parameters) are related to the insertion points as [39]

$$q_\alpha = z_{\alpha+1}/z_\alpha\,; \quad \text{for} \quad \alpha = 1,\ldots,r\,, \tag{3.21}$$

the remaining coupling associated to the special node $\tilde{0}$ is just $1/z$ and $z_1 = 1$.

In (3.20) besides standard AGT $U(1)$ factors an extra power of z responsible for scale transformation (with scaling factor z) mapping the insertion points shown in Fig.3.1b to those of the conformal block (3.17). Inserting (3.20) into (3.18) and replacing CFT parameters by their gauge theory counterparts we'll find a differential equation satisfied by the partition function. After tedious but straightforward transformations it is possible to represent this equation as (for more details on calculations of this kind see [38])

$$\sum_{\alpha=0}^{r+1} (-)^\alpha \chi_\alpha(-\epsilon_2\, z\partial_z; \hat{u}_1,\ldots,\hat{u}_{r+1}) z^{-\alpha-a_{0,1}/\epsilon_2} Z_{inst} = 0 \tag{3.22}$$

where

$$\hat{u}_1 = -\epsilon_1\epsilon_2 \sum_{\alpha=2}^{r+1} z_\alpha\partial_{z_\alpha}\,; \qquad \hat{u}_\alpha = \epsilon_1\epsilon_2 z_\alpha \partial_{z_\alpha} \qquad \text{for} \qquad \alpha = 2,\ldots,r+1 \tag{3.23}$$

and $\chi_\alpha(v; u_1,\ldots,u_{r+1})$ are quadratic in v and linear in $u_1,\ldots,u_{r+1}$ polynomials (we use notation

$\epsilon = \epsilon_1 + \epsilon_2$)

$$\chi_\alpha(v; u_1, \dots, u_{r+1}) = \sum_{1 \le k_1 < \cdots < k_\alpha \le r+1} \left(\prod_{\beta=1}^{\alpha} z_{k_\beta} \right) \Bigg(y_0(v + \alpha\epsilon + (\alpha - \delta_{k_1,1})\epsilon_1)$$
$$- \sum_{\beta=1}^{\alpha} \big(y_{k_\beta - 1}(v + (\alpha - \beta + 1)\epsilon + (\alpha - \delta_{k_1,1})\epsilon_1) - y_{k_\beta}(v + (\alpha - \beta)\epsilon + (\alpha - \delta_{k_1,1})\epsilon_1)$$
$$+ u_{k_\beta} + (c_{0,1} - c_{k_\beta - 1,1})(c_{k_\beta - 1,1} - c_{k_\beta,1}) \big)$$
$$+ \sum_{1 \le \beta < \gamma \le \alpha} (c_{k_\beta - 1,1} - c_{k_\beta,1})(c_{k_\gamma - 1,1} - c_{k_\gamma,1}) \Bigg) , \tag{3.24}$$

where for $\alpha = 0, 1, \dots, r+1$

$$y_\alpha(v) = (v - a_{\alpha,1})(v - a_{\alpha,2})v^2 - c_{\alpha,1}v + c_{\alpha,2} \, . \tag{3.25}$$

We set by definition

$$\chi_0(v) = y_0(v) \tag{3.26}$$

and for the other extreme value $\alpha = r+1$ it is easy to see that

$$\chi_{r+1}(v) = y_{r+1}(v) \prod_{\beta=1}^{r} z_\beta \, . \tag{3.27}$$

Representing Z_{inst} as a power series in $1/z$,

$$Z_{inst} = \sum_{v \in a_{0,1} + \epsilon_2 \mathbb{Z}} Q(v) z^{-(v - a_{0,1})/\epsilon_2} \tag{3.28}$$

from eq. (3.22) for the coefficients $Q(v)$ we get the relation

$$\sum_{\alpha=0}^{r+1} (-)^\alpha \chi_\alpha(v; \hat{u}_1, \dots, \hat{u}_{r+1}) Q(v - \alpha\epsilon_2) = 0 \, , \tag{3.29}$$

which is valid for infinitely many values $v \in a_{0,1} + \epsilon_2\mathbb{Z}$. Since Z_{inst} is regular at $z = \infty$, in fact we have nontrivial equations only for $v_k = a_{0,1} + \epsilon_2 k$, with $k \ge 0$.

Remind now that as discussed in previous section, due to eqs. (3.9), (3.10), Z_{inst} of the A_{r+1} theory up to a simple factor is the same as VEV of the quantity $\mathbf{Q}_{\vec{Y}_1}$ (3.10) calculated in the framework of A_r gauge theory (i.e. in theory without the dashed circle in Fig.3.1a). Explicitly

$$Q(v_k) = C \prod_{u=1}^{2} \frac{\epsilon_2^{(a_{0,1}-v_k)/\epsilon_2}}{\Gamma\left(\frac{v_k-a_{0,u}}{\epsilon_2}+1\right)} \langle \mathbf{Q}_{\vec{Y}_1}(v_k)\rangle_{A_r}\,, \tag{3.30}$$

where the constant C takes the value

$$C = \prod_{u=1}^{2} \frac{\Gamma\left(\frac{a_{1,u}-a_{0,1}}{\epsilon_2}\right)\Gamma\left(\frac{a_{0,1}-a_{0,u}}{\epsilon_2}+1\right)}{(-\epsilon_2)^{\frac{a_{0,1}-a_{0,u}}{\epsilon_2}}}\,, \tag{3.31}$$

if one adopts conventional unit normalization for both partition function and the conformal block. The right hand side of this equation can be calculated by means of gauge theory for arbitrary $v \in \mathbb{C}$. There are all reasons to believe that also for generic values of v the equation (3.29) still holds. Indeed, for a given instanton order, the equation (3.29) states, that some combination of rational functions‡ of v vanish for all values $v = v_k$, but this is possible only if this combination vanishes identically.

A simple inspection ensures that the equation (3.29) in Nekrasov-Shatashvili limit completely agrees with the analogous difference equation investigated in details in [38].

3.4 Summary

Thus we made an explicit link between the insertion of the $\mathbf{Q}$ operator in $\mathcal{N} = 2$ gauge theory and insertion of simplest degenerate field in AGT dual 2d CFT.

In the special case of the gauge groups $U(2)$ we found analog of the Baxter's $T - Q$ equation, previously known only in the Nekrasov-Shatashvili limit of the Ω-background [32,35–37,62].

‡Evidently, by multiplying with suitable gamma and exponential functions it is easy to get rid of non-rational prefactors of (3.6), (3.30).

To conclude let us mention that a "microscopic" proof of this statement e.g. along the line presented in [42] to prove qq-character identities of [41] would be highly desirable.

Another important contribution would be generalization of our analysis to the case of arbitrary $U(n)$ or other choices of gauge groups.

Chapter 4

RG domain wall for the N=1 minimal superconformal models

4.1 Introduction

If there exists a RG flow between two CFT's then it suggests that these theories could be connected by a non-trivial interface, which encodes the map from the UV observables to the IR ones [77, 78]. In particular in [78], for the $N = 2$ superconformal models by using matrix factorization technique such an interface (called RG domain wall) was constructed.

Later on an algebraic construction of a RG domain wall for the unitary minimal CFT models was proposed in [79]. It was shown that the results agree with A. Zamolodchikovs leading order perturbative analysis performed in [80].

It was shown in [81] that for the wider class of local fields including non-primary ones, the leading order perturbative calculation of the mixing coefficients again are in an impressive agreement with the RG domain wall approach.

The higher order perturbative calculations (see [82, 83]) further confirm the validity of this construction.

Gaiotto suggests that a similar construction is valid also for more general coset CFT models (see [79]). Among these cosets. are the $N = 1$ minimal superconformal CFT models

[84–86], which are the main subject of this paper.

in [87] the Renormalization Group (RG) flow between minimal $N = 1$ superconformal models SM_p and SM_{p-2} initialized by the perturbation with the top component of the Neveu-Schwarz superfield $\Phi_{1,3}$ in leading order of the perturbation theory has been investigated (see also [88, 89]).

In [90] by extending the technique developed in [82] for the minimal models to the supersymmetric case, the analysis of this RG flow has been sharpened even further by including also the next to leading order corrections.

In this article we apply Gaiotto's proposal for the case of the minimal N=1 SCFT models. We use a method which is based directly on the current algebra construction thus in this sense it is more general than the one originally employed by Gaiotto for the case of minimal models (he heavily exploited the fact that the product of successive minimal models can be alternatively represented as a product of $N = 1$ superconformal and Ising models). After that the mixing coefficients for several classes of fields is explicitly calculated by us. We also compare the results with the perturbative analysis of [87, 90] and find a complete agreement.

4.2 N=1 superconformal field theory

In any conformal field theory the energy-momentum tensor has two nonzero components: the holomorphic field $T(z)$ with conformal dimension $(2, 0)$ and its anti-holomorphic counterpart $\bar{T}(\bar{z})$ with dimensions $(0, 2)$. In $N = 1$ superconformal field theories one has in addition superconformal currents $G(z)$ and $\bar{G}(\bar{z})$ with dimensions $(3/2, 0)$ and $(0, 3/2)$ respectively. These fields satisfy the OPE rules

$$
\begin{aligned}
T(z)T(0) &= \frac{c}{2z^4} + \frac{2T(0)}{z^2} + \frac{T'(0)}{z} + \cdots , && (4.1)\\
T(z)G(0) &= \frac{3G(0)}{2z^2} + \frac{G'(0)}{z} + \cdots , && (4.2)\\
G(z)G(0) &= \frac{2c}{3z^3} + \frac{2T(0)}{z} + \cdots . && (4.3)
\end{aligned}
$$

The corresponding expressions for the anti-chiral fields look exactly the same. One should simply substitute z by $\bar{z}$. Further on we'll mainly concentrate on the holomorphic part assuming similar expressions for anti-holomorphic quantities implicitly. We can expand $T(z)$ in Laurent series

$$T(z) = \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}}, \tag{4.4}$$

where L_n's are the Virasoro generators. Due to the fermionic nature of the super current, there are two distinct possibilities for its behavior under the rotation of the argument around 0 by the angle 2π

$$G(e^{2\pi i} z) = G(z) \qquad \textit{Neveu - Schwarz sector (NS)}, \tag{4.5}$$

$$G(e^{2\pi i} z) = -G(z) \qquad \textit{Ramond sector (R)}. \tag{4.6}$$

The space of fields $\mathcal{A}$ of the superconformal theory decomposes into a direct sum

$$\mathcal{A} = \{NS\} \oplus \{R\}, \tag{4.7}$$

where the subspaces $\{NS\}$ and $\{R\}$ consist of the Neveu-Shwarz and the Ramond fields respectively. By definition, the monodromy of $G(z)$ around a Neveu-Schwarz field is trivial (the case of eq. (4.5)) and its monodromy around a Ramond field produces a minus sign (the case of eq. (4.6)). Because of these two possibilities the Laurent expansions for the super-current will be

$$\begin{aligned} G(z) &= \sum_{k \in Z+1/2} \frac{G_k}{z^{k+3/2}} \qquad \textit{Neveu-Schwarz sector (NS)}, \\ G(z) &= \sum_{k \in Z} \frac{G_k}{z^{k+3/2}} \qquad \textit{Ramond sector (R)}. \end{aligned}$$

The OPE's (4.1), (4.2), (4.3) are equivalent to the Neveu-Schwarz-Ramond algebra relations

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}\,, \\
[L_n, G_k] &= \frac{1}{2}(n-2k)G_{n+k}\,, \\
\{G_k, G_l\} &= 2L_{k+l} + \frac{c}{3}\left(k^2-1/4\right)\delta_{k+l,0}\,,
\end{aligned} \tag{4.8}$$

where $\{,\}$ denotes the anticommutator. In this paper we'll deal with minimal super-conformal series denoted as SM_p $(p = 3, 4, 5\ldots)$ corresponding to the choice of the central charge

$$c_p = \frac{3}{2}\left(1 - \frac{8}{p(p+2)}\right)\,. \tag{4.9}$$

The main distinctive mark of the minimal super-conformal theories is that they have finitely many super primary fields. These fields are numerated by two integers $n \in \{1, 2, \cdots, p-1\}$, $m \in \{1, 2, \cdots, p+1\}$ and will be denoted as $\phi_{n,m}$. It is assumed that $\phi_{p-n,p+2-m} \equiv \phi_{n,m}$, hence the number of super primaries is equal to $[p^2/2]$ ([x] is the integer part of x). $\phi_{p-1,p+1} \equiv \phi_{1,1}$ is the identity operator. For even (odd) $n-m$ the super-conformal classes $[\phi_{n,m}]$ form irreducible representations of the Neveu-Schwarz (Ramond) algebra. The fields $\phi_{n,m}$ have dimensions

$$h_{n,m} = \frac{((p+2)n - pm)^2 - 4}{8p(p+2)} + \frac{1}{32}(1 - (-)^{n-m})\,. \tag{4.10}$$

4.3 Current algebra and the coset construction

We apply the coset construction [91, 92] of super-minimal models for $\widehat{SU}(2)_k$ WZNW models [93, 94].

WZNW models are endowed with spin one holomorphic currents, thus the OPE relations of these currents specified for the case of $\widehat{SU}(2)_k$ read:

$$\begin{aligned} J^0(z)J^0(0) &= \frac{k/2}{z^2} + reg\,, \\ J^0(z)J^{\pm}(0) &= \pm\frac{J^{\pm}(0)}{z} + reg\,, \\ J^+(z)J^-(0) &= \frac{k}{z^2} + \frac{2J^0(0)}{z} + reg\,, \end{aligned} \tag{4.11}$$

where k is the level. The isotopic indices $\pm, 0$, which will be handy for the later use, are related to the usual Euclidean indices as:

$$J^0 \equiv J^3 \qquad \text{and} \qquad J^{\pm} \equiv J^1 \pm iJ^2\,. \tag{4.12}$$

The Laurent expansion is

$$J^a(z) = \sum_{n\in Z} \frac{J^a_n}{z^{n+1}}\,. \tag{4.13}$$

The OPE rules (4.11) imply that the current algebra generators obey the Kač $-$ *Moody* algebra commutation relations

$$\begin{aligned} \left[J^{\pm}_n, J^{\pm}_m\right] &= 0\,, \\ \left[J^{+}_n, J^{-}_m\right] &= kn\delta_{n+m,0} + 2J^0_{n+m}\,, \\ \left[J^{0}_n, J^{\pm}_m\right] &= \pm J^{\pm}_{n+m}\,, \\ \left[J^{0}_n, J^{0}_m\right] &= \frac{kn}{2}\delta_{n+m,0}\,. \end{aligned} \tag{4.14}$$

Note the subalgebra generated by J^a_0 is the Lie algebra $su(2)$.

With the help of the Sugawara construction, the energy momentum tensor is expressed through the currents

$$T(z) = \frac{1}{k+2}\left(J^0J^0 + \frac{1}{2}J^+J^- + \frac{1}{2}J^-J^+\right)\,. \tag{4.15}$$

We assumed above and in what follows that any product of local fields taken at coinciding points is regularized subtracting singular parts of the respective OPE. One can compute the central charge of the Virasoro algebra by using (4.15). The result is:

$$c_k = \frac{3k}{k+2}\,. \tag{4.16}$$

The primary fields of the theory $\phi_{j,m}$ and the corresponding states $|j,m\rangle$ are labeled by the spin of the representation $j = 0, 1/2, 1, \ldots, k/2$ and its projection $m = -j, -j+1, \ldots, j$, whose conformal dimensions are given by

$$h = \frac{j(j+1)}{k+2}\,. \tag{4.17}$$

The zero modes of the currents (4.15) act on the states $|j,m\rangle$ as *

$$\begin{aligned} J^{\pm}|j,m\rangle &= \sqrt{j(j+1) - m(m\pm 1)}|j, m\pm 1\rangle\,, \\ J^{0}|j,m\rangle &= m|j,m\rangle\,. \end{aligned} \tag{4.18}$$

As is known the explicit form of the $su(2)$ WZNW modular matrices is

$$S^{(k)}_{n,m} = \sqrt{\frac{2}{k+2}} \sin\frac{\pi n m}{k+2}\,. \tag{4.19}$$

The $N = 1$ super-minimal models can be represented as a coset [91, 92]

$$\mathcal{SM}_{k+2} = \frac{su(2)_k \times su(2)_2}{su(2)_{k+2}}\,.$$

Thus the energy momentum tensor of $\mathcal{SM}_{k+2}$ is given by

$$T_{(su(2)_k \times su(2)_2)/su(2)_{k+2}} = T_{su(2)_k} + T_{su(2)_2} - T_{su(2)_{k+2}}\,. \tag{4.20}$$

*Note that a consistent with eq. (4.18) conjugation rule for the primary fields would be $\phi^{\dagger}_{j,m} = (-)^{j-m}\phi_{j,-m}$

Indeed, one can easily check that the combination of the central charges (4.16) corresponding to these three terms matches with the central charge of the super-minimal models (4.9).

The construction of the super-current G is more subtle; it involves the primary fields $\phi_{1,m}$ of the level $k = 2$ WZNW theory (we denote the currents of this theory as K^a):

$$G(z) = C_a J^a(z)\phi_{1,-a}(z) + D_a K^a_{-1}\phi_{1,-a}(z)\,, \tag{4.21}$$

where summation over the index $a = \pm, 0$ is assumed. One can be fixed C_a, D_a by requiring the respective state to be the highest weight state of the diagonal current algebra $J + K$ i.e. both $J^+_0 + K^+_0$ and $J^+_1 + K^+_1$ annihilate the state

$$C_a J^a_{-1}|0\rangle|1, -a\rangle + D^a|0\rangle K^a_{-1}|1, -a\rangle\,. \tag{4.22}$$

Up to an overall constant κ one obtains

$$\begin{aligned} D_+ &= \frac{\kappa}{\sqrt{2}}\,, & D_0 &= \kappa\,, & D_- &= -\frac{\kappa}{\sqrt{2}}\,, \\ C_+ &= -\frac{3\kappa\sqrt{2}}{k}\,, & C_0 &= -\frac{6\kappa}{k}\,, & C_- &= \frac{3\kappa\sqrt{2}}{k}\,. \end{aligned} \tag{4.23}$$

Of course the value of κ can be determined using the normalization condition of the the super-current fixed by the OPE (4.3)

$$\kappa = \sqrt{\frac{(k+2)(k+4)}{(k+6)(5k+54)}}\,, \tag{4.24}$$

but this will not be importance for our goals.

4.4 Perturbative RG flows and domain walls

A. Zamolodchikov [80] has investigated the RG flow from minimal model $\mathcal{M}_p$ to $\mathcal{M}_{p-1}$ initiated by the relevant field $\phi_{1,3}$. By applying the leading order perturbation theory valid for $p >> 1$, for several classes of local fields, he calculated the mixing coefficients specifying the UV - IR map.

It was shown in [87] that a similar RG trajectory connecting $\mathcal{N} = 1$ super-minimal models $\mathcal{SM}_p$ to $\mathcal{SM}_{p-2}$ exists. In this case the RG flow is initiated by the top component of the Neveu-Schwartz superfield $\Phi_{1,3}$. For our purposes it is important that also in this case a detailed analysis of some classes of fields has been carried out.

It became clear later [89,95], that the above two examples are just the first simplest cases of more general RG flows. Under perturbation by the relevant field $\phi = \phi_{1,1}^{Adj}$, a wide class of CFT coset models

$$\mathcal{T}_{UV} = \frac{\hat{g}_l \times \hat{g}_m}{\hat{g}_{l+m}}, \qquad m > l \tag{4.25}$$

flow to the theories

$$\mathcal{T}_{IR} = \frac{\hat{g}_l \times \hat{g}_{m-l}}{\hat{g}_m}, \tag{4.26}$$

in the IR limit.

In [79] a nontrivial conformal interface between successive minimal CFT models is constructed and a is proposed that this interface (RG domain wall) encodes the UV - IR map resulting through the RG flow discussed above. Of course this proposal is in agreement with the leading order perturbative analysis of [80].

Generalization of leading order calculations to a wider class of local fields [81] as well as next to leading order calculations [82,83] further confirm the validity of this construction.

In [79] Gaiotto also suggests a candidate for RG domain wall for the much more general RG flow between (4.25) and (4.26). We will recall the construction briefly. Because a conformal interface between two CFT models is equivalent to some conformal boundary for the direct product of these theories (folding trick), it is natural to consider the product theory $\mathcal{T}_{UV} \times \mathcal{T}_{IR}$

$$\frac{\hat{g}_l \times \hat{g}_m}{\hat{g}_{m+l}} \times \frac{\hat{g}_l \times \hat{g}_{m-l}}{\hat{g}_m} \sim \frac{\hat{g}_{m-l} \times \hat{g}_l \times \hat{g}_l}{\hat{g}_{l+m}} . \tag{4.27}$$

Note that has a natural $\mathbb{Z}_2$ automorphism do to the appearance of two identical factors $\hat{g}_l$ above. Gaiottos proposal states that the boundary of the theory

$$\mathcal{T}_B = \frac{\hat{g}_l \times \hat{g}_l \times \hat{g}_{m-l}}{\hat{g}_{l+m}}, \qquad m > l \tag{4.28}$$

acts as a $\mathbb{Z}_2$ twisting mirror. More precisely the RG boundary condition is the image of the $\mathbb{Z}_2$ twisted $\mathcal{T}_B$ brane

$$|\tilde{B}\rangle = \sum_{s,t} \sqrt{S^{(m-l)}_{1,t} S^{(m+l)}_{1,s}} \sum_d |t, d, d, s; \mathcal{B}, Z_2\rangle\rangle, \tag{4.29}$$

where the indices t, d, s refer to the representations of $\hat{g}_{m-l}$, $\hat{g}_l$, $\hat{g}_{l+m}$ respectively and $S^{(k)}_{1,r}$ are the modular matrices of the $\hat{g}_k$ WZNW model.

In the coming sections we will examine the case of RG flow between $\mathcal{N} = 1$ super-minimal models, in details. We apply a method which directly explores the current algebra representation in contrary to the analysis in [79], where a specific representation applicable only for the unitary minimal series was used.

4.5 RG domain walls for super minimal models

In the case of the $\mathcal{N} = 1$ super-minimal models needs to consider

$$\frac{\widehat{su}(2)_k \times \widehat{su}(2)_2}{\widehat{su}(2)_{k+2}} \times \frac{\widehat{su}(2)_{k-2} \times \widehat{su}(2)_2}{\widehat{su}(2)_k} \sim \frac{\widehat{su}(2)_{k-2} \times \widehat{su}(2)_2 \times \widehat{su}(2)_2}{\widehat{su}(2)_{k+2}} , \tag{4.30}$$

where the first coset on the left hand side corresponds to the UV super conformal model $\mathcal{SM}_{k+2}$ and the second one to the IR theory $\mathcal{SM}_k$. We denote by $K(z)$ and $\widetilde{K}(z)$ the WZNW currents of $\widehat{su}(2)_2$ entering in the cosets of the IR and UV theories respectively. The current of $\widehat{su}(2)_{k-2}$

WZNW theory will be denoted as $J(z)$. Using (4.20) and the Sugawara construction, for the energy-momentum tensor of the IR theory (the second factor of the left hand sides of (4.30)) one obtains

$$T_{ir}(z) = \frac{1}{k}J(z)J(z) + \frac{1}{4}K(z)K(z) - \frac{1}{k+2}(K(z)+J(z))^2,$$

this can be rewritten as

$$T_{ir}(z) = \frac{2}{2k+k^2}J(z)J(z) - \frac{2}{2+k}J(z)K(z) + \frac{k-2}{4(k+2)}K(z)K(z). \tag{4.31}$$

In the same way the energy-momentum tensor for the UV theory is equal to

$$\begin{aligned} T_{uv}(z) = & \frac{2}{(2+k)(4+k)}J(z)J(z) + \frac{2}{(2+k)(4+k)}K(z)K(z) \\ & -\frac{2}{4+k}K(z)\widetilde{K}(z) + \frac{k}{4(k+4)}\widetilde{K}(z)\widetilde{K}(z) \\ & +\frac{4}{(2+k)(4+k)}J(z)K(z) - \frac{2}{4+k}J(z)\widetilde{K}(z)\,. \end{aligned} \tag{4.32}$$

To get the one-point functions of the theory $\mathcal{SM}_{k+2} \times \mathcal{SM}_k$ in the presence of RG boundary, we need explicit expressions of the states corresponding to fields $\phi^{IR}\phi^{UV}$ in terms of the states of the coset theory

$$\mathcal{T}_B = \frac{\widehat{su}(2)_{k-2} \times \widehat{su}(2)_2 \times \widehat{su}(2)_2}{\widehat{su}(2)_{k+2}}\,. \tag{4.33}$$

We denote the highest weight representation spaces of the current algebras $J(z)$, $K(z)$ and $\widetilde{K}(z)$ as $V_j^{(J)}$, $V_k^{(K)}$ and $V_{\widetilde{k}}^{(\widetilde{K})}$ respectively. The lower indices specify the spins of the highest weight states. It is convenient to fix a unique representative of a state of the coset $\mathcal{T}_B$ in the space $V_j^{(J)} \otimes V_k^{(K)} \otimes V_{\widetilde{k}}^{(\widetilde{K})}$ requiring that the state under consideration be a highest weight state of the diagonal current $J+K+\widetilde{K}$. Let us analyze the states corresponding to $\phi^{IR}_{n,n}\phi^{UV}_{n,n}$. Since

$$h^{ir}_{n,n} = \frac{n^2-1}{4k} - \frac{n^2-1}{4(k+2)}\,,$$
$$h^{uv}_{n,n} = \frac{n^2-1}{4(k+2)} - \frac{n^2-1}{4(k+4)}\,,$$

the total dimension of the product field is

$$h^{ir}_{n,n} + h^{uv}_{n,n} = \frac{n^2-1}{4k} - \frac{n^2-1}{4(k+4)}\,, \tag{4.34}$$

thus the corresponding state is readily identified with

$$|\frac{n-1}{2},\frac{n-1}{2}\rangle|0,0\rangle|0,0\rangle \in V^{(J)}_{\frac{n-1}{2}} \otimes V^{(K)}_0 \otimes V^{(\widetilde{K})}_0\,. \tag{4.35}$$

where $|j,m\rangle$ denotes a primary state of spin j and projection m, so the state above is a spin $\frac{n-1}{2}$ highest weight state of the combined current $J+K+\widetilde{K}$ and its $\mathcal{T}_B$ dimension

$$h^{(J)}_{\frac{n-1}{2}} + h^{(K)}_0 + h^{(\widetilde{K})}_0 - h^{(J+K+\widetilde{K})}_{\frac{n-1}{2}}$$

coincides with (4.34). Note that the permutation of the second and third factors i.e. the $\mathbb{Z}_2$ action on this state is trivial. Hence the overlap of this state with its $\mathbb{Z}_2$ image is equal to 1 and from (4.29)

$$\langle\phi^{IR}_{n,n}\phi^{UV}_{n,n}|RG\rangle = \frac{\sqrt{S^{(k-2)}_{1,n}S^{(k+2)}_{1,n}}}{S^{(k)}_{1,n}}\,. \tag{4.36}$$

For large k and for $n\sim O(1)$ this gives $1+3/k^2+O(1/k^3)$. Our conclusion is that up to $1/k^2$ terms, the fields $\phi^{UV}_{n,n}$ flow to $\phi^{IR}_{n,n}$ without mixing with other fields, which is in a complete agreement with both leading order [87] and next to leading order [90] perturbative calculations.

We next examine a more interesting case that of Ramond fields $\phi^{UV}_{n,n\pm1}$ which are expected to flow to certain combinations of the fields $\phi^{IR}_{n\pm1,n}$ [87].

From (4.10) for the state corresponding to $\phi^{ir}_{n-1,n}\phi^{uv}_{n,n-1}$, we get

$$h^{ir}_{n-1,n} = \frac{3}{16} + \frac{(n-1)^2-1}{4k} - \frac{n^2-1}{4(k+2)}\,, \tag{4.37}$$

$$h^{uv}_{n,n-1} = \frac{3}{16} - \frac{(n-1)^2-1}{4(k+4)} + \frac{n^2-1}{4(k+2)}\,. \tag{4.38}$$

Thus the conformal dimension of this product field will be

$$h^{ir}_{n-1,n} + h^{uv}_{n,n-1} = \frac{3}{8} + \frac{(n-1)^2-1}{4k} - \frac{(n-1)^2-1}{4(k+4)}\,. \tag{4.39}$$

There are three primaries in $su(2)_2$ WZNW theory with $j = 0, 1, 2$ representations and conformal dimensions 0, $\frac{3}{16}$ and $\frac{1}{2}$ respectively. Hence to get the right dimension one should choose a combination of states $|\frac{n}{2}-1, m\rangle|\frac{1}{2},\alpha\rangle|\frac{1}{2},\beta\rangle$. More this combination must be the spin $\frac{n}{2}-1$ highest weight state of $J+K+\widetilde{K}$, this is to match with the last, negative term of (4.39). In that way we are lead to

$$C_{\alpha\beta}|\frac{n}{2}-1, \frac{n}{2}-1-\alpha-\beta\rangle|\frac{1}{2},\alpha\rangle|\frac{1}{2},\beta\rangle, \tag{4.40}$$

where a summation over the indices $\alpha, = \pm 1/2$ is assumed. The highest weight condition which is to say that the operator $J^+_0 + K^+_0 + \widetilde{K}_0$ annihilates this state, implies

$$\sqrt{n-2}C_{++} + C_{-+} + C_{+-} = 0\,.$$

A further constraint

$$C_{++} - \sqrt{n-2}C_{-+} = 0\,,$$

one obtains imposing the condition that this state should be an eigenstate of the Virasoro operator L^{IR}_0 constructed from the energy-momentum tensor T_{ir} (4.31) with eigenvalue $h^{ir}_{n,n-1}$

(4.37). Thus we get

$$C_{++} = \sqrt{n-2}\, C_{-+}\,, \qquad C_{+-} = -(n-1)\, C_{-+}$$

Note that the undefined overall multiplier could be fixed from the normalization condition. By taking normalized scalar product of the state (4.40) with its $\mathbb{Z}_2$ image we find

$$\langle \phi^{ir}_{n-1,n}\phi^{uv}_{n,n-1}|RG\rangle = -\frac{1}{n-1}\frac{\sqrt{S^{(k-2)}_{1,n-1}S^{(k+2)}_{1,n-1}}}{S^{k}_{1,n}}\,. \tag{4.41}$$

Consideration of the product $\phi^{ir}_{n+1,n}\phi^{uv}_{n,n+1}$ fields is quite similar and leads to the state

$$C_{\alpha\beta}|\frac{n}{2},\frac{n}{2}-\alpha-\beta\rangle|\frac{1}{2},\alpha\rangle|\frac{1}{2},\beta\rangle\,,$$

with the coefficients

$$C_{+-} = 0\,, \qquad C_{++} = -\frac{1}{\sqrt{n}}C_{-+}\,.$$

So, in this case

$$\langle \phi^{ir}_{n+1,n}\phi^{uv}_{n,n+1}|RG\rangle = \frac{1}{n+1}\frac{\sqrt{S^{(k-2)}_{1,n+1}S^{(k+2)}_{1,n+1}}}{S^{k}_{1,n}}\,. \tag{4.42}$$

Constructing the states corresponding to $\phi^{ir}_{n-1,n}\phi^{uv}_{n,n+1}$ and $\phi^{ir}_{n+1,n}\phi^{uv}_{n,n-1}$ is even simpler and one easily gets $|\frac{n}{2}-1,\frac{n}{2}-1\rangle|\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle$ and $|\frac{n}{2},\frac{n}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle$ respectively. In both cases the $\mathbb{Z}_2$ action is trivial, hence

$$\langle \phi^{ir}_{n-1,n}\phi^{uv}_{n,n+1}|RG\rangle = \frac{\sqrt{S^{(k-2)}_{1,n-1}S^{(k+2)}_{1,n+1}}}{S^{k}_{1,n}}\,, \tag{4.43}$$

$$\langle \phi^{ir}_{n+1,n}\phi^{uv}_{n,n-1}|RG\rangle = \frac{\sqrt{S^{(k-2)}_{1,n+1}S^{(k+2)}_{1,n-1}}}{S^{k}_{1,n}}\,. \tag{4.44}$$

In the large k limit we get

$$\begin{aligned}
\langle \phi^{ir}_{n+1,n}\phi^{uv}_{n,n+1}|RG\rangle &= \frac{1}{n}+O(1/k^2)\,, &(4.45)\\
\langle \phi^{ir}_{n+1,n}\phi^{uv}_{n,n-1}|RG\rangle &= \frac{\sqrt{n^2-1}}{n}+O(1/k^2)\,, &(4.46)\\
\langle \phi^{ir}_{n-1,n}\phi^{uv}_{n,n+1}|RG\rangle &= \frac{\sqrt{n^2-1}}{n}+O(1/k^2)\,, &(4.47)\\
\langle \phi^{ir}_{n-1,n}\phi^{uv}_{n,n-1}|RG\rangle &= -\frac{1}{n}+O(1/k^2)\,, &(4.48)
\end{aligned}$$

they agree with the second order perturbation theory results [90].

We also analyzed the more complicated case of mixing of the primary Neveu-Schwartz superfields $\Phi_{n,n\pm2}$ and the descendant superfield $\mathbf{D}\bar{\mathbf{D}}\Phi_{n,n}$, here $\mathbf{D}$ and $\bar{\mathbf{D}}$ are the super-derivatives. One can find the details of our calculations in the appendix. Here are the final results:

$$\langle \psi^{ir}_{n+2,n}\psi^{uv}_{n,n+2}|RG\rangle = \frac{2}{(n+1)(n+2)}\frac{\sqrt{S^{(k-2)}_{1,n+2}S^{(k+2)}_{1,n+2}}}{S^{(k)}_{1,n}}\,, \tag{4.49}$$

$$\langle \phi^{ir}_{n+2,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = \frac{2}{n+1}\frac{\sqrt{S^{(k-2)}_{1,n+2}S^{(k+2)}_{1,n}}}{S^{(k)}_{1,n}}\,, \tag{4.50}$$

$$\langle \psi^{ir}_{n+2,n}\psi^{uv}_{n,n-2}|RG\rangle = \frac{\sqrt{S^{(k-2)}_{1,n+2}S^{(k+2)}_{1,n-2}}}{S^{(k)}_{1,n}}\,, \tag{4.51}$$

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}\phi^{uv}_{n,n+2}|RG\rangle = \frac{2}{n+1}\frac{\sqrt{S^{(k-2)}_{1,n}S^{(k+2)}_{1,n+2}}}{S^{(k)}_{1,n}}\,, \tag{4.52}$$

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = \frac{n^2-5}{n^2-1}\frac{\sqrt{S^{(k-2)}_{1,n}S^{(k+2)}_{1,n}}}{S^{(k)}_{1,n}}\,, \tag{4.53}$$

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}\phi^{uv}_{n,n-2}|RG\rangle = -\frac{2}{n-1}\frac{\sqrt{S^{(k-2)}_{1,n}S^{(k+2)}_{1,n-2}}}{S^{(k)}_{1,n}}\,, \tag{4.54}$$

$$\langle \psi^{ir}_{n-2,n}\psi^{uv}_{n,n+2}|RG\rangle = \frac{\sqrt{S^{(k-2)}_{1,n-2}S^{(k+2)}_{1,n+2}}}{S^{(k)}_{1,n}}\,, \tag{4.55}$$

$$\langle \phi^{ir}_{n-2,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = -\frac{2}{n-1}\frac{\sqrt{S^{(k-2)}_{1,n-2}S^{(k+2)}_{1,n}}}{S^{(k)}_{1,n}}\,, \tag{4.56}$$

$$\langle \phi^{ir}_{n-2,n}\phi^{uv}_{n,n-2}|RG\rangle = \frac{2}{(n-1)(n-2)}\frac{\sqrt{S^{(k-2)}_{1,n-2}S^{(k+2)}_{1,n-2}}}{S^{k}_{1,n}}\,. \tag{4.57}$$

At the large k limit we get

$$\langle \psi^{ir}_{n+2,n}\psi^{uv}_{n,n+2}|RG\rangle = \frac{2}{n(n+1)} + O(1/k^2)\,, \tag{4.58}$$

$$\langle \phi^{ir}_{n+2,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = \frac{2}{n+1}\sqrt{\frac{n+2}{n}} + O(1/k^2)\,, \tag{4.59}$$

$$\langle \psi^{ir}_{n+2,n}\psi^{uv}_{n,n-2}|RG\rangle = \frac{\sqrt{n^2-4}}{n} + O(1/k^2)\,, \tag{4.60}$$

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}\phi^{uv}_{n,n+2}|RG\rangle = \frac{2}{n+1}\sqrt{\frac{n+2}{n}} + O(1/k^2)\,, \tag{4.61}$$

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = \frac{n^2-5}{n^2-1} + O(1/k^2)\,, \tag{4.62}$$

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}\phi^{uv}_{n,n-2}|RG\rangle = -\frac{2}{n-1}\sqrt{\frac{n-2}{n}} + O(1/k^2)\,, \tag{4.63}$$

$$\langle \psi^{ir}_{n-2,n}\psi^{uv}_{n,n+2}|RG\rangle = \frac{\sqrt{n^2-4}}{n} + O(1/k^2)\,, \tag{4.64}$$

$$\langle \phi^{ir}_{n-2,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = -\frac{2}{n-1}\sqrt{\frac{n-2}{n}} + O(1/k^2)\,, \tag{4.65}$$

$$\langle \phi^{ir}_{n-2,n}\phi^{uv}_{n,n-2}|RG\rangle = \frac{2}{n(n-1)} + O(1/k^2)\,. \tag{4.66}$$

Again, the results are in complete agreement with the next to leading order perturbative calculations of [90].

4.6 Mixing of the fields $\Phi_{n,n\pm 2}$ and the descendant $\mathbf{D}\bar{\mathbf{D}}\Phi_{n,n}$

Let us start with the product field $\phi^{ir}_{n-2,n}\phi^{uv}_{n,n-2}$. The corresponding dimensions are

$$h^{ir}_{n-2,n} = \frac{1}{2} + \frac{(n-2)^2-1}{4k} - \frac{n^2-1}{4(k+2)}\,, \tag{4.67}$$

$$h^{uv}_{n,n-2} = \frac{1}{2} - \frac{(n-2)^2-1}{4(4+k)} + \frac{n^2-1}{4(k+2)}\,, \tag{4.68}$$

hence

$$h^{ir}_{n-2,n} + h^{uv}_{n,n-2} = 1 + \frac{(n-2)^2-1}{4k} - \frac{(n-2)^2-1}{4(4+k)}\,. \tag{4.69}$$

A careful examination shows that the required state should be chosen among the combinations

$$\sum_{\alpha,\beta\in\{-1,0,1\}} C_{\alpha,\beta}|\frac{n-3}{2},\frac{n-3}{2}-\alpha-\beta\rangle|1,\alpha\rangle|1,\beta\rangle\,. \tag{4.70}$$

Indeed the other candidates such as $J^a_{-1}|\frac{n-3}{2},\frac{n-3}{2}-a\rangle|0\rangle|0\rangle$, $K^a_{-1}|\frac{n-3}{2},\frac{n-3}{2}-a\rangle|0\rangle|0\rangle$ or $\widetilde{K}^\alpha_{-1}|\frac{n-3}{2},\frac{n-3}{2}-a\rangle|0\rangle|0\rangle$ though have a correct total dimension, can not be combined to get the required IR dimension (4.67). This can be easily seen by examining the zero mode of the IR current

$$T^{ir} = \frac{1}{k}J^2 - \frac{1}{k+2}(J+K)^2 + \frac{1}{4}K^2\,. \tag{4.71}$$

The only way to get the term 1/2 of (4.67) is to choose $j=1$ representation of the current K (see the last term of (4.71)).

To get correct IR dimension one should impose the condition that the zero mode of $(J+K)^2$ on the state (4.70) must acquire the eigenvalue $\frac{n-1}{2}\frac{n+1}{2}$. Together with our usual requirement of being a highest weight state of the $J+K+\widetilde{K}$ algebra this fixes the coefficients up to an overall multiplier

$$
\begin{aligned}
C_{+0} &= \sqrt{\frac{n-3}{2}}C_{00}\,, & C_{++} &= -\sqrt{\frac{n-3}{2}}\frac{\sqrt{n-4}}{n-2}C_{00}\,, \\
C_{+-} &= \frac{1-n}{2}C_{00}\,, & C_{0+} &= -\frac{2}{n-2}\sqrt{\frac{n-3}{2}}C_{00}\,, \\
C_{-+} &= -\frac{1}{n-2}C_{00}\,, & C_{-0} &= C_{0-} = C_{--} = 0\,.
\end{aligned}
$$

This leads to the one point function

$$
\langle \phi^{ir}_{n-2,n}\phi^{uv}_{n,n-2}|RG\rangle = \frac{2}{(n-1)(n-2)}\frac{\sqrt{S^{(k-2)}_{1,n-2}S^{(k+2)}_{1,n-2}}}{S^{k}_{1,n}}\,. \tag{4.72}
$$

In the same way we construct the state corresponding to $\phi^{ir}_{n+2,n}\phi^{uv}_{n,n+2}$

$$
C_{\alpha\beta}|\frac{n+1}{2},\frac{n+1}{2}-\alpha-\beta\rangle|1,\alpha\rangle|1,\beta\rangle\,,
$$

where

$$
C_{++} = -\frac{1}{\sqrt{n}}\,C_{00}, \quad C_{-+} = -\sqrt{\frac{n+1}{2}}\,C_{00}, \qquad C_{0+} = C_{00} \tag{4.73}
$$

(all other $C_{\alpha\beta}$ vanish) and

$$
\langle \psi^{ir}_{n+2,n}\psi^{uv}_{n,n+2}|RG\rangle = \frac{2}{(n+1)(n+2)}\frac{\sqrt{S^{(k-2)}_{1,n+2}S^{(k+2)}_{1,n+2}}}{S^{(k)}_{1,n}}\,. \tag{4.74}
$$

The state corresponding to $\psi^{ir}_{n+2,n}\psi^{uv}_{n,n-2}$ is simply $|\frac{n+1}{2},\frac{n+1}{2}\rangle|1,-1\rangle|1,-1\rangle$ and

$$
\langle \psi^{ir}_{n+2,n}\psi^{uv}_{n,n-2}|RG\rangle = \frac{\sqrt{S^{(k-2)}_{1,n+2}S^{(k+2)}_{1,n-2}}}{S^{(k)}_{1,n}}\,. \tag{4.75}
$$

Similarly for $\psi^{ir}_{n-2,n}\psi^{uv}_{n,n+2}$ the state is $|\frac{n-3}{2},\frac{n-3}{2}\rangle|1,1\rangle|1,1\rangle$ and

$$\langle\psi^{ir}_{n-2,n}\psi^{uv}_{n,n+2}|RG\rangle = \frac{\sqrt{S^{(k-2)}_{1,n-2}S^{(k+2)}_{1,n+2}}}{S^{(k)}_{1,n}}\,. \tag{4.76}$$

Let us now consider states corresponding to the descendant field $G^{ir}_{-1/2}\psi^{ir}_{n,n}\psi^{uv}_{n,n+2}$. Partial dimensions of the field $\phi^{ir}_{n,n}\phi^{uv}_{n,n+2}$ are

$$\begin{aligned}
h^{ir}_{n,n} &= \frac{n^2-1}{4k} - \frac{n^2-1}{4(k+2)}\,,\\
h^{uv}_{n,n+2} &= \frac{1}{2} + \frac{n^2-1}{4(k+2)} - \frac{(n+2)^2-1}{4(k+4)}\,,\\
h^{ir}_{n,n} + h^{uv}_{n,n+2} &= \frac{1}{2} + \frac{n^2-1}{4k} - \frac{(n+2)^2-1}{4(k+4)}\,.
\end{aligned}$$

Evidently the correct representative of the respective state is

$$|\frac{n-1}{2},\frac{n-1}{2}\rangle|0\rangle|1,1\rangle\,. \tag{4.77}$$

Using the expression (4.21) its is straightforward to find the result of the action of the supercurrent mode $G^{ir}_{-1/2}$ on this state:

$$\begin{aligned}
G^{ir}_{-\frac{1}{2}}|\frac{n-1}{2},\frac{n-1}{2}\rangle|0\rangle|1,1\rangle &= C_a J^a_0|\frac{n-1}{2},\frac{n-1}{2}\rangle|1,-a\rangle|1,1\rangle\\
&\quad + D_a K^a_0|\frac{n-1}{2},\frac{n-1}{2}\rangle|1,-a\rangle|1,1\rangle\,,
\end{aligned} \tag{4.78}$$

where the coefficients C_a, D_a are given by (4.23) (one should replace k by $k-2$). The final result is:

$$\begin{aligned}
G^{ir}_{-\frac{1}{2}}|\frac{n-1}{2},\frac{n-1}{2}\rangle|0\rangle|1,1\rangle &= -\frac{3(n-1)}{k-2}|\frac{n-1}{2},\frac{n-1}{2}\rangle|1,0\rangle|1,1\rangle\\
&\quad + \frac{6}{k-2}\sqrt{\frac{n-1}{2}}|\frac{n-1}{2},\frac{n-3}{2}\rangle|1,1\rangle|1,1\rangle\,.
\end{aligned} \tag{4.79}$$

Thus for the one-point function we get

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}\phi^{uv}_{n,n+2}|RG\rangle = \frac{2}{n+1}\frac{\sqrt{S^{(k-2)}_{1,n}S^{(k+2)}_{1,n+2}}}{S^{(k)}_{1,n}}\,. \tag{4.80}$$

Consideration of the remaining cases do not involve new ingredients and we will simply list the results.

- The state corresponding to $\phi^{ir}_{n,n}\phi^{uv}_{n,n-2}$ is:

$$-\frac{1}{\sqrt{n-2}}|\frac{n-1}{2},\frac{n-5}{2}\rangle|0\rangle|1,1\rangle + |\frac{n-1}{2},\frac{n-3}{2}\rangle|0\rangle|1,0\rangle \\ -\sqrt{\frac{n-1}{2}}|\frac{n-1}{2},\frac{n-1}{2}\rangle|0\rangle|1,-1\rangle\,.$$

The result of $G^{ir}_{-\frac{1}{2}}$ action on this state looks ugly:

$$|\frac{n-1}{2},\frac{n-3}{2}\rangle|1,-1\rangle|1,1\rangle + \frac{n-5}{2\sqrt{n-2}}|\frac{n-1}{2},\frac{n-5}{2}\rangle|1,0\rangle|1,1\rangle \\ -\sqrt{\frac{3n-9}{2n-4}}|\frac{n-1}{2},\frac{n-7}{2}\rangle|1,1\rangle|1,1\rangle - \sqrt{\frac{n-1}{2}}|\frac{n-1}{2},\frac{n-1}{2}\rangle|1,-1\rangle|1,0\rangle \\ -\frac{n-3}{2}|\frac{n-1}{2},\frac{n-3}{2}\rangle|1,0\rangle|1,0\rangle + \sqrt{n-2}|\frac{n-1}{2},\frac{n-5}{2}\rangle|1,1\rangle|1,0\rangle \\ +\Big(\frac{n-1}{2}\Big)^{\frac{3}{2}}|\frac{n-1}{2},\frac{n-1}{2}\rangle|1,0\rangle|1,-1\rangle - \frac{n-1}{2}|\frac{n-1}{2},\frac{n-3}{2}\rangle|1,1\rangle|1,-1\rangle$$

multiplied by an overall factor $\frac{6}{k-2}$. The corresponding one-point function simply is:

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}\phi^{uv}_{n,n-2}|RG\rangle = -\frac{2}{n-1}\frac{\sqrt{S^{(k-2)}_{1,n}S^{(k+2)}_{1,n-2}}}{S^{(k)}_{1,n}}\,. \tag{4.81}$$

- In the $\phi^{ir}_{n-2,n}\phi^{uv}_{n,n}$ case the corresponding state is

$$|\frac{n-3}{2},\frac{n-3}{2}\rangle|1,1\rangle|0\rangle\,. \tag{4.82}$$

Now we must act on this state by the operator $G^{uv}_{-1/2}$

$$G^{uv}_{-1/2}|\frac{n-3}{2},\frac{n-3}{2}\rangle|1,1\rangle|0\rangle = \Big(C_a(K_0^a+J_0^a)+D_a\widetilde{K}_0^a\Big)\,|\frac{n-3}{2},\frac{n-3}{2}\rangle|1,-a\rangle|0\rangle$$
$$= -\frac{3(n-1)}{k}|\frac{n-3}{2},\frac{n-3}{2}\rangle|1,1\rangle|1,0\rangle + \frac{6}{k}|\frac{n-3}{2},\frac{n-3}{2}\rangle|1,0\rangle|1,1\rangle$$
$$+\frac{6}{k}\sqrt{\frac{n-3}{2}}|\frac{n-3}{2},\frac{n-5}{2}\rangle|1,1\rangle|1,1\rangle\,.$$

The one point function:

$$\langle\phi^{ir}_{n-2,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = -\frac{2}{n-1}\frac{\sqrt{S^{(k-2)}_{1,n-2}S^{(k+2)}_{1,n}}}{S^{(k)}_{1,n}}\,. \tag{4.83}$$

- The state corresponding to the field $\phi^{ir}_{n+2,n}\phi^{uv}_{n,n}$ is

$$-\frac{1}{\sqrt{n}}|\frac{n+1}{2}\frac{n-3}{2}\rangle|1,1\rangle|0\rangle + |\frac{n+1}{2},\frac{n-1}{2}\rangle|1,0\rangle|0\rangle$$
$$-\sqrt{\frac{n+1}{2}}|\frac{n+1}{2},\frac{n+1}{2}\rangle|1,-1\rangle|0\rangle\,. \tag{4.84}$$

Acting by $G^{uv}_{-1/2}$ on this state we get

$$\frac{n-1}{2\sqrt{n}}|\frac{n+1}{2},\frac{n-3}{2}\rangle|1,1\rangle|1,0\rangle + \sqrt{\frac{n+1}{2}}(\frac{n-1}{2})|\frac{n+1}{2},\frac{n+1}{2}\rangle|1,-1\rangle|1,0\rangle$$
$$-\sqrt{\frac{3n-3}{2n}}|\frac{n+1}{2},\frac{n-5}{2}\rangle|1,1\rangle|1,1\rangle + \frac{n-1}{\sqrt{n}}|\frac{n+1}{2},\frac{n-3}{2}\rangle|1,0\rangle|1,1\rangle$$
$$-\frac{n-1}{2}|\frac{n+1}{2},\frac{n-1}{2}\rangle|1,0\rangle|1,0\rangle - \frac{n-1}{2}|\frac{n+1}{2},\frac{n-1}{2}\rangle|1,-1\rangle|1,1\rangle$$

multiplied by $\frac{6}{k}$. The result for one-point function:

$$\langle\phi^{ir}_{n+2,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = \frac{2}{n+1}\frac{\sqrt{S^{(k-2)}_{1,n+2}S^{(k+2)}_{1,n}}}{S^{(k)}_{1,n}}\,. \tag{4.85}$$

- Finally, the state corresponding to the field $G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}$ is

$$(C_a J_0^a + D_a K_0^a)(C_b(K_0^b + J_0^b) + D_b \widetilde{K}_0^b)|\frac{n-1}{2},\frac{n-1}{2}\rangle|1,-a\rangle|1,-b\rangle \tag{4.86}$$

which after some algebra becomes

$$\begin{aligned}
&\Big(\frac{n-1}{2}\Big)^2|\frac{n-1}{2},\frac{n-1}{2}\rangle|1,0\rangle|1,0\rangle - \sqrt{\frac{n-1}{2}}\frac{n-1}{2}|\frac{n-1}{2},\frac{n-3}{2}\rangle|1,0\rangle|1,1\rangle \\
&-\frac{n-1}{2}|\frac{n-1}{2},\frac{n-1}{2}\rangle|1,1\rangle|1,-1\rangle - \sqrt{\frac{n-1}{2}}\frac{n-3}{2}|\frac{n-1}{2},\frac{n-3}{2}\rangle|1,1\rangle|1,0\rangle \\
&\qquad\qquad + \sqrt{\frac{n-1}{2}}\sqrt{n-2}|\frac{n-1}{2},\frac{n-5}{2}\rangle|1,1\rangle|1,1\rangle
\end{aligned}$$

multiplied by $\frac{36}{k(k+2)}$. The respective one-point function is equal to

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = \frac{n^2-5}{n^2-1}\frac{\sqrt{S^{(k-2)}_{1,n}S^{(k+2)}_{1,n}}}{S^{(k)}_{1,n}}\,. \tag{4.87}$$

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