INSTITUTE OF APPLIED PROBLEMS OF PHYSICS NAS RA

Grigoryan Ashot Khachik

Some phenomena of quantum theory in condensed matter physics

Specialty: 01.04.07 - "Physics of Condensed State"

Thesis

presented for the degree of Doctor of Philosophy

in physical and mathematical sciences

Scientific supervisor: Doctor of phys. math. sci.

Professor A. A. Saharian

YEREVAN - 2018

Contents

INTRODUCTION

1	INDUCED FERMIONIC CHARGE AND CURRENT DENSITIES IN TWO-				
	DIN	MENSIONAL RINGS	13		
	1.1	Problem setup	14		
	1.2	Fermionic modes	17		
	1.3	Charge density	22		
	1.4	Azimuthal current	37		
	1.5	Charge and current densities in P- and T-symmetric models with applications			
		to graphene rings	47		
	1.6	Summary	51		
2	FERMIONIC CONDENSATE AND THE CASIMIR EFFECT IN A CON-				
	ICA	AL DEFECT SPACETIME	55		
	2.1	Geometry of the problem and the mode functions	56		
	2.2	Fermionic condensate	60		
	2.3	Energy-momentum tensor	71		
	2.4	Casimir force	83		
	2.5	Summary	85		
3	CASIMIR EFFECT FOR PARALLEL METALLIC PLATES IN CONICAL				
	DE	FECT SPACETIME	88		
	3.1	Mode functions for electromagnetic field	89		
	3.2	VEVs for the electric and magnetic field squared	92		
	3.3	An alternative representation of the electric field squared and the Casimir-Polder			
		potential	100		
	3.4	Energy-momentum tensor and the Casimir force	104		
	3.5	Summary	113		

3

CONCLUSIONS	115
LIST OF PAPERS PUBLISHED ON THE TOPIC OF THE THESIS	117
ACKNOWLEDGMENTS	118
REFERENCES	119

INTRODUCTION

The contemporary physics of elementary particles is based on the standard model. The latter is a quantum field theory of leptons and quarks interacting through gauge fields. The use of quantum field theory for the description of the electromagnetic, weak and strong interactions was extremely successful and still there are no experimental facts that are not explained by the standard model. At the present time, there is no universally recognized theory of quantum gravity and the gravitational field is not included in the standard model scheme. Related to that, the influence of the gravity on matter quantum fields is studied within the framework of semiclassical theory where the gravity is treated as a classical curved background. Within this approach a number of interesting effects were obtained, including the vacuum polarization and particle creation by black holes and at the early stages of the Universe expansion (for reviews see [1]-[4]). In particular, this type of quantum effects may solve the problem of singularities in general relativity.

Recent years have seen an explosion of applications of field theoretical methods in condensed matter physics (for various aspects of these applications see [5]-[22]). Being a theory of system with a large number of degrees of freedom, quantum field theory provides powerful tools for the investigations of different condensed matter systems. They include electronic subsystems in metals and in graphene like two-dimensional structures, critical phenomena and phase transitions, Bose-Einstein condensation, superfluidity and superconductivity, various types of integrable models, quantum Hall effect, various kinds of spin states (spin-liquid states, chiral spin states, anyons), topological defects, topological insulators and topological phases, bulk-toboundary correspondence.

Field theories for the number of spatial dimensions D other than 3 have attracted a great deal of attention. For D > 3 this was mainly motivated by the importance of the Kaluza-Klein and braneworld type models, as frameworks for the unification of the fundamental physical interactions. Extra dimensions are an inherent feature in string theories and in supergravity. There has been a growing interest in recent years in models formulated on backgrounds with the number of spatial dimensions D < 3. Aside from their role as simplified models in particle physics, field theories in lower dimensions serve as effective theories describing the long-wavelength properties of a number of condensed matter systems. Examples for the latter are high-temperature superconductors, d-density-wave states, Weyl semimetals, graphene (and graphene related materials) and topological insulators. For these systems, the long-wavelength dynamics of excitations is formulated in terms of the Dirac-like theory living in (2+1)-dimensional spacetime where the role of the velocity of light is played by the Fermi velocity. In topological insulators, 2Dmassless fermionic excitations appear as edge states on the surface of a 3D topological insulator. (2+1)-dimensional models also appear as high temperature limits of 4-dimensional field theories.

Among interesting features in (2+1)-dimensional models are flavour symmetry breaking, parity violation, fractionalization of quantum numbers, the possibility of the excitations with fractional statistics. Important new possibilities appear in gauge theories. In particular, the topologically gauge invariant terms in the action provide masses for the gauge fields. This leads to a natural infrared cutoff in the theory and to the solution for the infrared problem without changing the physics in the ultraviolet range [23]. A possible mechanism for the generation of gauge invariant topological mass terms is provided by quantum corrections [24]. The corresponding theories provide a natural framework for the investigation of the quantum Hall effect. In models with fermions coupled to the Chern-Simons gauge field, there are states with nonzero magnetic field and with the energy lower than the lowest energy state in the absence of the magnetic field [25]. As a consequence of this, the Lorentz invariance is spontaneously broken [26]-[31]. Among the most interesting topics in the studies of (2+1)-dimensional theories is the parity and chiral symmetry-breaking. In particular, it has been shown that a background magnetic field can serve as a catalyst for the dynamical symmetry breaking [32, 33],[34]-[39]. In addition, the background gauge fields give rise to the polarization of the ground state for quantum fields with the generation of various types of quantum numbers [24], [40]-[44]. In particular, charge and current densities are induced [45]-[53]. The persistent currents of this kind in toroidal carbon nanotubes, threaded by a magnetic flux, were investigated in [54]-[58]. Note that the persistent currents in normal metal rings have been recently measured in [59, 60]. The measurements have been done for both a single ring and arrays of rings. The corresponding results agree well with calculations based on a model of non-interacting electrons.

In a number of field theoretical models, including the ones describing the condensed matter systems at large length scales, additional boundary conditions are imposed on the field operator. These conditions can have different physical origins. For example, in graphene nanotubes and nanoloops, because of the compactification of one or two spatial dimensions, the Dirac equation is supplemented by quasiperiodicity conditions along compact dimensions with phases depending on the wrapping direction (chirality of the nanotube). Another type of graphene made structures in which additional boundary conditions are imposed on the field wavefunctions are graphene nanoribbons, geometrically terminated single layers of graphite (see, for instance, [61, 62]). The edge effects play a crucial role in electronic properties of nanoribbons. In particular, depending on the boundary conditions, a nonzero band gap may be generated. An important new thing is the possibility for the appearance of dispersive edge states.

Among the most interesting physical consequences originating from the spatial confinement of a quantum field is the Casimir effect [63]-[67]. The boundary conditions modify the spectrum of zero-point fluctuations and, as a consequence of that, the vacuum expectation values (VEVs) of physical observables are shifted. The physical quantities, most popular in the investigations of the Casimir effect, are the vacuum energy and stresses. By using these quantities, the forces acting on the constraining boundaries can be evaluated. These forces are presently under active experimental investigations. For charged fields, among the most important characteristics of the ground state are the expectation values of the charge and current densities. Similarly to the vacuum energy and stresses, the VEVs of these quantities are influenced by the change of spatial topology or by the presence of boundaries. The vacuum currents in spaces with nontrivial topology and with quasiperiodic boundary conditions on the field operator along compact dimensions have been investigated in [68, 69] for the flat background geometry and in [70] and [71] for locally de Sitter and anti-de Sitter backgrounds. For the special case D = 2, the general results were applied to cylindrical and toroidal graphene nanotubes, within the framework of the effective Dirac theory. The influence of additional boundaries on the vacuum currents along compact dimensions has been discussed in [72, 73] and [74, 75] for locally Minkowski and anti-de Sitter backgrounds. The finite temperature effects on the fermionic condensate and current densities in models with an arbitrary number of toroidally compact spatial dimensions are discussed in [76].

The properties of the ground state in both field-theoretical and condensed matter systems, in addition to boundaries, are sensitive also to the topology of the background geometry. Among the sources of the nontrivial topology are the various kinds of topological defects, formed as a result of symmetry breaking phase transitions [77]-[80]. Depending on the topology of the vacuum manifold they can be planar, linear and point-like defects. Among the well-known examples of planar defects are the domain walls appearing in both the condensed matter and cosmological contexts. Examples of linear defects in condensed matter physics are the disclinations in crystals. Topologically stable linear defects may have been created by phase transitions in the early Universe [77]. These defects are called cosmic strings and they are candidates for the generation of a variety of interesting physical effects, including gravitational lensing, anisotropies in the cosmic microwave background radiation, the generation of gravitational waves, high-energy cosmic rays, and gamma ray bursts. In [81] it has been demonstrated that the defects in condensed matter physics and in field theories can be dealt with the same geometric approach. In the simplest theoretical model describing the infinite straight linear defect the spacetime is locally flat outside the defect core. The only effect is the generation of angle deficit in the plane perpendicular to the defect.

Among the examples of condensed matter systems where topological defects may be formed by phase transitions are the superfluid He³, superfluid He⁴, superconductors and liquid crystals [80, 82, 83]. In particular, nematic liquid crystals consist of rod-like molecules which are oriented randomly above the phase transition temperature. As temperature decreases they tend to align and topological defects are expected to form in such a symmetry breaking transition. They include strings (disclination lines) and monopoles. In the case of strings, the rods point away from the core of the string. The laboratory experiments allowing to investigate defect formation in the course of phase transitions have been discussed in [80, 82]. With these studies, one is able to clarify the nature of cosmological phase transitions. In particular, because the superfluid He³ exhibits many similarities with our Universe, the superfluid condensate may be considered as a quantum vacuum with various types of quasiparticles and topological defects. Consequently, the condensate provides a test system for the experimental investigation of a number of problems in cosmology that are otherwise difficult or even impossible to investigate experimentally [80, 84]. The laboratory simulation of cosmic string formation in the early Universe by using superfluid He³ has been discussed in [85, 86, 87]. In string theories, one of the mechanisms for the formation of linear topological defects is the brane collisions and annihilations. In [88] laboratory experiments were reported simulating brane annihilation using the coherent phase boundary between the two phases of superfluid He³. By creating a brane-antibrane pair in superfluid He³ and subsequently annihilating it, defects were indeed created. This confirms the concept of defect formation after brane annihilation in the early Universe.

In quantum field theory the corresponding non-trivial topology induces non-zero VEVs for physical observables. In this context, the VEVs of the energy-momentum tensor have been evaluated for scalar, fermionic, and electromagnetic fields [89]-[106]. The analysis of the Casimir effect in the cosmic string spacetime have been developed for scalar [107], fermionic [108, 109], and electromagnetic fields [110, 111] for the geometry of a coaxial cylindrical boundary. The Casimir force for massless scalar fields subject to Dirichlet and Neumann boundary conditions in the setting of the conical piston has been recently investigated in [112]. The Casimir densities for a scalar field induced by a flat boundary perpendicular to the string have been considered in [113]. The combined effects of the topology, induced by a conical defect, and coaxial boundaries on the vacuum currents have been studied in [109, 114]. The finite temperature charge and current densities in the geometry of a (3 + 1)-dimensional conical defect with magnetic flux are studied in [115]. In [116] the finite temperature effects are investigated on the fermion condensate and on the expectation values of the charge and current densities for a massive fermionic field with nonzero chemical potential in a (2+1)-dimensional conical spacetime in the presence of a magnetic flux located at the cone apex.

The aim of the present thesis is the investigation of the effects from boundaries and nontrivial topology on the quantum properties of the ground state for fermionic and electromagnetic fields. We have studied:

• VEVs of the charge and current densities for a charged fermionic field in two-dimensional planar rings, threaded by an external magnetic flux,

- Fermionic condensate, the VEV of the energy-momentum tensor for a fermionic field and the forces acting on the plates perpendicular to a linear conical defect,
- VEVs of the electric and magnetic field squared, of the energy-momentum tensor for the electromagnetic field, the Casimir and Casimir-Polder forces in the geometry of parallel conducting plates perpendicular to a conical defect.

Scientific novelty

Closed analytical expressions are obtained for the charge and current densities of a fermionic field in two dimensional rings with bag boundary conditions on the edges. The edge-induced parts are extracted explicitly and their dependence on the geometrical characteristics and on the magnetic flux. Formulas are derived for the fermion condensate and for the VEV of the energy-momentum tensor of a massive fermionic field in (3+1)-dimensional spacetime with a linear conical defect and in the presence of parallel plates perpendicular to the defect axis. The Casimir forces acting on the boundaries are investigated and it has been shown that they are attractive. The combined effects of a linear topological defect and conducting plates on the quantum properties of the electromagnetic vacuum are studied. The expressions for the VEVs of the electric and magnetic field squared, for the energy-momentum tensor, for the Casimir and Casimir-Polder forces are derived. The features of these forces are investigated in dependence of the problem parameters.

The results obtained in the thesis can be applied for the investigations of the effects from boundaries on the charge and currend densities induced by a magnetic flux in systems with the long-wavelength excitations described by the Dirac model, as well as to the systems of condensed matter physics with linear topological defects. The methods for the evaluation of the VEVs of charge and currend densities in planar rings may be applied to the investigation of these quantities in two-dimensional conical structures, in particular, to graphene nanocones.

Basic theses to be defended

1. The magnetic flux threading a two-dimensional circular planar ring induces the charge and azimuthal current densities in the ground state of a quantum fermionic field. They are odd periodic continuous functions of the magnetic flux with the period equal to the flux quantum.

- 2. As functions of the energy gap, the charge and current densities exhibit quite different features for two inequivalent representations of the Clifford algebra. On the outer edge of the ring the current density is equal to the charge density whereas on the inner edge they have opposite signs. For a fixed values of the other parameters, both the charge and current densities decrease by the modulus with decreasing outer radius.
- 3. In the geometry of two parallel plates perpendicular to the conical defect the topological parts in the fermionic condensate and the ground state energy-momentum tensor vanish on the plates as a consequence of the cancellation between the boundary-free and boundary-induced parts. The fermionic condensate is positive near the defect and negative at large distances from the defect. Both the boundary-free and boundary-induced contributions to the energy density are negative.
- 4. The fermionic Casimir pressures on the plates vanish at the points where the defect crosses the plates and is negative at other points. The corresponding forces are attractive.
- 5. In the geometry of parallel conducting plates perpendicular to a conical defect, the expectation value of the electric field squared is negative near the defect and positive near the plates. The mean value of the magnetic field squared is negative everywhere. The boundary-induced part in the expectation value of the energy-momentum tensor for the electromagnetic field is nonzero in the region between the plates only and it dooes not depend on the coordinate along the defect axis. The corresponding energy density is positive near the defect and negative at large distances.
- 6. The electromagnetic Casimir pressures on the plates depend on the distance from the defect and this dependence is not monotonic. The corresponding forces are always attractive. The Casimir-Polder forces acting on a polarziable microparticle are repulsive with respect to the defect and attractive with respect to the closer plate.

The structure of the thesis is the following. In Chapter 1 we investigate the expectation

value of the fermionic charge and current densities induced by a magnetic flux in a spatial region of (2+1)-dimensional spacetime bounded by two concentric circles. On the circles, bag boundary conditions are imposed. We assume that the flux is located inside the inner boundary and, consequently, its effect on the vacuum properties is of the Aharonov-Bohm type. In Section 1.1 we specify the bulk and boundary geometries and the boundary conditions imposed on the fermionic field in the problem under consideration. In Section 1.2, the complete set of positive-and negative-energy mode functions is determined in the geometry of a finite width ring. These mode functions are used in Section 1.3 for the evaluation of the VEV of the charge density. Two equivalent representations are provided with the explicitly separated boundary contributions. The VEV of the azimuthal current density is investigated in Section 1.4. In Section 1.5, based on the results from previous sections, the induced charge and current densities are discussed in parity and time-reversal symmetric models and applications are given for graphene rings. The main results of the Chapter 1 are summarized in Section 1.6.

In Chapter 2 we are interested in the influence of the nontrivial topology due to the conical defect on the fermionic Casimir densities and the Casimir force in the geometry of two parallel plates with the MIT bag boundary condition. In order to have an exactly solvable problem, we will consider the idealized model for the defect with flat spacetime everywhere except on the defect where the curvature tensor has delta-type singularity. In this simplified model the exterior spacetime has a conical structure with the planar angle deficit related to the linear mass density. In Section 2.1 we specify the bulk and boundary geometries and the boundary conditions imposed on the fermion field. The complete set of the positive- and negative-energy wavefunctions is presented. By making use of these modes, in Section 2.2 we evaluate the fermionic condensate in the region between the plates. Various asymptotic limits of the general expression are discussed. Similar considerations for the expectation value of the energy-momentum tensor are presented in section 2.3. The Casimir forces acting on the plates are investigated in section 2.4. The main results of the chapter are summarized in Section 2.5.

In Chapter 3 we evaluate the expectation values of the electric and magnetic field squared and the energy-momentum tensor of the electromagnetic field for two parallel conducting plates perpendicular to the conical defect axis. These quantities are among the most important local characteristics of the electromagnetic vacuum. Though the corresponding operators are local, due to the global nature of the vacuum state, they contain an important information about the topology of the background spacetieme. In addition, the VEV of the energy-momentum tensor plays an important role in modelling a self-consistent dynamics involving the gravitational field. In Section 3.1 we consider the mode functions for the electric and magnetic fields in the region between two conducting plates. In Section 3.2, these mode functions are used for the evaluation of the expectation values of the electric and magnetic fields squared. By making use of the Abel-Plana summation formula, the latter are decomposed as the sum of boundary-free, single plate-induced, and second plate-induced parts. The Casimir-Polder force on a polarizable particle is discussed in Section 3.3. The expectation value of the energy-momentum tensor and the Casimir forces acting on the plates are investigated in Section 3.4. Finally, in section 3.5 the main results are summarized.

Chapter 1: INDUCED FERMIONIC CHARGE AND CURRENT DENSITIES IN TWO-DIMENSIONAL RINGS

In this chapter, for a massive quantum fermionic field, we investigate the vacuum expectation values (VEVs) of the charge and current densities induced by an external magnetic flux in a two-dimensional circular ring. Both the irreducible representations of the Clifford algebra are considered. On the ring edges the bag (infinite mass) boundary conditions are imposed for the field operator. This leads to the Casimir type effect on the vacuum characteristics. The radial current vanishes. The charge and the azimuthal current are decomposed into the boundary-free and boundary-induced contributions. Both these contributions are odd periodic functions of the magnetic flux with the period equal to the flux quantum. An important feature that distinguishes the VEVs of the charge and current densities from the VEV of the energy density is their finiteness on the ring edges. The current density is equal to the charge density for the outer edge and has the opposite sign on the inner edge. The VEVs are peaked near the inner edge and, as functions of the field mass, exhibit quite different features for two inequivalent representations of the Clifford algebra. We show that, unlike the VEVs in the boundary-free geometry, the vacuum charge and the current in the ring are continuous functions of the magnetic flux and vanish for half-odd integer values of the flux in units of the flux quantum. Combining the results for two irreducible representations, we also investigate the induced charge and current in parity and time-reversal symmetric models. The corresponding results are applied to graphene rings with the electronic subsystem described in terms of the effective Dirac theory with the energy gap. If the energy gaps for two valleys of the graphene hexagonal lattice are the same, the charge densities corresponding to the separate valleys cancel each other, whereas the azimuthal current is doubled.

1.1 Problem setup

Dirac-like fermions play an important role in modern condensed matter physics. In a large number of systems, including graphene, d-wave superconductors, topological insulators and Weyl semimetals, the properties of the electronic subsystems is well described by the Dirac equation (for a short review see [20]). Important materials science and practical implications are based on the understanding of Dirac particles in two and three spatial imensions. Although the microscopic physics that gives rise to the massless Dirac fermions in each of the examples is different, the low-energy properties are governed by the same Dirac field theory.

In a background geometry described by the metric tensor $g_{\mu\nu}$ and in the presence of an external electromagnetic field with the vector potential A_{μ} the Dirac equation for a quantum fermion field $\psi(x)$ is presented as

$$(i\gamma^{\mu}D_{\mu} - sm)\psi(x) = 0, \qquad (1.1)$$

where $D_{\mu} = \partial_{\mu} + \Gamma_{\mu} + ieA_{\mu}$ is the gauge extended covariant derivative, Γ_{μ} is the spin connection and e is the charge of the field quanta. The Dirac matrices γ^{μ} obey the Clifford algebra $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$. As the background geometry, we consider (2+1)-dimensional flat spacetime described in polar coordinates $x^{\mu} = (t, r, \phi)$ with the metric tensor $g_{\mu\nu} = \text{diag}(1, -1, -r^2)$. It is known that in odd number of spacetime dimensions the Clifford algebra has two inequivalent irreducible representations (with 2×2 Dirac matrices in (2+1) dimensions). Firstly we will discuss the case of a fermionic field realizing the irreducible representation of the Clifford algebra. The parameter s in Eq. (1.1), with the values s = +1 and s = -1, corresponds to two different representations (for more details see below). With these representations, the mass term violates both the parity (P-) and time-reversal (T-) invariances. We will discuss the charge and current densities in parity and time-reversal symmetric models in section 1.5. In the long wavelength description of the graphene, s labels two Dirac cones corresponding to \mathbf{K}_{+} and \mathbf{K}_{-} valleys of the hexagonal lattice.

We assume that the field is confined in the spatial region bounded by two concentric circles which have radii a and b, a < b (two-dimensional ring, see Fig. 1). On the edges of this region



Figure 1: Two-dimensional ring threaded by a magnetic flux.

the field operator obeys the MIT bag boundary conditions

$$(1 + in_{\mu}\gamma^{\mu})\psi(x) = 0, \ r = a, b, \tag{1.2}$$

with n_{μ} being the outward pointing unit vector normal to the boundaries. In the region $a \leq r \leq b$ one has $n_{\mu} = n_u \delta^1_{\mu}$ for the boundary at r = u with

$$n_a = -1, \ n_b = 1. \tag{1.3}$$

As a consequence of the boundary conditions (1.2), we get $n_{\mu}\bar{\psi}\gamma^{\mu}\psi = 0$ and the normal component of the fermionic current vanishes on the edges. Here and in what follows, $\bar{\psi} = \psi^{\dagger}\gamma^{0}$ is the Dirac adjoint and the dagger denotes Hermitian conjugation. This shows that the boundaries are impenetrable for the fermionic field. The boundary condition of the type (1.2) was used in the bag model of hadrons for the confinement of quarks [117, 118]. The analog boundary condition in graphene physics is referred as the infinite mass boundary condition. It has been employed in (2+1)-dimensional Dirac theory for the breaking of time-reversal symmetry without magnetic fields [119]. Mainly motivated by the graphene physics, more general conditions for the Dirac equation, ensuring the absence of current normal to the boundary, have been discussed in [61, 62],[120]-[124]. In (2+1)-dimensions, the most general energy-independent boundary condition contains four parameters.

For the further consideration we need to specify the representation for the Dirac matrices. In Cartesian coordinates, we take $\gamma^{(0)} = \sigma_3$, $\gamma^{(l)} = i\sigma_l$, l = 1, 2, where σ_1 , σ_2 , σ_3 are Pauli matrices. The Dirac matrices γ^{μ} in cylindrical coordinates are obtained in the standard way, using the corresponding tetrad fields $e^{\mu}_{(k)}$, k = 0, 1, 2, satisfying the relation $e^{\mu}_{(k)}e^{\nu}_{(l)}\eta^{kl} = g^{\mu\nu}$, with η^{kl} being the Minkowski spacetime metric tensor in Cartesian coordinates. In terms of the tetrad fields one has $\gamma^{\mu} = e^{\mu}_{(k)}\gamma^{(k)}$. For the tetrads we use the representation

$$e_{(0)}^{\mu} = (1,0,0), \ e_{(1)}^{\mu} = (0,\cos\phi, -\sin\phi/r), \ e_{(2)}^{\mu} = (0,\sin\phi,\cos\phi/r).$$
 (1.4)

With this representation one gets $\gamma^0 = \sigma_3$ and

$$\gamma^{1} = i \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}, \quad \gamma^{2} = \frac{1}{r} \begin{pmatrix} 0 & e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix}.$$
 (1.5)

The corresponding spin connection vanishes, $\Gamma_{\mu} = 0$. For the vector potential we will consider a configuration corresponding to the presence of a magnetic flux located in the region r < a. In the region under consideration, $a \leq r \leq b$, for the covariant components in the coordinates (t, r, ϕ) we have

$$A_{\mu} = (0, 0, A_2). \tag{1.6}$$

Note that the physical azimuthal component for the vector potential is given by $A_{\phi} = -A_2/r$ and for the magnetic flux threading the ring we have $\Phi = -2\pi A_2$. Though the magnetic field strength for (1.6) vanishes, the magnetic flux enclosed by the ring gives rise to Aharonov-Bohmlike effects on physical observable, particularly for the VEVs. Note that the distribution of the magnetic flux in the region r < a can be arbitrary. Since the boundary r = a is impenetrable for the fermionic field, the effect of the gauge field is purely topological and depends on the total flux only. From this point of view, the inner boundary can be regarded as a simplified model for a finite radius magnetic flux with the reflecting wall.

The zero-point fluctuations of the fermionic field $\psi(x)$ in the region $a \leq r \leq b$ are influenced by the magnetic flux threading the ring. As a consequence, the VEVs of physical quantities depend on the flux. This gives rise to a number of interesting physical phenomena, such as parity anomalies, formation of fermionic condensate and generation of quantum numbers. In this chapter we are interested in the VEV of the fermionic current $j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$. It appears as the source in the semiclassical Maxwell equations and, therefore, it plays an important role in selfconsistent dynamics involving the electromagnetic field. Let $S_{ik}^{(1)}(x, x') = \langle 0 | [\psi_i(x), \bar{\psi}_k(x')] | 0 \rangle$ be the fermion two-point function, where *i* and *k* are spinor indices and $|0\rangle$ denotes the vacuum state (ground state in condensed matter systems). The VEV of the current density is expressed in terms of this function as

$$\langle j^{\mu}(x)\rangle \equiv \langle 0|j^{\mu}(x)|0\rangle = -\frac{e}{2}\mathrm{Tr}(\gamma^{\mu}S^{(1)}(x,x)), \qquad (1.7)$$

with the trace over spinor indices understood. The expression in the right-hand side can be presented in the form of the sum over a complete set of positive- and negative-energy fermionic modes $\{\psi_{\sigma}^{(+)}(x), \psi_{\sigma}^{(-)}(x)\}$, specified by a set of quantum numbers σ . The mode functions $\psi_{\sigma}^{(\kappa)}(x), \kappa = \pm$, obey the Dirac equation (1.1) and the boundary conditions (1.2). Expanding the field operator in terms of $\psi_{\sigma}^{(\kappa)}(x)$ and using the commutation relations for the annihilation and creation operators, for the VEV of the current density the following mode sum is obtained:

$$\langle j^{\mu} \rangle = -\frac{e}{2} \sum_{\sigma} \sum_{\kappa = -, +} \kappa \bar{\psi}_{\sigma}^{(\kappa)}(x) \gamma^{\mu} \psi_{\sigma}^{(\kappa)}(x).$$
(1.8)

We need to find the complete set of modes and to use this mode sum for the evaluation of the vacuum charge and current densities.

From the perspective of the renormalization of the VEVs, the important point here is that, owing to the flatness of the background spacetime and the zero field tensor for the external electromagnetic field in the region under consideration, for points outside the boundaries the structure of divergences is the same as for the (2+1)-dimensional boundary-free Minkowski spacetime in the absence of the magnetic flux. Consequently, the renormalization is reduced to the subtraction from the VEVs of the corresponding Minkowskian quantity. In problems of the quantum field theory with boundaries, one of the most efficient ways to extract from the VEVs the boundary-free part in a regularization independent way is based on the application of the Abel-Plana-type summation formulae to the corresponding mode sums (for the applications of the Abel-Plana formula and its generalizations in the theory of the Casimir effect see [63]-[67],[125]).

1.2 Fermionic modes

For the evaluation of the VEV in accordance with Eq. (1.8) we need to have the mode functions $\psi_{\sigma}^{(\pm)}(x)$ obeying the boundary conditions (1.2). Decomposing the spinor $\psi_{\sigma}^{(\kappa)}(x)$, $\kappa = \pm$, into

upper and lower components, φ_+ and φ_- , respectively, from Eq. (1.1) one gets the equations

$$(\partial_0 \pm ism) \,\varphi_{\pm} \pm i e^{\mp i\phi} \left[\partial_1 \mp i(\partial_2 + i\alpha)/r\right] \varphi_{\mp} = 0, \tag{1.9}$$

with the notation

$$\alpha = eA_2 = -\Phi/\Phi_0,\tag{1.10}$$

where $\Phi_0 = 2\pi/e$ is the elementary flux or the flux quantum. From Eq. (1.9) we find the second-order differential equations for the separate components:

$$\left[\partial_0^2 - \partial_1^2 - \frac{1}{r}\partial_1 - \frac{1}{r}(\partial_2 + i\alpha)^2 + m^2\right]\varphi_{\pm} = 0.$$
 (1.11)

Note that this equation is the same for both components.

Let us present the solutions of these equations in the form

$$\varphi_{\pm} = \chi_{\pm}(r)e^{-\kappa i E t + i(j_{\pm} - 1/2)\phi},\tag{1.12}$$

where E > 0 and $j_{\pm} = \ldots, -3/2, -1/2, 1/2, 3/2, \ldots$ For the functions $\chi_{\pm}(r)$ one gets the equations

$$(-\kappa E \pm sm) \,\chi_{\pm} e^{i(j_{\pm} \pm 1 - j_{\mp})\phi} \pm \left(\partial_1 \pm \frac{j_{\pm} + \alpha - 1/2}{r}\right) \chi_{\mp} = 0. \tag{1.13}$$

and

$$\left[\partial_1^2 + \frac{1}{r}\partial_1 + \gamma^2 - \frac{1}{r^2}\left(j_{\pm} + \alpha - 1/2\right)^2\right]\chi_{\pm}(r) = 0, \qquad (1.14)$$

where $\gamma = \sqrt{E^2 - m^2}$. From Eq. (1.13) it follows that $j_- = j_+ + 1$.

With this choice and denoting $j_+ = j$, we take the solution for the function with the upper sign as

$$\chi_{+}(r) = Z_{\beta_j}(\gamma r) = c_1 J_{\beta_j}(\gamma r) + c_2 Y_{\beta_j}(\gamma r), \qquad (1.15)$$

where $J_{\nu}(x)$ and $Y_{\nu}(x)$ are the Bessel and Neumann functions and for the order we have used the notation

$$\beta_j = |j + \alpha| - \epsilon_j/2. \tag{1.16}$$

Here, $\epsilon_j = 1$ for $j > -\alpha$, $\epsilon_j = -1$ for $j \leq -\alpha$. Note that $\epsilon_j \beta_j = j + \alpha - 1/2$.

With the upper component of the spinor from (1.15), the lower component is found from Eq. (1.13) with the lower sign:

$$\chi_{-}(r) = -\frac{1}{\kappa E + sm} \left(\partial_{1} - \frac{j_{+} + \alpha - 1/2}{r} \right) \chi_{+}(r).$$
 (1.17)

By using the well-known properties of the cylindrical functions one gets

$$\chi_{-}(r) = \frac{\epsilon_{j}\gamma}{\kappa E + sm} Z_{\beta_{j} + \epsilon_{j}}(\gamma r).$$
(1.18)

The ratio of the coefficients in the linear combination (1.15) is determined from the boundary condition (1.2) at r = a. Substituting $\psi(x) = (\varphi_+, \varphi_-)^T$, where T stands for the transponation, and using (1.12), the boundary condition is reduced to

$$e^{-iEt+i(j-1/2)\phi} \begin{pmatrix} Z_{\beta_j}(\gamma a) + \frac{\epsilon_j \gamma}{E+sm} Z_{\beta_j+\epsilon_j}(\gamma a) \\ \left[Z_{\beta_j}(\gamma a) + \frac{\epsilon_j \gamma}{\kappa E+sm} Z_{\beta_j+\epsilon_j}(\gamma a) \right] e^{i\phi} \end{pmatrix} = 0.$$
(1.19)

From here it follows that

$$Z_{\beta_j}(\gamma a) + \frac{\epsilon_j \gamma}{E + sm} Z_{\beta_j + \epsilon_j}(\gamma a) = 0.$$
(1.20)

By taking into account (1.15), for the ratio of the coefficients in the linear combination one obtains

$$\frac{c_2}{c_1} = -\frac{J_{\beta_j}(\gamma a) + \frac{\epsilon_n \gamma}{E+sm} J_{\beta_j + \epsilon_j}(\gamma a)}{Y_{\beta_j}(\gamma a) + \frac{\epsilon_n \gamma}{E+sm} Y_{\beta_j + \epsilon_j}(\gamma a)}.$$
(1.21)

Using the recurrence relations for the Bessel and Neumann function we can see that

$$f_{\beta_j + \epsilon_j}(\gamma x) = -\epsilon_j f'_{\beta_j}(\gamma x) + \frac{\beta_j}{\gamma x} f_{\beta_j}(\gamma x), \qquad (1.22)$$

with f = J, Y. As a consequence, the ratio is written in an equivalent form

$$\frac{c_2}{c_1} = -\frac{J_{\beta_j}^{(a)}(\gamma a)}{Y_{\beta_i}^{(a)}(\gamma a)},$$
(1.23)

where and in what follows, for the Bessel and Neumann functions, we use the notation defined as

$$f_{\beta_{j}}^{(u)}(x) = x f_{\beta_{j}}'(x) + [n_{u}(\kappa \sqrt{x^{2} + m_{u}^{2}} + sm_{u}) - \epsilon_{j}\beta_{j}]f_{\beta_{j}}(x)$$

$$= n_{u}(\kappa \sqrt{x^{2} + m_{u}^{2}} + sm_{u})f_{\beta_{j}}(x) - \epsilon_{j}x f_{\beta_{j} + \epsilon_{j}}(x),$$
(1.24)

with u = a, b, and $m_u = mu$. Hence, the mode functions are written in the form

$$\psi_{\sigma}^{(\kappa)}(x) = C_{\kappa} e^{-\kappa i E t + i j \phi} \begin{pmatrix} g_{\beta_j, \beta_j} \left(\gamma a, \gamma r\right) e^{-i\phi/2} \\ \frac{\epsilon_j \gamma e^{i\phi/2}}{\kappa E + sm} g_{\beta_j, \beta_j + \epsilon_j} \left(\gamma a, \gamma r\right) \end{pmatrix}, \qquad (1.25)$$

where

$$g_{\beta_j,\mu}(x,y) = Y_{\beta_j}^{(a)}(x) J_{\mu}(y) - J_{\beta_j}^{(a)}(x) Y_{\mu}(y).$$
(1.26)

It can be checked that the modes (1.25) are eigenfunctions of the total angular momentum operator $\hat{J} = -i(\partial_2 + ieA) + \sigma_3/2$, for the eigenvalues $j + \alpha$:

$$\hat{J}\psi_{\sigma}^{(\kappa)}(x) = (j+\alpha)\psi_{\sigma}^{(\kappa)}(x).$$
(1.27)

Here, the part $\sigma_3/2$ corresponds to the pseudospin.

Similarly, from the boundary condition (1.2) at r = b we get

$$\frac{c_2}{c_1} = -\frac{J_{\beta_j}^{(b)}(\gamma b)}{Y_{\beta_i}^{(b)}(\gamma b)}.$$
(1.28)

Combining this with Eq. (1.23), we conclude that the eigenvalues for γ in the region $a \leq r \leq b$ are the roots of the equation

$$C_{\beta_j}(\eta, \gamma a) \equiv J_{\beta_j}^{(a)}(\gamma a) Y_{\beta_j}^{(b)}(\gamma b) - J_{\beta_j}^{(b)}(\gamma b) Y_{\beta_j}^{(a)}(\gamma a) = 0, \qquad (1.29)$$

with $\eta = b/a$. The positive solutions of this equation with respect to γa will be denoted by z_l , $l = 1, 2, ..., z_l < z_{l+1}$. For the eigenvalues of γ one has $\gamma = \gamma_l = z_l/a$. In this way, the mode functions are specified by the quantum numbers $\sigma = (l, j)$. Note that, the roots z_l depend on the value of j as well. In order to simplify the expressions below we do not write this dependence explicitly. For a given j, the equations (1.29) for the eigenvalues γ_l of the positiveand negative-energy modes differ by the change of the energy sign, $E \rightarrow -E$ (through κ in (1.24)). For the energy one has $E = \sqrt{\gamma_l^2 + m^2}$. In the case of a massless field, as we see, the finite size effects induce a gap in the energy spectrum. The gap can be controlled by the geometrical characteristics of the model. Note that the gap generated by the finite size effects plays a crucial role in graphene made nanoribbons. In the problem at hand the size of the energy gap is determined by the minimal value of γ_l . For example, in the case $\alpha = 1/3$, j = 1/2for the first root one has $z_1 \approx 1.59$ for b/a = 2 and $z_1 \approx 0.81$ for b/a = 3. The root increases as b/a decreases and j increases.

The function $C_{\beta_j}(\eta, x)$ in (1.29) can also be written in terms of the Hankel functions $H_{\beta_j}^{(1,2)}(x)$. We can obtain that expression if, we search the general solution for $\chi_+(r)$ in the form of

$$\chi_{+}(r) = \sum_{l=1,2} c_{l}^{(\pm)} H_{\beta_{j}}^{(l)}(\gamma r) , \qquad (1.30)$$

then we only need to perform the same steps as we did for the previous case, giving

$$C_{\nu}(\eta, x) = (i/2) \left[H_{\beta_j}^{(2a)}(x) H_{\beta_j}^{(1b)}(\eta x) - H_{\beta_j}^{(1a)}(x) H_{\beta_j}^{(2b)}(\eta x) \right],$$
(1.31)

with the notations defined as Eq. (1.24). With this representation, we can see that for a massless field the equation $C_{\beta_j}(\eta, \gamma a) = 0$ is reduced to the one given in [126] for graphene rings described by the Dirac model with the infinite mass boundary condition on the edges. Note that in order to obtain an analytical approximation of the spectrum, in [126] the asymptotic form of the Hankel functions for large arguments was used. This approximation is valid for rings with the radius much larger than the width. As it will be shown below, the use of the generalized Abel-Plana formula allows us to obtain closed analytic expressions for the VEVs in the general case of geometrical characteristics of the ring.

For the complete specification of the mode functions (1.25) it remains to determine the coefficient C_{κ} . The latter is found from the orthonormalization condition

$$\int_{a}^{b} dr \int_{0}^{2\pi} d\phi \, r \psi_{\sigma}^{(\kappa)\dagger}(x) \psi_{\sigma'}^{(\kappa)}(x) = \delta_{jj'} \, \delta_{ll'}. \tag{1.32}$$

Substituting the modes, for the product $\psi_{\sigma}^{(\kappa)\dagger}(x)\psi_{\sigma}^{(\kappa)}(x)$ we get

$$\psi_{\sigma}^{(\kappa)\dagger}(x)\psi_{\sigma}^{(\kappa)}(x) = |C_{\kappa}|^2 \left(g_{\beta_j,\beta_j}^2\left(\gamma a,\gamma r\right) + \frac{\gamma^2}{\left(\kappa E + sm\right)^2}g_{\beta_j,\beta_j+\epsilon_j}^2\left(\gamma a,\gamma r\right)\right).$$
(1.33)

Therefore the orthonormalization condition is reduced to

$$2\pi |C_{\kappa}|^2 \int_a^b dr \, r \left(g_{\beta_j,\beta_j}^2 \left(\gamma a, \gamma r \right) + \frac{\gamma^2}{\left(\kappa E + sm \right)^2} g_{\beta_j,\beta_j+\epsilon_j}^2 \left(\gamma a, \gamma r \right) \right) = 1. \tag{1.34}$$

Here we use the integration formula valid for any cylinder function $Z_{\nu}(\gamma r)$:

$$\int_{a}^{b} dr r Z_{\nu}^{2}(\gamma r) = \frac{r^{2}}{2} \left[Z_{\nu}^{\prime 2}(\gamma r) + \left(1 - \nu^{2} / r^{2} \gamma^{2}\right) Z_{\nu}^{2}(\gamma r) \right]_{a}^{b}, \qquad (1.35)$$

(see [127]). After long but straightforward calculations, with the help of this formula one finds

$$|C_{\kappa}|^{2} = \frac{\pi z_{l}}{16a^{2}} \frac{E + \kappa sm}{E} T^{ab}_{\beta_{j}}(\eta, z_{l}).$$
(1.36)

Here, we have introduced the notation

$$T_{\beta_j}^{ab}(\eta, z) = z \left[D_b J_{\beta_j}^{(a)2}(z) / J_{\beta_j}^{(b)2}(\eta z) - D_a \right]^{-1}, \qquad (1.37)$$

with

$$D_u = u^2 \frac{E + \kappa sm}{E} \left[E \left(E + \kappa n_u \frac{\epsilon_j \beta_j}{u} \right) + \kappa n_u \frac{E - \kappa sm}{2u} \right].$$
(1.38)

As it has been already mentioned, the eigenvalue equations (1.29) for the positive- and negative-energy modes are obtained from each other by the change of the energy sign. Redefining the azimuthal quantum number in accordance with $j \rightarrow -j$, the negative-energy modes can also be written in the form

$$\psi_{\sigma}^{(-)} = C'_{-}e^{iEt-ij\phi} \begin{pmatrix} \frac{\epsilon_{j}^{-}\gamma e^{-i\phi/2}}{E+sm}g_{\beta_{j}^{-},\beta_{j}^{-}+\epsilon_{\lambda_{n}^{-}}}(\gamma a,\gamma r)\\ g_{\beta_{j}^{-},\beta_{j}^{-}}(\gamma a,\gamma r)e^{i\phi/2} \end{pmatrix},$$
(1.39)

where $\beta_j^- = |j - \alpha| - \epsilon_j^-/2$, $\epsilon_j^- = 1$ for $j > \alpha$ and $\epsilon_j^- = -1$ for $j \leq \alpha$. In the definition (1.26) of the function $g_{\beta_j^-,\mu}(x,y)$ the notations $f_{\beta_j^-}^{(u)}(x)$ are defined as Eq. (1.24) with $\kappa = 1$ (as in the case of the positive-frequency functions). With the modes (1.39), the eigenvalues for γ are solutions of the equation

$$J_{\beta_{j}^{-}}^{(a)}(\gamma a) Y_{\beta_{j}^{-}}^{(b)}(\gamma b) - Y_{\beta_{j}^{-}}^{(a)}(\gamma a) J_{\beta_{j}^{-}}^{(b)}(\gamma b) = 0.$$
(1.40)

Since now the functions $f_{\nu}^{(u)}(x)$ are the same for the positive- and negative-energy modes, Eq. (1.40) differs from the corresponding equation (1.29) for the positive-energy modes just by the sign of α , $\alpha \to -\alpha$. One can see that the normalization constant C'_{-} is given by the expression in the right-hand side of Eq. (1.36) with $\kappa = 1$ and with $z_l = \gamma a$ being the roots of Eq. (1.40). Of course, in the evaluation of the VEVs we can use both types of the modes.

1.3 Charge density

We start our consideration of the VEVs with the charge density. Substituting the mode functions (1.25) into the mode-sum formula (1.8) we get

$$\langle j^0 \rangle = -\frac{\pi e}{32a^2} \sum_j \sum_{l=1}^{\infty} \sum_{\kappa=\pm} \kappa T^{ab}_{\beta_j}(\eta, z_l) h_0(z_l), \qquad (1.41)$$

where \sum_{j} stands for the summation over $j = \pm 1/2, \pm 3/2, \ldots$, and

$$h_0(z) = \frac{z}{E} \left[(E + \kappa sm) g_{\beta_j,\beta_j}^2(z, zr/a) + (E - \kappa sm) g_{\beta_j,\beta_j+\epsilon_j}^2(z, zr/a) \right], \qquad (1.42)$$

with $E = \sqrt{z^2/a^2 + m^2}$. In Eq. (1.41), the eigenvalues z_l are given implicitly, as roots of Eq. (1.29), and this representation is not convenient for the further evaluation of the VEV.

Another disadvantage of the representation (1.41) is that the separate terms in the series for large values of the quantum numbers are highly oscillating.

These difficulties are overcome by using for the summation over l the Abel-Plana-type formula

$$\sum_{l=1}^{\infty} h(z_l) T_{\beta_j}^{ab}(\eta, z_l) = \frac{4}{\pi^2} \int_0^\infty dz \, \frac{h(z)}{J_{\beta_j}^{(a)2}(z) + Y_{\beta_j}^{(a)2}(z)} \\ - \frac{2}{\pi} \operatorname{Res}_{z=0} \left[\frac{h(z) H_{\beta_j}^{(1b)}(\eta z)}{C_{\beta_j}(\eta, z) H_{\beta_j}^{(1a)}(z)} \right] - \frac{1}{\pi} \int_0^\infty dz \, \sum_{p=+,-} \Omega_{a\beta_j}^{(p)}(z, \eta z) h(z e^{pi\frac{\pi}{2}}),$$
(1.43)

where

$$\Omega_{a\beta_j}^{(p)}(z,\eta z) = \frac{K_{\beta_j}^{(bp)}(\eta z) / K_{\beta_j}^{(ap)}(z)}{K_{\beta_j}^{(ap)}(z) I_{\beta_j}^{(bp)}(\eta z) - I_{\beta_j}^{(ap)}(z) K_{\beta_j}^{(bp)}(\eta z)}.$$
(1.44)

Here we have introduced the notation

$$f_{\beta_j}^{(up)}(x) = x f_{\beta_j}'(x) + \left\{ n_u \left[\kappa \sqrt{(x e^{p\pi i/2})^2 + m_u^2} + s m_u \right] - \epsilon_j \beta_j \right\} f_{\beta_j}(x) , \qquad (1.45)$$

with f = I, K for the modified Bessel functions $I_{\beta_j}(x)$ and $K_{\beta_j}(x)$. The formula (1.43) is valid for a function h(z) analytic in the complex half-plane Re z > 0, z = x + iy, and obeying the condition $|h(z)| < \varepsilon(x)e^{c|y|}$, where $c < 2(\eta - 1)$ and $\varepsilon(x)/x \to 0$ for $x \to +\infty$. On the imaginary axis the function h(z) may have branch points. The summation formula (1.43) is obtained from the generalized Abel-Plana formula [125] (see also [107]). Note that for the square root in Eq. (1.45) we have

$$\sqrt{\left(ze^{p\pi i/2}\right)^2 + m_u^2} = \begin{cases} \sqrt{m_u^2 - z^2}, & z < m_u \\ pi\sqrt{z^2 - m_u^2}, & z > m_u \end{cases}$$
(1.46)

Particularly, it is seen that $f_{\beta_j}^{(u+)}(z) = f_{\beta_j}^{(u-)}(z)$ for $z < m_u$. From here it follows that $\Omega_{a\beta_j}^{(-)}(z,\eta z) = \Omega_{a\beta_j}^{(+)}(z,\eta z)$ for z < ma.

For the charge density the function h(z) in the summation formula (1.43) is specified by Eq. (1.42). This function has branch points $z = \pm ima$ on the imaginary axis. For z < ma one has the relation $h_0(ze^{-\pi i/2}) = -h_0(ze^{\pi i/2})$ and, hence, in the last integral of Eq. (1.43) the part over the region [0, ma] becomes zero. For z > ma we find

$$h_{0}(ze^{pi\frac{\pi}{2}}) = \frac{4z}{\pi^{2}} \left[\left(\frac{\kappa sm}{\sqrt{z^{2}/a^{2} - m^{2}}} + pi \right) G^{(ap)2}_{\beta_{j},\beta_{j}}(z, zr/a) + \left(\frac{\kappa sm}{\sqrt{z^{2}/a^{2} - m^{2}}} - pi \right) G^{(ap)2}_{\beta_{j},\beta_{j} + \epsilon_{j}}(z, zr/a) \right],$$
(1.47)

with the function

$$G_{\beta_{j},\mu}^{(ap)}(x,y) = K_{\beta_{j}}^{(ap)}(x) I_{\mu}(y) - (-1)^{\mu-\beta_{j}} I_{\beta_{j}}^{(ap)}(x) K_{\mu}(y) .$$
(1.48)

Substituting Eq. (1.47) into Eq. (1.43) and then putting this into Eq. (1.41), we see that

$$\begin{split} \langle j^0 \rangle &= \langle j^0 \rangle_a + \frac{e}{8\pi^2 a^2} \sum_j \sum_{\kappa=\pm 1} \\ &\times \int_{ma}^{\infty} dz \, z \sum_{p=+,-} \kappa p \Omega_{a\beta_j}^{(p)}(z,\eta z) \left[\left(\frac{s\kappa pm}{\sqrt{z^2/a^2 - m^2}} + i \right) G_{\beta_j,\beta_j}^{(ap)2}(z,zr/a) \right. \tag{1.49} \\ &+ \left(\frac{s\kappa pm}{\sqrt{z^2/a^2 - m^2}} - i \right) G_{\beta_j,\beta_j+\epsilon_j}^{(ap)2}(z,zr/a) \right], \end{split}$$

with

$$\langle j^0 \rangle_a = -\frac{e}{8\pi a^2} \sum_j \sum_{\kappa=\pm 1} \int_0^\infty dz \, \frac{\kappa h_0(z)}{J_{\beta_j}^{(a)2}(z) + Y_{\beta_j}^{(a)2}(z)}.$$
 (1.50)

It is seen that in the right-hand side of Eq. (1.49) κ and p enter into the form of the product κp . From here it follows

$$\sum_{\kappa=\pm 1} \sum_{p=+,-} \kappa p f(\kappa p) = \sum_{p=+,-} p \left[f(p) - f(-p) \right] = 2 \sum_{p=+,-} p f(p).$$
(1.51)

Making use of this relation the expression Eq. (1.49) can be cast into the following form

$$\langle j^{0} \rangle = \langle j^{0} \rangle_{a} + \frac{e}{2\pi^{2}} \sum_{j} \int_{m}^{\infty} dx \, x \left\{ \frac{sm}{\sqrt{x^{2} - m^{2}}} \right.$$

$$\times \operatorname{Re} \left[\Omega_{a\beta_{j}}(ax, bx) \left(G_{\beta_{j},\beta_{j}}^{(a)2}(ax, rx) + G_{\beta_{j},\beta_{j}+\epsilon_{j}}^{(a)2}(ax, rx) \right) \right]$$

$$-\operatorname{Im} \left[\Omega_{a\beta_{j}}(ax, bx) \left(G_{\beta_{j},\beta_{j}}^{(a)2}(ax, rx) - G_{\beta_{j},\beta_{j}+\epsilon_{j}}^{(a)2}(ax, rx) \right) \right] \right\},$$

$$(1.52)$$

The new notations are defined as

$$\Omega_{a\beta_j}(ax, bx) = \frac{K_{\beta_j}^{(b)}(bx)/K_{\beta_j}^{(a)}(ax)}{K_{\beta_j}^{(a)}(ax) I_{\beta_j}^{(b)}(bx) - I_{\beta_j}^{(a)}(ax)K_{\beta_j}^{(b)}(bx)},$$
(1.53)

and

$$G_{\beta_{j},\mu}^{(u)}(x,y) = K_{\beta_{j}}^{(u)}(x) I_{\mu}(y) - (-1)^{\mu-\beta_{j}} I_{\beta_{j}}^{(u)}(x) K_{\mu}(y), \qquad (1.54)$$

with u = a, b (the function $G^{(b)}_{\beta_j,\mu}(x, y)$ is used below). For the modified Bessel functions now we use the notations

$$f_{\beta_{j}}^{(u)}(z) = z f_{\beta_{j}}'(z) + \left[n_{u} \left(i \sqrt{z^{2} - m_{u}^{2}} + s m_{u} \right) - \epsilon_{j} \beta_{j} \right] f_{\beta_{j}}(z)$$

$$= \delta_{f} z f_{\beta_{j} + \epsilon_{j}}(z) + n_{u} (i \sqrt{z^{2} - m_{u}^{2}} + s m_{u}) f_{\beta_{j}}(z) ,$$
(1.55)

where $f = I, K, \delta_I = 1$ and $\delta_K = -1$.

Let us present the parameter α from Eq. (1.10), related to the magnetic flux threading the ring, in the form

$$\alpha = N + \alpha_0, \ |\alpha_0| \leqslant 1/2, \tag{1.56}$$

where N is an integer. Redefining the summation variable j in accordance with $j + N \rightarrow j$ we see that the charge density does not depend on the integer part N. Then, separating the summations over negative and positive values of j, making the replacement $j \rightarrow -j$ in the part with negative j and introducing a new summation variable n = j - 1/2, the charge density is presented as

$$\langle j^{0} \rangle = \langle j^{0} \rangle_{a} + \frac{e}{2\pi^{2}} \sum_{n=0}^{\infty} \sum_{p=\pm} p \int_{m}^{\infty} dx \, x \left\{ \frac{sm}{\sqrt{x^{2} - m^{2}}} \right.$$

$$\times \operatorname{Re} \left[\Omega_{an_{p}}(ax, bx) \left(G_{n_{p}, n_{p}}^{(a)2}(ax, rx) + G_{n_{p}, n_{p}+1}^{(a)2}(ax, rx) \right) \right]$$

$$+ \operatorname{Im} \left[\Omega_{an_{p}}(ax, bx) \left(G_{n_{p}, n_{p}+1}^{(a)2}(ax, rx) - G_{n_{p}, n_{p}}^{(a)2}(ax, rx) \right) \right] \right\},$$

$$(1.57)$$

with

$$n_p = n + p\alpha_0, \tag{1.58}$$

and now the notation (2.51) in the definitions (1.53) and (1.54) for $\Omega_{an_p}(ax, bx)$ and $G_{n_p,\mu}^{(a)}(ax, rx)$ is specified to

$$f_{n_p}^{(u)}(z) = \delta_f z f_{n_p+1}(z) + n_u (sm_u + i\sqrt{z^2 - m_u^2}) f_{n_p}(z), \qquad (1.59)$$

for f = I, K. The representation (1.57) explicitly shows that the last term is an odd function of the fractional part α_0 . The property that the VEVs do not depend on the integer part of the flux, in units of the flux quantum, is a general feature in the Aharonov-Bohm effect and is a consequence of the fact that the flux enters through the phase of the wavefunction.

The last term in Eq. (1.57) vanishes in the limit $b \to \infty$ (for large values of b the function $\Omega_{an_p}(ax, bx)$ falls off as e^{-2bx}). From here it follows that the part (1.50) is the charge density in the region $r \ge a$ for the geometry of a single boundary at r = a. In order to extract from that part the boundary-induced effects we further transform the expression (1.50), with $h_0(z)$ form (1.42), by using the relation

$$\frac{g_{\beta_j,\lambda}^2(z,y)}{J_{\beta_j}^{(a)2}(z) + Y_{\beta_j}^{(a)2}(z)} = J_{\lambda}^2(y) - \frac{1}{2} \sum_{l=1,2} \frac{J_{\beta_j}^{(a)}(z)}{H_{\beta_j}^{(la)}(z)} H_{\lambda}^{(l)2}(y),$$
(1.60)

with $\lambda = \beta_j, \beta_j + \epsilon_j$. This results in

$$\langle j^{0} \rangle_{a} = \langle j^{0} \rangle_{0} + \frac{e}{16\pi a^{2}} \sum_{j} \sum_{\kappa=\pm 1} \kappa \int_{0}^{\infty} dz \, z \sum_{l=1,2} \frac{J_{\beta_{j}}^{(a)}(z)}{H_{\beta_{j}}^{(la)}(z)} \\ \times \left[\left(1 + \frac{s\kappa m}{E} \right) H_{\beta_{j}}^{(l)2}(zr/a) + \left(1 - \frac{s\kappa m}{E} \right) H_{\beta_{j}+\epsilon^{j}}^{(l)2}(zr/a) \right],$$

$$(1.61)$$

with

$$\langle j^0 \rangle_0 = -\frac{e}{8\pi} \sum_j \sum_{\kappa=\pm 1} \kappa \int_0^\infty dx \, x \\ \times \left[\left(1 + \frac{s\kappa m}{\sqrt{x^2 + m^2}} \right) J_{\beta_j}^2(xr) + \left(1 - \frac{s\kappa m}{\sqrt{x^2 + m^2}} \right) J_{\beta_j + \epsilon_j}^2(xr) \right].$$
(1.62)

In the part with the Hankel functions, we rotate the contour of the integration over z by the angles $\pi/2$ and $-\pi/2$ for the terms with l = 1 and l = 2, respectively. Introducing the modified Bessel functions in accordance with

$$J_{\nu}^{(a)}(e^{\pm\pi i/2}z) = e^{\pm\pi i\nu/2}I_{\nu}^{(a\pm)}(z),$$

$$H_{\nu}^{(1a)}(e^{\pi i/2}z) = \frac{2}{\pi i}e^{-\pi i\nu/2}K_{\nu}^{(a+)}(z),$$

$$H_{\nu}^{(2a)}(e^{-\pi i/2}z) = -\frac{2}{\pi i}e^{\pi i\nu/2}K_{\nu}^{(a-)}(z),$$
(1.63)

we get

$$\int_{0}^{i\infty} dz \, z \frac{J_{\beta_{j}}^{(a)}(z)}{H_{\beta_{j}}^{(a,1)}(z)} \left[\left(1 + \frac{s\kappa m}{E} \right) H_{\beta_{j}}^{(1)2}(zr/a) + \left(1 - \frac{s\kappa m}{E} \right) H_{\beta_{j}+\epsilon^{j}}^{(1)2}(zr/a) \right] \\
= \frac{2}{\pi i} \int_{0}^{\infty} dz \, z \frac{I_{\beta_{j}}^{(a+)}(z)}{K_{\beta_{j}}^{(a+)}(z)} \left[\left(\frac{is\kappa m}{\sqrt{z^{2}/a^{2} - m^{2}}} - 1 \right) K_{\beta_{j}}^{2}(zr/a) + \left(1 + \frac{is\kappa m}{\sqrt{z^{2}/a^{2} - m^{2}}} \right) K_{\beta_{j}+\epsilon^{j}}^{2}(zr/a) \right],$$
(1.64)

for the first part and

$$\int_{0}^{-i\infty} dz \, z \frac{J_{\beta_{j}}^{(a)}(z)}{H_{\beta_{j}}^{(a,2)}(z)} \left[\left(1 + \frac{s\kappa m}{E} \right) H_{\beta_{j}}^{(2)2}(zr/a) + \left(1 - \frac{s\kappa m}{E} \right) H_{\beta_{j}+\epsilon^{j}}^{(2)2}(zr/a) \right] \\
= \frac{2}{\pi i} \int_{0}^{\infty} dz \, z \frac{I_{\beta_{j}}^{(a-)}(z)}{K_{\beta_{j}}^{(a-)}(z)} \left[\left(\frac{is\kappa m}{\sqrt{z^{2}/a^{2} - m^{2}}} - 1 \right) K_{\beta_{j}}^{2}(zr/a) + \left(\frac{is\kappa m}{\sqrt{z^{2}/a^{2} - m^{2}}} - 1 \right) K_{\beta_{j}+\epsilon^{j}}^{2}(zr/a) \right],$$
(1.65)

for the second part. By taking into account that $f_{\beta_j}^{(a+)}(z) = f_{\beta_j}^{(a-)}(z)$ for z < ma, we see that the parts in the integrals over the segments $[0, im_a]$ and $[0, -im_a]$ cancel each other. As a result, for the part of the charge density $\langle j^0 \rangle_a$ one obtains the representation

$$\langle j^0 \rangle_a = \langle j^0 \rangle_0 + \langle j^0 \rangle_a^{(b)}, \tag{1.66}$$

where

$$\langle j^{0} \rangle_{a}^{(b)} = \frac{e}{2\pi^{2}} \sum_{j} \int_{m}^{\infty} dx \, x \left\{ sm \frac{K_{\beta_{j}}^{2}(rx) + K_{\beta_{j}+\epsilon_{j}}^{2}(rx)}{\sqrt{x^{2} - m^{2}}} \operatorname{Re}\left[\frac{I_{\beta_{j}}^{(a)}(ax)}{K_{\beta_{j}}^{(a)}(ax)}\right] + \left[K_{\beta_{j}+\epsilon_{j}}^{2}(rx) - K_{\beta_{j}}^{2}(rx)\right] \operatorname{Im}\left[\frac{I_{\beta_{j}}^{(a)}(ax)}{K_{\beta_{j}}^{(a)}(ax)}\right] \right\},$$

$$(1.67)$$

with the notations (2.51). For the representation s = 1, this expression for a single boundaryinduced part coincides with the one given in Ref. [109]-[115], [128] (the sign difference is related to that in [109]-[115], [128], for the evaluation of the VEVs, the analog of the negative-energy mode functions (1.39) for the geometry with a single boundary was used with α replaced by $-\alpha$; therefore, in comparing the formulas here with the results of [109]-[115], [128], the replacements $\alpha \to -\alpha$ and $\alpha_0 \to -\alpha_0$ should be made).

In order to give a physical interpretation of the separate terms in Eq. (1.66) we consider the limit $a \rightarrow 0$. Note that the radius of the magnetic flux should also be taken to zero. By using the asymptotic expressions for the modified Bessel functions for small arguments, in the range $j + \alpha > 0$ one gets

$$\frac{I_{\beta_j}^{(a)}(ax)}{K_{\beta_j}^{(a)}(ax)} = -\frac{xI_{|j+\alpha|+1/2}(ax) - (i\sqrt{x^2 - m^2} + sm)I_{|j+\alpha|-1/2}(ax)}{xK_{|j+\alpha|+1/2}(ax) + (i\sqrt{x^2 - m^2} + sm)K_{|j+\alpha|-1/2}(ax)} \\
\approx \frac{i\sqrt{x^2 - m^2} + sm}{x} \frac{2(ax/2)^{2(j+\alpha)}}{\Gamma^2(j+\alpha+1/2)}.$$
(1.68)

In a similar way, for $j + \alpha < 0$ we find

$$\frac{I_{\beta_{j}}^{(a)}(ax)}{K_{\beta_{j}}^{(a)}(ax)} = -\frac{xI_{|j+\alpha|-1/2}(ax) - (i\sqrt{x^{2}-m^{2}}+sm)I_{|j+\alpha|+1/2}(ax)}{xK_{|j+\alpha|-1/2}(ax) + (i\sqrt{x^{2}-m^{2}}+sm)K_{|j+\alpha|+1/2}(ax)} \\
\approx \frac{i\sqrt{x^{2}-m^{2}}-sm}{x}\frac{2(ax/2)^{2|j+\alpha|}}{\Gamma^{2}(|j+\alpha|+1/2)}.$$
(1.69)

Combining these two results, for $j + \alpha \neq 0$ one obtains

$$\frac{I_{\beta_j}^{(a)}(ax)}{K_{\beta_j}^{(a)}(ax)} \approx \frac{i\sqrt{x^2 - m^2} + \epsilon_j sm}{x} \frac{2(ax/2)^{2|j+\alpha|}}{\Gamma^2(|j+\alpha|+1/2)}.$$
(1.70)

This shows that the corresponding part in (1.67) vanishes as $a^{2|j+\alpha|}$. For half-odd integer values of α , that is for $j + \alpha = 0$, the exceptional case corresponds to the mode with $j = -\alpha$. For this mode $\beta_j = 1/2$, $\epsilon_j = -1$, and we get

$$\frac{I_{\beta_{j}}^{(a)}(ax)}{K_{\beta_{j}}^{(a)}(ax)} = -\frac{xI_{-1/2}(ax) - (i\sqrt{x^{2} - m^{2}} + sm)I_{1/2}(ax)}{xK_{-1/2}(ax) + (i\sqrt{x^{2} - m^{2}} + sm)K_{1/2}(ax)} \\
\approx -\frac{2}{\pi}\frac{x}{i\sqrt{x^{2} - m^{2}} + sm},$$
(1.71)

or, equivalently,

$$\frac{I_{\beta_j}^{(a)}(ax)}{K_{\beta_j}^{(a)}(ax)} \approx \frac{2}{\pi} \frac{i\sqrt{x^2 - m^2} - sm}{x}.$$
(1.72)

Note that for this special value of β_j , in Eq. (1.67) the coefficient of the term with the imaginary part of Eq. (1.72) vanishes.

Therefore, if α is not a half-odd integer we have

$$\lim_{a \to 0} \langle j^0 \rangle_a = \langle j^0 \rangle_0. \tag{1.73}$$

In this case, the part (1.67) in the VEV of the charge density is induced by the presence of the boundary at r = a, whereas $\langle j^0 \rangle_0$ gives the charge density in the boundary-free geometry with a point like magnetic flux at r = 0. For half-odd integer values of α , using (1.72), from Eq. (1.67) we get

$$\lim_{a \to 0} \langle j^0 \rangle_a = \langle j^0 \rangle_0 + \frac{2em^2}{\pi^3} \int_m^\infty dx \, \frac{K_{1/2}^2(rx)}{\sqrt{x^2 - m^2}} = \langle j^0 \rangle_0 + \frac{em}{\pi^2 r} \int_1^\infty dx \, \frac{1}{\sqrt{x^2 - 1}} \frac{1}{x} e^{-2mrx},$$
(1.74)

or

$$\lim_{a \to 0} \langle j^0 \rangle_a = \langle j^0 \rangle_0 + \frac{em}{\pi^2 r} \int_0^\infty dy \, \frac{e^{-2mr\sqrt{y^2 + 1}}}{y^2 + 1}.$$
(1.75)

For a massless field the last term in this expression vanishes and we come to the same interpretation of the separate terms in Eq. (1.66).

The boundary-induced contribution (1.67) does not depend on the integer part N in Eq. (1.56).

Redefining the summation variable in Eq. (1.67), this contribution is rewritten in the form

$$\langle j^{0} \rangle_{a}^{(b)} = \frac{e}{2\pi^{2}} \sum_{n=0}^{\infty} \sum_{p=-,+} p \int_{m}^{\infty} dx \, x$$

$$\times \left\{ sm \frac{K_{n_{p}}^{2}(rx) + K_{n_{p}+1}^{2}(rx)}{\sqrt{x^{2} - m^{2}}} \operatorname{Re}\left[\frac{I_{n_{p}}^{(a)}(ax)}{K_{n_{p}}^{(a)}(ax)}\right] + \left[K_{n_{p}+1}^{2}(rx) - K_{n_{p}}^{2}(rx)\right] \operatorname{Im}\left[\frac{I_{n_{p}}^{(a)}(ax)}{K_{n_{p}}^{(a)}(ax)}\right] \right\},$$
(1.76)

with n_p given by Eq. (1.58). This explicitly shows that the single boundary-induced charge density is an odd function of α_0 . Using Eq. (1.59), the ratio of the modified Bessel functions in Eq. (1.76) can be explicitly written as

$$\frac{I_{\nu}^{(u)}(z)}{K_{\nu}^{(u)}(z)} = \frac{I_{\nu}(z)}{K_{\nu}(z)} \frac{zI_{\nu+1}(z)/I_{\nu}(z) + n_u(sm_u + i\sqrt{z^2 - m_u^2})}{-zK_{\nu+1}(z)/K_{\nu}(z) + n_u(sm_u + i\sqrt{z^2 - m_u^2})}.$$
(1.77)

Now given the Wronskian relation $I_{\nu+1}(z) K_{\nu}(z) + K_{\nu+1}(z) I_{\nu}(z) = 1/z$, written in the form

$$z\frac{I_{\nu+1}(z)}{I_{\nu}(z)} + z\frac{K_{\nu+1}(z)}{K_{\nu}(z)} = \frac{1}{I_{\nu}(z)K_{\nu}(z)},$$
(1.78)

one can transform Eq. (1.77) as

$$\frac{I_{\nu}^{(u)}(z)}{K_{\nu}^{(u)}(z)} = \frac{1}{K_{\nu}^{2}(z)} \left[I_{\nu}(z) K_{\nu}(z) - \frac{zK_{\nu+1}(z)/K_{\nu}(z) - sn_{u}m_{u} + in_{u}\sqrt{z^{2} - m_{u}^{2}}}{(zK_{\nu+1}(z)/K_{\nu}(z) - sn_{u}m_{u})^{2} + z^{2} - m_{u}^{2}} \right].$$
 (1.79)

Hence, the real and imaginary parts of the ratio in (1.76) are explicitly read from

$$\frac{I_{n_p}^{(u)}(z)}{K_{n_p}^{(u)}(z)} = \frac{W_{n_p}^{(u)}(z) - in_u\sqrt{1 - m_u^2/z^2}}{z[K_{n_p+1}^2(z) + K_{n_p}^2(z)] - 2sn_um_uK_{n_p}(z)K_{n_p+1}(z)},$$
(1.80)

with u = a, b and with the function

$$W_{\nu}^{(u)}(z) = z \left[I_{\nu}(z) K_{\nu}(z) - I_{\nu+1}(z) K_{\nu+1}(z) \right] + n_u sm_u \left[I_{\nu+1}(z) K_{\nu}(z) - I_{\nu}(z) K_{\nu+1}(z) \right].$$
(1.81)

Note that for $z \ge m_u$ the denominator in Eq. (1.80) is positive. For a massless field and at large distances from the boundary, $r \gg a$, the dominant contribution to the boundary-induced part (1.76) comes from the term n = 0 and this part decays as $(a/r)^{3-2|\alpha_0|}$ with the sign $\operatorname{sgn}(\alpha_0)\langle j^0\rangle_a^{(\mathrm{b})}/e < 0$. For a massive field and for $r \gg a, m^{-1}$, the dominant contribution in Eq. (1.76) comes from the region near the lower limit of the integration and the boundaryinduced charge density is suppressed by the factor $e^{-2mr}/r^{3/2}$.

From these considerations it follows that, if $|\alpha_0| \neq 1/2$, the part $\langle j^0 \rangle_0$ can be interpreted as the charge density in boundary-free two-dimensional space with a special type of boundary condition on the magnetic flux line at r = 0. Namely, we impose the bag boundary condition at finite radius which is then taken to zero. Consequently, the part (1.76) is interpreted as the contribution induced in the region $a \leq r < \infty$ by the boundary r = a. The last term in Eq. (1.57) is the contribution in the charge density induced when we add the boundary at r = bto the geometry with a single boundary at r = a. Therefore, in this sense, this part can be termed as the second boundary-induced contribution.

The expression (1.62) for the boundary-free part admits of further simplification. The first terms in the brackets of the coefficients of the functions $J^2_{\beta_j}(xr)$ and $J^2_{\beta_j+\epsilon_j}(xr)$ are canceled for the contributions coming from the positive- and negative-energy modes. For the remaining part we get

$$\langle j^0 \rangle_0 = \frac{esm}{4\pi} \sum_j \int_0^\infty dx \, x \frac{J_{\beta_j + \epsilon_j}^2(xr) - J_{\beta_j}^2(xr)}{\sqrt{x^2 + m^2}}.$$
 (1.82)

This can be further simplified by the following procedure. Let us assume that $|\alpha| < 1/2$. Then one can write

$$\sum_{j} J_{\beta_{j}}^{2}(xr) = \sum_{j=1/2}^{+\infty} \left[J_{j-\alpha+1/2}^{2}(xr) + J_{j+\alpha-1/2}^{2}(xr) \right], \qquad (1.83)$$

and

$$\sum_{j} J_{\beta_{j}+\epsilon_{j}}^{2}(xr) = \sum_{j=1/2}^{\infty} \left[J_{j-\alpha+1/2}^{2}(xr) + J_{j+\alpha-1/2}^{2}(xr) \right] + J_{-\alpha}^{2}(xr) - J_{\alpha}^{2}(xr).$$
(1.84)

Combining these two results we get

$$\sum_{j} \left[J_{\beta_{j}|}^{2}(xr) - J_{\beta_{j}+\epsilon_{j}}^{2}(xr) \right] = J_{\alpha}^{2}(xr) - J_{-\alpha}^{2}(xr), \qquad (1.85)$$

and, hance, $\langle j^0 \rangle_0$ becomes

$$\langle j^0 \rangle_0 = \frac{esm}{4\pi} \int_0^\infty dx \, x \frac{J^2_{-\alpha_0}(xr) - J^2_{\alpha_0}(xr)}{\sqrt{x^2 + m^2}}.$$
 (1.86)

This expression with s = 1 was obtained in [45, 47]. After the rotation of the integration contour, it can also be presented in the form [45, 47]

$$\langle j^0 \rangle_0 = \frac{esm^2}{\pi^3} \sin(\pi\alpha_0) \int_1^\infty dx \frac{xK_{\alpha_0}^2(mrx)}{\sqrt{x^2 - 1}}.$$
 (1.87)

An equivalent expression for the boundary-free part is provided in [109]-[115], [128]:

$$\langle j^0 \rangle_0 = \frac{esm}{2\pi^2 r} \sin\left(\pi\alpha_0\right) \int_0^\infty dx \, \frac{\cosh(2\alpha_0 x)}{\cosh x} e^{-2mr\cosh x}.$$
 (1.88)

Similarly to the boundary-induced contributions, this part is an odd function of the parameter α_0 . For a massless field the boundary-free contribution in the charge density vanishes for $r \neq 0$. In the case of a massive field, at large distances, the charge density $\langle j^0 \rangle_0$ falls off as $e^{-2mr}/r^{3/2}$, whereas at the origin it diverges as 1/r. At large distances, the decaying factor for a massive field is the same as that for the boundary-induced contribution (1.76).

The charge density (1.87) for the boundary-free geometry with a point like magnetic flux corresponds to a special boundary condition on the fermion field at the location of the flux. Generally, the self-adjoint extension procedure for the Dirac Hamiltonian leads to a one-parameter family of boundary conditions [129]-[131]. The value of the parameter in the boundary condition is determined by the physical details of the magnetic field distribution inside a more realistic finite radius flux tube (see, e.g., [132]-[137] for models with finite radius magnetic flux).

Combining all the results given above, we conclude that the total charge density $\langle j^0 \rangle$ in the region $a \leq r \leq b$ is a periodic odd function of the magnetic flux threading the ring, with the period equal to the flux quantum. As a function of the parameter α (magnetic flux in units of the flux quantum), the charge density (1.88) in the boundary-free geometry is discontinuous at the half-odd integer values $\alpha = N + 1/2$:

$$\lim_{\alpha_0 \to \pm 1/2} \langle j^0 \rangle_0 = \pm \frac{esm}{2\pi^2 r} K_0(2mr).$$
(1.89)

In the geometry with a single boundary at r = a, we can see that for the boundary-induced contribution in the region $r \ge a$ we have $\lim_{\alpha_0 \to \pm 1/2} \langle j^0 \rangle_a^{(b)} = -\lim_{\alpha_0 \to \pm 1/2} \langle j^0 \rangle_0$. This means that, in this geometry, the total charge density vanishes for $\alpha = N + 1/2$, $\lim_{\alpha_0 \to \pm 1/2} \langle j^0 \rangle_a = 0$, and it is a continuous function of the magnetic flux everywhere. We can chack that, in the expression (1.57) for the total charge in the ring geometry the last term vanishes in the limits $\alpha_0 \to \pm 1/2$ (the real and imaginary parts in Eq. (1.57) vanish separately) and, hence, similarly to the single boundary part, the total charge density $\langle j^0 \rangle$ is a continuous function at the half-odd integer values of α . Having discussed general features of the charge density, for the further clarification of the dependence on the parameters of the model, let us consider numerical examples. In the left panel of Fig. 2, for the geometry of two boundaries at r = a, b, we have plotted the charge density as a function of the radial coordinate for a massless fermionic field and for $\alpha_0 = 1/4$. The numbers near the curves are the values of the ratio b/a. The dashed curve presents the charge density in the geometry of a single boundary at r = a, namely, the quantity $10^3 a^2 \langle j^0 \rangle_a^{(b)}/e$. In the right panel of Fig. 2, by the full curves, the charge density is plotted as a function of α_0 for fixed values ma = 0.1, b/a = 8, r/a = 2. The numbers near the curves are the values of the parameter s. The dashed curve is the corresponding charge density for a massless field with the same values of the ring. The ratio $\langle j^0 \rangle / e$ is negative near the inner edge and positive near the outer edge. In the geometry of a single boundary at r = a this ratio is negative for $\alpha_0 > 0$. We have already mentioned that the charge density vanishes for $|\alpha_0| = 1/2$, a feature seen from Fig. 2.

In Fig. 3 the charge density is displayed versus the ratio b/a for fixed values $\alpha_0 = 1/4$, r/a = 1.5. The numbers near the curves are the values of the parameter ma. The dashed lines correspond to the current density outside a single boundary of radius a. For the left and right panels s = 1 and s = -1, respectively. Note that the scaling factors for these panels are different. The charge density for the representation s = -1 is essentially larger. The general feature seen from Fig. 3 is that the presence of the outer edge leads to the decrease of the absolute value of the charge density. For large values of b/a, the approach of the charge density in the ring geometry to the corresponding quantity in the geometry with a single edge is quicker with increasing mass.

From figures 2 and 3 we see that the behavior of the charge density when the parameter ma increases from ma = 0 is essentially different for the representations s = 1 and s = -1. With the initial increase of ma, the modulus of the charge density decreases for the former case and increases for the latter one. Naturally, we expect that for $ma \gg 1$ the charge density will be suppressed for both cases. This is seen from Fig. 4. It presents the dependence of the charge density on the mass of the field for irreducible representations s = 1 (left panel) and s = -1



Figure 2: Charge density in the ring as a function of the radial coordinate for a massless field (left panel) and as a function of the parameter α_0 (right panel). The left panel is plotted for the magnetic flux parameter $\alpha_0 = 1/4$ and the numbers near the curves are the values of b/a. The dashed curve in that panel corresponds to the charge density outside a single boundary at r = a. The full curves in the right panel are plotted for ma = 0.1, b/a = 8, r/a = 2 and the numbers near the curves are the values of s. The dashed curve in the right panel corresponds to a massless field.



Figure 3: The dependence of the charge density on the ratio b/a for irreducible representations s = 1 (left panel) and s = -1 (right panel). The graphs are plotted for $\alpha_0 = 1/4$, r/a = 1.5 and the numbers near the curves are the values for ma. The horizontal dashed curves correspond to the charge density in the geometry of a single boundary at r = a.

(right panel). The graphs are plotted for $\alpha_0 = 1/4$, b/a = 8, r/a = 2. The dashed curves correspond to the charge density in the geometry of a single boundary at r = a. The dotted line in the right panel is the charge density for s = -1 in the boundary-free problem. The corresponding charge density for s = 1 differs by the sign. The suppression of the VEV with increasing ma is stronger in the case s = 1.



Figure 4: Charge density as a function of the field mass (in units of 1/a) for fields with s = 1 (left panel) and s = -1 (right panel). For the values of the parameters one has taken $\alpha_0 = 1/4$, b/a = 8, r/a = 2. The dashed curves present the charge density outside a single boundary at r = a. The dotted line corresponds to the charge density in the boundary-free geometry.

The representation (1.57) for the charge density in the ring is not symmetric with respect to the inner and outer edges. An alternative representation, with the extracted outer boundary part is obtained from Eq. (1.49) using the relation

$$\frac{I_{n_p}^{(a)}(ax)}{K_{n_p}^{(a)}(ax)}K_{\mu}^2(rx) = \frac{K_{n_p}^{(b)}(bx)}{I_{n_p}^{(b)}(bx)}I_{\mu}^2(rx) + \sum_{u=a,b}n_u\Omega_{un_p}(ax,bx)G_{n_p,\mu}^{(u)2}(ux,rx),$$
(1.90)

where

$$\Omega_{bn_p}(ax, bx) = \frac{I_{n_p}^{(a)}(ax)/I_{n_p}^{(b)}(bx)}{K_{n_p}^{(a)}(ax)I_{n_p}^{(b)}(bx) - I_{n_p}^{(a)}(ax)K_{n_p}^{(b)}(bx)}.$$
(1.91)

The expression for the charge density takes the form

$$\langle j^{0} \rangle = \langle j^{0} \rangle_{b} + \frac{e}{2\pi^{2}} \sum_{n=0}^{\infty} \sum_{p=\pm} p \int_{m}^{\infty} dx \, x \left\{ \frac{sm}{\sqrt{x^{2} - m^{2}}} \right. \\ \left. \times \operatorname{Re} \left[\Omega_{bn_{p}}(ax, bx) \left(G_{n_{p}, n_{p}}^{(b)2}(bx, rx) + G_{n_{p}, n_{p}+1}^{(b)2}(bx, rx) \right) \right] \right.$$

$$\left. + \operatorname{Im} \left[\Omega_{bn_{p}}(ax, bx) \left(G_{n_{p}, n_{p}+1}^{(b)2}(bx, rx) - G_{n_{p}, n_{p}}^{(b)2}(bx, rx) \right) \right] \right\}.$$

$$(1.92)$$

Here, the first term in the right-hand side is decomposed as

$$\langle j^0 \rangle_b = \langle j^0 \rangle_0 + \langle j^0 \rangle_b^{(b)}, \tag{1.93}$$

with

$$\langle j^{0} \rangle_{b}^{(b)} = \frac{e}{2\pi^{2}} \sum_{n=0}^{\infty} \sum_{p=\pm} p \int_{m}^{\infty} dx \, x \left\{ sm \frac{I_{n_{p}}^{2}(rx) + I_{n_{p}+1}^{2}(rx)}{\sqrt{x^{2} - m^{2}}} \operatorname{Re}\left[\frac{K_{n_{p}}^{(b)}(bx)}{I_{n_{p}}^{(b)}(bx)}\right] + \left[I_{n_{p}+1}^{2}(rx) - I_{n_{p}}^{2}(rx)\right] \operatorname{Im}\left[\frac{K_{n_{p}}^{(b)}(bx)}{I_{n_{p}}^{(b)}(bx)}\right] \right\},$$

$$(1.94)$$

and with the notations defined according to Eq. (1.59). For the ratio under the signs of the real and imaginary parts in Eq. (1.94) we can get an explicit expression:

$$\frac{K_{n_p}^{(u)}(z)}{I_{n_p}^{(u)}(z)} = \frac{-n_u z K_{n_p+1}(z) + s m_u K_{n_p}(z) + i \sqrt{z^2 - m_u^2} K_{n_p}(z)}{n_u z I_{n_p+1}(z) + s m_u I_{n_p}(z) + i \sqrt{z^2 - m_u^2} I_{n_p}(z)} \\
= \frac{z \left[K_{n_p}(z) I_{n_p}(z) - I_{n_p+1}(z) K_{n_p+1}(z) \right]}{z [I_{n_p+1}^2(z) + z I_{n_p}^2(z)] + 2n_u s m_u I_{n_p}(z) I_{n_p+1}(z)} \\
+ \frac{n_u s m_u \left[I_{n_p+1}(z) K_{n_p}(z) - I_{n_p}(z) K_{n_p+1}(z) \right] + i n_u \sqrt{1 - m_u^2/z^2}}{z [I_{n_p+1}^2(z) + z I_{n_p}^2(z)] + 2n_u s m_u I_{n_p}(z) I_{n_p+1}(z)} \tag{1.95}$$

which can be presented as

$$\frac{K_{n_p}^{(u)}(z)}{I_{n_p}^{(u)}(z)} = \frac{W_{n_p}^{(u)}(z) + in_u\sqrt{1 - m_u^2/z^2}}{z[I_{n_p+1}^2(z) + I_{n_p}^2(z)] + 2n_u sm_u I_{n_p}(z) I_{n_p+1}(z)}.$$
(1.96)

The denominator in this expression is positive for $z \ge m_u$. Relatively simple expressions for single boundary parts (1.76) and (1.94) are obtained for a massless field.

If $|\alpha_0| \neq 1/2$, in the limit $a \to 0$ one has

$$\Omega_{bn_p}(az, bz) \approx \frac{I_{n_p}^{(a)}(az)}{K_{n_p}^{(a)}(az) I_{n_p}^{(b)2}(bz)},$$
(1.97)

with the ratio of the modified Bessel functions given by Eq. (1.70). From here it follows that in this limit the last term in Eq. (1.92) vanishes. This means that the part (1.93) is the charge density in the region $0 \le r \le b$ for the geometry of a single boundary at r = b. The contribution (1.94) is induced by the latter. Note that, for $|\alpha_0| \ne 1/2$, this contribution is also obtained from Eq. (1.57) in the limit $a \rightarrow 0$. Therefore, we conclude that the last term in Eq. (1.92) is the contribution in the charge density induced by adding the boundary at r = a in the geometry with a single boundary at r = b (the second boundary-induced part). In the limit $r \rightarrow 0$ the dominant contribution in Eq. (1.94) for the single boundary contribution comes from the term with n = 0, $p = -\text{sgn}(\alpha_0)$, and the boundary-induced charge density behaves as $1/r^{2|\alpha_0|}$. For a massive field the boundary-free contribution diverges like 1/r and it dominates in the total VEV for $|\alpha_0| < 1/2$. All the separate terms in the representation (1.92) are discontinuous at half-odd integer values of α . However, as it has been already emphasized before, the total charge density is a continuous function of the magnetic flux everywhere.

Fig. 5 presents the charge density in the region $r \leq b$ for the geometry with a single boundary at r = b. For the corresponding magnetic flux we have taken $\alpha_0 = 1/4$. In the left panel the charge density is plotted versus the radial coordinate for a massless field. The right panel displays the boundary-induced part in the charge density as a function of the field mass in the cases s = 1 and s = -1 (numbers near the curves) and for r/a = 0.5. The dashed curve is for the charge density in the boundary-free geometry for s = 1.



Figure 5: The boundary-induced part in the charge density inside a single boundary with radius b as a function of the radial coordinate (left panel) and of the field mass (right panel) for $\alpha_0 = 1/4$. The left panel is plotted for a massless field. For the right panel r/b = 0.5 and the numbers near the curves specify the irreducible representations denoted by s. The dashed curve presents the charge density in the boundary-free geometry for representation s = 1.

As it has been mentioned, the last terms in the representations (1.57) and (1.92) are induced when we add the second edge to the geometry with a single boundary. These second boundaryinduced contributions are further simplified on the edges r = a and r = b respectively. The
corresponding expressions can be written in a combined form

$$\langle j^0 \rangle = \langle j^0 \rangle_u + \frac{e}{\pi^2} \sum_{n=0}^{\infty} \sum_{p=\pm} p \int_m^\infty dx \, \frac{x}{\sqrt{x^2 - m^2}} \operatorname{Re}\left[(sm + i\sqrt{x^2 - m^2})\Omega_{un_p}(ax, bx) \right], \quad (1.98)$$

on the edge r = u with u = a, b. The second term in the right-hand side presents the charge induced on the edge r = u by the other edge.

It is important to mention at this point that the VEV of the charge density is finite on the edges (note that for the evaluation of the corresponding limiting value of the single boundaryinduced parts $\langle j^0 \rangle_u^{(b)}$, u = a, b, we cannot directly put r = u in the representations (1.76) and (1.94) as the separate integrals in the summation over p logarithmically diverge in the upper limit). From the theory of the Casimir effect it is known that in quantum field theory with boundary conditions imposed on the field operator the VEVs of local physical observables, in general, diverge on the boundaries. For example, the latter is the case for the VEV of the energy-momentum tensor or for the fermion condensate in problems with fermionic fields. The appearance of surface divergences in this type of quantities is a consequence of the idealization replacing the physical interaction by the imposition of boundary conditions and indicates that a more realistic physical model should be employed. For instance, the microstructure of the boundary on small scales can introduce a physical cutoff needed to produce finite values for surface quantities.

1.4 Azimuthal current

Now we turn to the spatial components of the fermionic current density. First let us consider the radial current. The corresponding mode sum is written as

$$\langle j^1 \rangle = -\frac{e}{2} \sum_{j} \sum_{l=1}^{\infty} \sum_{\kappa=-,+}^{\infty} \kappa \psi_{\gamma_l n}^{(\kappa)\dagger}(x) \gamma^0 \gamma^1 \psi_{\gamma_l n}^{(\kappa)}(x), \qquad (1.99)$$

where for the product of the gamma matrices one has

$$\gamma^{0}\gamma^{1} = i \begin{pmatrix} 0 & e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix}.$$
 (1.100)

Making use of the mode functions (1.25) it is easy to see that the expression under the mode-sum (1.8) is identically zero:

$$\psi_{\gamma_l n}^{(\kappa)\dagger}(x)\gamma^0\gamma^1\psi_{\gamma_l n}^{(\kappa)}(x) = 0.$$
(1.101)

Hence, the radial component of the current density vanishes, $\langle j^1 \rangle = 0$, and the only nonzero component corresponds to the azimuthal current.

The mode-sum for the physical component $\langle j_{\phi} \rangle = r \langle j^2 \rangle$ of the azimuthal current can be written as

$$\langle j_{\phi} \rangle = -\frac{er}{2} \sum_{n=-\infty}^{+\infty} \sum_{l=1}^{\infty} \sum_{\kappa=-,+}^{\infty} \kappa \psi_{\gamma_l n}^{(\kappa)\dagger}(x) \gamma^0 \gamma^2 \psi_{\gamma_l n}^{(\kappa)}(x), \qquad (1.102)$$

where

$$\gamma^0 \gamma^2 = \frac{1}{r} \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}.$$
 (1.103)

Substituting the eigenfunctions for the positive- and negative-energy modes the corresponding current density is written in the form

$$\langle j_{\phi} \rangle = -\frac{\pi e}{16a^3} \sum_{j} \sum_{\kappa=\pm} \epsilon_j \sum_{l=1}^{\infty} T^{ab}_{\beta_j}(\eta, z_l) h_2(z_l).$$
(1.104)

with the function

$$h_2(z) = z^2 \frac{g_{\beta_j,\beta_j}(z, zr/a)}{\sqrt{z^2/a^2 + m^2}} g_{\beta_j,\beta_j + \epsilon_j}(z, zr/a).$$
(1.105)

As in the previous case, the terms $\kappa = -$ and $\kappa = +$ are the contributions from the negativeand positive-energy modes.

For the separation of the effects induced by the boundaries we apply to the series over l the summation formula (1.43) with $h(z) = h_2(z)$. For the function (1.105) one gets

$$h_{2}(xe^{p\pi i/2}) = -\frac{x^{2}}{\sqrt{(xe^{\beta\pi i/2})^{2}/a^{2} + m^{2}}}g_{\beta_{j},\beta_{j}}(xe^{p\pi i/2}, xe^{p\pi i/2}r/a)$$

$$\times g_{\beta_{j},\beta_{j}+\epsilon_{j}}(xe^{p\pi i/2}, xe^{p\pi i/2}r/a).$$
(1.106)

given that

$$g_{\beta_{j},\beta_{j}}(ze^{i\pi/2}, ze^{i\pi/2}r/a) = -\frac{2}{\pi}G^{(+)}_{\beta_{j},\beta_{j}}(z, zr/a)$$

$$= -\frac{2}{\pi}\left[K^{(a+)}_{\beta_{j}}(z)I_{\beta_{j}}(zr/a) - I^{(a+)}_{\beta_{j}}(z)K_{\beta_{j}}(zr/a)\right]$$
(1.107)

and that

$$g_{\beta_{j},\beta_{j}+\epsilon_{j}}(ze^{i\pi/2}, ze^{i\pi/2}r/a) = -\frac{2}{\pi}e^{\epsilon_{j}\pi i/2}G^{(+)}_{\beta_{j},\beta_{j}+\epsilon_{j}}(z, zr/a)$$

$$= -\frac{2}{\pi}e^{\epsilon_{j}\pi i/2}\left[K^{(a+)}_{\beta_{j}}(z)I_{\beta_{j}+\epsilon_{j}}(zr/a) + I^{(a+)}_{\beta_{j}}(z)K_{\beta_{j}+\epsilon_{j}}(zr/a)\right].$$
(1.108)

where

$$G_{\nu,\mu}^{(p)}(x,y) = K_{\nu}^{(ap)}(x) I_{\mu}(y) - (-1)^{\mu-\nu} I_{\nu}^{(ap)}(x) K_{\mu}(y)$$
(1.109)

we find

$$h_2(ze^{pi\frac{\pi}{2}}) = -\frac{4}{\pi^2} \frac{pi\epsilon_j z^2 G^{(ap)}_{\beta_j}(z, zr/a)}{\sqrt{(ze^{p\pi i/2})^2/a^2 + m^2}} G^{(ap)}_{\beta_j + \epsilon_j}(z, zr/a).$$
(1.110)

For the series over l in the expression of the current density $\langle j_{\phi} \rangle$ one gets

$$\sum_{l=1}^{\infty} T_{\beta_j}^{ab}(\eta, z_l) h_2(z_l) =$$

$$= \frac{4}{\pi^2} \int_0^\infty dx \, \frac{h_2(x)}{J_{\beta_j}^{(a)2}(x) + Y_{\beta_j}^{(a)2}(x)} - \frac{1}{\pi} \sum_{p=\pm} \int_0^\infty dx \, \Omega_{a\beta_j}^{(p)}(x, \eta x) h_2(x e^{p\pi i/2}), \tag{1.111}$$

or

$$\sum_{l=1}^{\infty} T_{\beta_j}^{ab}(\eta, z_l) h_2(z_l) = = \frac{4}{\pi^2} \int_0^\infty dx \, \frac{h_2(x)}{J_{\beta_j}^{(a)2}(x) + Y_{\beta_j}^{(a)2}(x)} + \frac{4i}{\pi^3} \sum_{p=\pm} p \int_0^\infty dx \, \frac{\epsilon_j x^2 \Omega_{a\beta_j}^{(p)}(x, \eta x)}{\sqrt{(xe^{p\pi i/2})^2/a^2 + m^2}} \times G_{\beta_j}^{(ap)}(x, xr/a) G_{\beta_j + \epsilon_j}^{(ap)}(x, xr/a).$$
(1.112)

From here, the current density is decomposed as

$$\langle j_{\phi} \rangle = \langle j_{\phi} \rangle_{a} - \frac{ie}{4\pi^{2}a^{3}} \sum_{j} \sum_{\kappa=-,+} \sum_{p=\pm} p \int_{0}^{\infty} dx \, \frac{x^{2} \Omega_{a\beta_{j}}^{(p)}(x,\eta x)}{\sqrt{(xe^{p\pi i/2})^{2}/a^{2} + m^{2}}}$$

$$\times G_{\beta_{j}}^{(ap)}(x,xr/a) G_{\beta_{j}+\epsilon_{j}}^{(ap)}(x,xr/a),$$

$$(1.113)$$

where

$$\langle j_{\phi} \rangle_{a} = -\frac{e}{4\pi a^{3}} \sum_{j} \sum_{\kappa=-,+} \epsilon_{j} \int_{0}^{\infty} dx \, \frac{h_{2}(x)}{J_{\beta_{j}}^{(a)2}(x) + Y_{\beta_{j}}^{(a)2}(x)}.$$
(1.114)

By using the relation 1.46, the formula 1.113 is presented in a simpler form

$$\langle j_{\phi} \rangle = \langle j_{\phi} \rangle_{a} - \frac{e}{2\pi^{2}a^{2}} \sum_{j} \sum_{p=\pm} \int_{ma}^{\infty} dx \, \frac{x^{2} \Omega_{a\beta_{j}}^{(p)}(x,\eta x)}{\sqrt{x^{2} - m^{2}a^{2}}} G_{\beta_{j}}^{(ap)}(x,xr/a) G_{\beta_{j}+\epsilon_{j}}^{(ap)}(x,xr/a).$$
(1.115)

By taking into account Eq. (1.46), we conclude that in the last integral of Eq. (1.43) for the integration range [0, ma] the terms with p = + and p = - cancel each other. For the integration range $[ma, \infty)$ of the remaining integral κ and p appear in the form of the product κp and, hence, the negative- and positive-energy modes give the same contribution to the last term. As a consequence we get

$$\langle j_{\phi} \rangle = \langle j_{\phi} \rangle_{a} - \frac{e}{\pi^{2}} \sum_{j} \int_{m}^{\infty} dx \, \frac{x^{2}}{\sqrt{x^{2} - m^{2}}}$$

$$\times \operatorname{Re} \left[\Omega_{a\beta_{j}}(ax, bx) G_{\beta_{j}}^{(a)}(ax, rx) G_{\beta_{j} + \epsilon_{j}}^{(a)}(ax, rx) \right].$$

$$(1.116)$$

The part (1.114) comes from the first term in the right-hand side of the summation formula (1.43). Taking into account that in the limit $b \to \infty$ the last term in Eq. (1.116) vanishes, we conclude that $\langle j_{\phi} \rangle_a$ is the current density in the region $r \ge a$ for the geometry with a single boundary at r = a.

For the separation of the boundary-induced effects in Eq. (1.114) we use the relation

$$\frac{g_{\beta_j,\beta_j}(z,y)g_{\beta_j,\beta_j+\epsilon_j}(z,y)}{J_{\beta_j}^{(a)2}(x)+Y_{\beta_j}^{(a)2}(x)} = J_{\beta_j}(y)J_{\beta_j+\epsilon_j}(y) - \frac{1}{2}\sum_{l=1,2}\frac{J_{\beta_j}^{(a)}(z)}{H_{\beta_j}^{(al)}(z)}H_{\beta_j}^{(l)}(y)H_{\beta_j+\epsilon_j}^{(l)}(y).$$
(1.117)

The further transformations are similar to that of the charge density. Substituting Eq. (1.117) into Eq. (1.114), in the part with the last term we rotate the integration contour by the angle $\pi/2$ for the term with l = 1 and by the angle $-\pi/2$ for the l = 2 term. The integrals over the intervals [0, ima] and [0, -ima] are canceled. As a result, the contribution (1.114) is presented in the decomposed form

$$\langle j_{\phi} \rangle_a = \langle j_{\phi} \rangle_0 + \langle j_{\phi} \rangle_a^{(b)}, \qquad (1.118)$$

where the separate terms are given by the expressions

$$\langle j_{\phi} \rangle_0 = -\frac{e}{2\pi} \sum_j \epsilon_j \int_0^\infty dx \, x^2 \frac{J_{\beta_j}(rx) J_{\beta_j + \epsilon_j}(rx)}{\sqrt{x^2 + m^2}},\tag{1.119}$$

and

$$\langle j_{\phi} \rangle_{a}^{(b)} = \frac{e}{\pi^{2}} \sum_{j} \int_{m}^{\infty} dx \, \frac{x^{2}}{\sqrt{x^{2} - m^{2}}} \operatorname{Re}\left[\frac{I_{\beta_{j}}^{(a)}(ax)}{K_{\beta_{j}}^{(a)}(ax)}\right] K_{\beta_{j}}(rx) K_{\beta_{j}+\epsilon_{j}}(rx).$$
(1.120)

For α different from a half-odd integer, the part $\langle j_{\phi} \rangle_{a}^{(b)}$ vanishes in the limit $a \to 0$ and, hence, in this case Eq. (1.119) is interpreted as the current density in two dimensional space without boundaries. Respectively, the part (1.120) presents the contribution induced in the region $r \leq a$ by a single boundary at r = a. In the special case s = 1, the single boundary-induced contribution coincides with the result previously obtained in [109]-[115], [128] for a more general geometry of a conical space (with the sign difference explained above).

The boundary-free part in the current density, given by Eq. (1.119), does not depend on the parameter *s* and, hence, on the irreducible representation of the Clifford algebra in (2+1)dimensional spacetime. A more convenient expression for this part is provided in Ref. [109]-[115], [128]

$$\langle j_{\phi} \rangle_{0} = \frac{e \sin(\pi \alpha_{0})}{4\pi^{2} r^{2}} \int_{0}^{\infty} dz \, \frac{\cosh(2\alpha_{0}z)}{\cosh^{3} z} \left(1 + 2mr \cosh z\right) e^{-2mr \cosh z}. \tag{1.121}$$

An alternative expression is given in [47]:

$$\langle j_{\phi} \rangle_{0} = -\frac{er}{\pi^{3}} \sin(\pi\alpha_{0}) \int_{m}^{\infty} dx \, x^{3} \frac{K_{\alpha_{0}}^{2}(rx) - K_{1-\alpha_{0}}(rx)K_{1+\alpha_{0}}(rx)}{\sqrt{x^{2} - m^{2}}}.$$
 (1.122)

Unlike the case of the charge density, the current density does not vanish for a massless field. For a massive field, at distances $mr \gg 1$, it behaves as $e^{-2mr}/r^{3/2}$. At the origin the current density diverges like $1/r^2$. Similar to the case of the charge density in the boundary-free geometry, $\langle j_{\phi} \rangle_0$ is discontinuous at half-odd integer values of α with the discontinuity $2\langle j_{\phi} \rangle_0|_{\alpha_0=1/2}$. Particularly, for a massless field for the discontinuity we have $e/(2\pi^2 r^2)$.

Decomposing the parameter α in accordance with Eq. (1.56) and redefining the summation variable j, the current density can be presented in the form

$$\langle j_{\phi} \rangle = \langle j_{\phi} \rangle_{0} + \langle j_{\phi} \rangle_{a}^{(\mathrm{b})} - \frac{e}{\pi^{2}} \sum_{n=0}^{\infty} \sum_{p=\pm} p \int_{m}^{\infty} dx \, \frac{x^{2}}{\sqrt{x^{2} - m^{2}}}$$

$$\times \operatorname{Re} \left[\Omega_{an_{p}}(ax, bx) G_{n_{p}}^{(a)}(ax, rx) G_{n_{p}+1}^{(a)}(ax, rx) \right], \qquad (1.123)$$

with the single boundary-induced part

$$\langle j_{\phi} \rangle_{a}^{(b)} = \frac{e}{\pi^{2}} \sum_{n=0}^{\infty} \sum_{p=\pm} p \int_{m}^{\infty} dx \, \frac{x^{2}}{\sqrt{x^{2} - m^{2}}} \operatorname{Re}\left[\frac{I_{n_{p}}^{(a)}(ax)}{K_{n_{p}}^{(a)}(ax)}\right] K_{n_{p}}(rx) K_{n_{p}+1}(rx).$$
(1.124)

This representation shows in an explicit manner that the current density does not depend on the integer part N and is an odd function of the fractional part α_0 . For a massless field and at large distances from the boundary, $r \gg a$, the single boundary-induced contribution (1.124) behaves as $(a/r)^{4-4|\alpha_0|}$, $|\alpha_0| < 1/2$, with the sign $\text{sgn}(\alpha_0)\langle j_{\phi}\rangle_a^{(b)}/e < 0$. In this limit, the total VEV in the geometry of a single boundary is dominated by the boundary-free part. In the case of a massive field, at distances $r \gg a, m^{-1}$, for the part (1.124) one has the suppression by the factor $e^{-2mr}/r^{3/2}$ and the boundary-induced contribution is of the same order as the boundary-free one.

Both the terms in the right-hand side of Eq. (1.118) are discontinuous at half-odd integer values of the parameter α . For the corresponding limiting values we have

$$\lim_{\alpha_0 \to \pm 1/2} \langle j_{\phi} \rangle_0 = -\lim_{\alpha_0 \to \pm 1/2} \langle j_{\phi} \rangle_a^{(b)} = \pm \frac{em}{2\pi^2 r} K_1(2mr).$$
(1.125)

Though the separate terms are discontinuous, the total current density in the region r > a for the geometry of a single boundary vanishes in the limits $\alpha_0 \to \pm 1/2$ and it is continuous. This is the case for the current density (1.123) in the geometry of the ring as well. It can be checked that the last term in the right-hand side of this formula vanishes for $\alpha_0 \rightarrow \pm 1/2$. Hence, we conclude that the current density in the ring is a continuous function of the magnetic flux including the points corresponding to the half-odd integer values of the magnetic flux in units of the flux quantum.

In order to clarify the dependence of the current density on the parameters of the problem, let us consider some numerical examples. The behavior of the current density in the region $a \leq r \leq b$ as a function of the radial coordinate and of the parameter α_0 is presented in figure 6. The left panel is plotted for a massless field and for the magnetic flux parameter $\alpha_0 = 1/4$. In this panel, the numbers near the full curves are the values of the ratio b/a and the dashed curve presents the current density in the geometry of a single boundary at r = a. The full curves in the right panel are plotted for b/a = 8, r/a = 2, ma = 0.1 and the numbers near them are the values of the parameter s. The dashed curve in the right panel corresponds to the current density for a massless field for the same values of b/a and r/a. Similarly to the VEV of the charge density, the current density is finite on the edges. As it has been already emphasized, the current density vanishes for half-odd integer values of the ratio of the magnetic flux to the flux quantum, corresponding to $|\alpha_0| = 1/2$. The graphs show that the current density is peaked near the inner edge and it decreases with decreasing the width of the ring.

In Fig. 7, the current density is displayed as a function of the relative location of the outer boundary for fixed values of the parameters $\alpha_0 = 1/4$, r/a = 1.5. The numbers near the curves are the values for ma. The left and right panels correspond to the representations with s = 1and s = -1, respectively. The dashed lines on both the panels present the current density in the geometry of a single boundary at r = a. Again, the graphs show that, for a fixed inner radius, the current density increases with increasing the width of the ring.

Similarly to the case of the charge density, we see an essentially different behavior of the current density for the representations s = 1 and s = -1, as a function of the field mass at relatively small values of the parameter b/a. The dependence of the current density on the mass is plotted in Fig. 8 for the irreducible representations s = 1 (left panel) and s = -1 (right panel) and for the values of the parameters $\alpha_0 = 1/4$, b/a = 8, r/a = 2. The dashed curves



Figure 6: Current density in the region between two boundaries as a function of the radial coordinate for a massless field (left panel) and as a function of the parameter α_0 (right panel). The left panel is plotted for the magnetic flux parameter $\alpha_0 = 1/4$ and the numbers near the curves are the values of b/a. The full curves in the right panel are plotted for ma = 0.1, b/a = 8, r/a = 2 and the numbers near the curves are the values of s. The dashed curve in the right panel corresponds to a massless field.



Figure 7: Current density versus the ratio b/a for irreducible representations s = 1 (left panel) and s = -1 (right panel). The graphs are plotted for $\alpha_0 = 1/4$, r/a = 1.5 and the numbers near the curves are the values of ma. The dashed curves present the current density in the geometry of a single boundary at r = a.

present the charge density in the geometry of a single boundary at r = a. The dotted line in the right panel is the current density in the boundary-free problem.



Figure 8: Current density versus the field mass for the fields with s = 1 (left panel) and s = -1 (right panel) and for $\alpha_0 = 1/4$, b/a = 8, r/a = 2. The dashed curves present the current density outside a single boundary at r = a and the dotted line is the current density in the boundary-free geometry.

In the representation (1.123) the current density for the geometry of a single boundary at r = a is explicitly separated. The representation with the outer edge contribution separated is obtained by making use of the identity

$$-\frac{I_{n_p}^{(a)}(ax)}{K_{n_p}^{(a)}(ax)}K_{n_p}(rx)K_{n_p+1}(rx) = \frac{K_{n_p}^{(b)}(bx)}{I_{n_p}^{(b)}(bx)}I_{n_p}(rx)I_{n_p+1}(rx) + \sum_{u=a,b}n_u\Omega_{un_p}(ax,bx)G_{n_p,n_p}^{(u)}(ux,rx)G_{n_p,n_p+1}^{(u)}(ux,rx).$$
(1.126)

With this relation, the current density in the region $a \leq r \leq b$ is presented as

$$\langle j_{\phi} \rangle = \langle j_{\phi} \rangle_b - \frac{e}{\pi^2} \sum_{n=0}^{\infty} \sum_{p=\pm} p \int_m^\infty dx \, \frac{x^2}{\sqrt{x^2 - m^2}}$$

$$\times \operatorname{Re} \left[\Omega_{bn_p}(ax, bx) G_{n_p, n_p}^{(b)}(bx, rx) G_{n_p, n_p+1}^{(b)}(bx, rx) \right].$$
(1.127)

Here

$$\langle j_{\phi} \rangle_b = \langle j_{\phi} \rangle_0 + \langle j_{\phi} \rangle_b^{(b)}, \qquad (1.128)$$

with

$$\langle j_{\phi} \rangle_{b}^{(b)} = -\frac{e}{\pi^{2}} \sum_{n=0}^{\infty} \sum_{p=\pm} p \int_{m}^{\infty} dx \, \frac{x^{2}}{\sqrt{x^{2} - m^{2}}} \operatorname{Re}\left[\frac{K_{n_{p}}^{(b)}(bx)}{I_{n_{p}}^{(b)}(bx)}\right] I_{n_{p}}(rx) I_{n_{p}+1}(rx).$$
(1.129)

By taking into account that for $|\alpha_0| \neq 1/2$ one has $\Omega_{bn_p}(az, bz) \sim a^{2n+1+2\alpha_0}$ for $a \to 0$, from Eq. (1.127) we conclude that the part $\langle j_{\phi} \rangle_b$ is the current density in the region $0 \leq r \leq b$ for the geometry of a single boundary at r = b and the contribution (1.129) is induced by the boundary. For points near the center of the disc $r \leq b$, the dominant contribution to the edge-induced part (1.129) comes from the term n = 0 and it behaves as $r^{1-2|\alpha_0|}$. Taking into account that the boundary-free part behaves like $1/r^2$, we see that near the center the current density is dominated by the boundary-free part.

For the magnetic flux parameter $\alpha_0 = 1/4$, the boundary-induced charge density (1.129) in the geometry of a single boundary at r = b is plotted in Fig. 9 versus the radial coordinate and the mass. The dashed curve corresponds to the current density in the boundary-free problem. The left panel is plotted for a massless field and the numbers near the curves on the right panel are the values of the parameter s and we have taken r/b = 0.5.



Figure 9: The boundary-induced contribution to the current density inside a single circular boundary with radius *b* versus the radial coordinate and the mass for $\alpha_0 = 1/4$. The left panel is plotted for a massless field. For the right panel r/b = 0.5 and the numbers near the curves are the values of *s*.

The last term in Eq. (1.127) is the current density induced in the region $a \leq r \leq b$ when we add the boundary at r = a to the geometry of the disc with the radius b. All the separate terms in the right-hand side of Eq. (1.127) have jumps at half-odd integer values of α . However, it can be chaked that the total current density is a continuous function of the magnetic flux and it vanishes for $\alpha = N + 1/2$. Similar to the case of the charge density, relatively simple expressions for the second edge-induced parts in the representations (1.123) and (1.127) are obtained on the edges r = a and r = b, respectively. The current density on the edge r = uwith u = a, b, is presented in the form

$$\langle j_{\phi} \rangle = \langle j_{\phi} \rangle_u + \frac{en_u}{\pi^2} \sum_{n=0}^{\infty} \sum_{p=\pm} p \int_m^\infty dx \, \frac{x}{\sqrt{x^2 - m^2}} \operatorname{Re}\left[(sm + i\sqrt{x^2 - m^2})\Omega_{un_p}(ax, bx) \right]. \quad (1.130)$$

Comparing with Eq. (1.98), we see that the second edge-induced contribution in the current density on the first edge is equal to the corresponding charge density for the outer edge and has the opposite sign on the inner edge. These relations between the charge and current densities on the ring edges hold for the total VEVs as well. If we formally put r = a, b in the mode sums (1.41) and (1.104), then, using the Wronskian relation for the Bessel and Neumann functions and Eq. (1.29) for the eigenvalues of the radial quantum number, we can see that

$$\langle j^0 \rangle_{r=u} = n_u \langle j_\phi \rangle_{r=u} = -\frac{e}{4\pi a^2} \sum_j \sum_{l=1}^\infty \sum_{\kappa=\pm} \kappa T^{ab}_{\beta_j}(\eta, z_l) \frac{E + \kappa sm}{E} z_l B_u(z_l).$$
(1.131)

with u = a, b and

$$B_{a}(x) = 1, \ B_{b}(x) = J_{\beta_{j}}^{(a)2}(x) / J_{\beta_{j}}^{(b)2}(xb/a), \qquad (1.132)$$

This feature is a consequence of the bag boundary conditions we have imposed on the edges (see also [138]).

In the discussion above for the charge and current densities we have assumed that the fermionic field is in the vacuum state. If the field is in thermal equilibrium at finite temperature, in addition to the vacuum parts we have considered here, the expectation values will receive contributions coming from particles and antiparticles (for finite temperature effects on the fermionic charge and current densities in topologically nontrivial spaces see, e.g., [76],[139]; see also [140] for a recent review of finite temperature field theoretical effects in toroidal topology). For a fermionic field with the chemical potential μ , obeying the condition $|\mu| < m$, the charge and current densities at zero temperature coincide with the VEVs we have investigated above. In the case $|\mu| > m$, the zero temperature expectation values in addition to the VEVs will contain contributions from particles or antiparticles (depending on the sign of the chemical potential) filling the states with the energies E in the range $m \leq E \leq |\mu|$.

1.5 Charge and current densities in P- and T-symmetric models with applications to graphene rings

Making use of the results from the previous sections we can obtain the vacuum densities in the parity and time-reversal symmetric massive fermionic models. In (2+1) dimensions the irreducible representations for the Clifford algebra are realized by 2×2 matrices. In cylindrical coordinates, we can choose the Dirac matrix γ^2 in two inequivalent ways: $\gamma^2 = \gamma_{(s)}^2 = -is\gamma^0\gamma^1/r$, where, as before, $s = \pm 1$. The gamma matrices (1.5), we have used in the discussion above, correspond to the representation with the upper sign. Two sets of Dirac matrices $\gamma_{(s)}^{\mu} = (\gamma^0, \gamma^1, \gamma_{(s)}^2)$ realize two inequivalent irreducible representations of the Clifford algebra. In these representations, the mass term in the Lagrangian density for a two-component spinor field,

$$L_s = \bar{\psi}_{(s)} (i\gamma^{\mu}_{(s)} D_{\mu} - m) \psi_{(s)}, \qquad (1.133)$$

is not invariant under the *P*- and *T*-transformations. Here we assume that both the fields $\psi_{(s)}$ obey the same boundary conditions

$$\left(1 + in_{\mu}\gamma^{\mu}_{(s)}\right)\psi_{(s)}(x) = 0, \qquad (1.134)$$

on the circular edges r = a, b. In order to recover the *P*- and *T*-invariances, let us consider the combined Lagrangian density $L = \sum_{s=\pm 1} L_s$. By appropriate transformations of the fields $\psi_{(-1)}$ and $\psi_{(+1)}$, this Lagrangian is invariant under *P*- and *T*-transformations (in the absence of magnetic fields).

In order to relate the fields $\psi_{(s)}$ to the ones we have considered in the evaluation of the vacuum densities, let us introduce new two-component fields $\psi'_{(s)}$ in accordance with $\psi'_{(+1)} = \psi_{(+1)}, \ \psi'_{(-1)} = \gamma^0 \gamma^1 \psi_{(-1)}$. In terms of these fields, the Lagrangian density is presented as

$$L = \sum_{s=\pm 1} \bar{\psi}'_{(s)} (i\gamma^{\mu} D_{\mu} - sm) \psi'_{(s)}, \qquad (1.135)$$

where $\gamma^{\mu} = \gamma^{\mu}_{(+1)}$. From Eq. (1.135) we conclude that the equations for the fields with s = -1and s = +1 differ by the sign of the mass term and coincide with Eq. (2.2). From the boundary conditions (1.134) it follows that the fields in Eq. (1.135) obey the conditions

$$(1 + i s n_{\mu} \gamma^{\mu}) \psi'_{(s)}(x) = 0, \qquad (1.136)$$

on r = a, b. As it is seen, the field $\psi'_{(+1)}$ obeys the condition (1.2), whereas the boundary condition for the field $\psi'_{(-1)}$ differs by the sign of the term with the normal to the boundary. As it has been already noticed in [119], this type of condition with the reversed sign is an equally acceptable boundary condition for the Dirac equation. Combining 2-component spinors ψ'_{+} and ψ'_{-} in a 4-component one, $\Psi = (\psi'_{+}, \psi'_{-})^{T}$, and introducing 4×4 Dirac matrices $\gamma^{\mu}_{(4)} = \sigma_{3} \otimes \gamma^{\mu}$, the Lagrangian density (1.135) is rewritten in the form

$$L = \bar{\Psi}(i\gamma^{\mu}_{(4)}D_{\mu} - m)\Psi.$$
(1.137)

In this reducible representation, the boundary conditions (1.136) are combined as

$$\left(1 + in_{\mu}\gamma^{\mu}_{(4)}\right)\Psi(x) = 0. \tag{1.138}$$

The latter has the form of the standard MIT bag condition for a 4-component spinor.

By taking into account that $\bar{\psi}_{(s)}\gamma^{\mu}_{(s)}\psi_{(s)} = \bar{\psi}'_{(s)}\gamma^{\mu}\psi'_{(s)}$, for the total VEV of the current density in the model with the combined Lagrangian $L = \sum_{s=\pm 1} L_s$, with L_s form (1.133), one gets

$$\langle J^{\mu} \rangle = \sum_{s=\pm 1} \langle \bar{\psi}_{(s)} \gamma^{\mu}_{(s)} \psi_{(s)} \rangle = \sum_{s=\pm 1} \langle \bar{\psi}'_{(s)} \gamma^{\mu} \psi'_{(s)} \rangle.$$
(1.139)

The charge and current densities for the field $\psi_{(+1)}$ are obtained from the expressions given in the previous sections with s = 1. In order to find the VEVs for the field $\psi'_{(-1)}$, we note that it obeys the field equation (2.2) with s = -1 and the boundary condition that differs from Eq. (1.2) by the sign of the term containing the normal to the boundaries. Consequently, the VEV $\langle \bar{\psi}'_{(-1)} \gamma^{\mu} \psi'_{(-1)} \rangle$ is obtained from the corresponding formulas given above taking s = -1and making the replacement

$$n_u \to -n_u, \ u = a, b \tag{1.140}$$

(see Eq. (1.3) for the definition of n_u). In the final formulas this replacement is made through the definition (1.59) for f = I, K. From here we conclude that the expressions for the VEVs $\langle \bar{\psi}'_{(s)} \gamma^{\mu} \psi'_{(s)} \rangle$, $\mu = 0, 2$, are given by the formulas in sections 1.3 and 1.4 where now we should take

$$f_{n_p}^{(u)}(z) = \delta_f z f_{n_p+1}(z) + n_u (m_u + si\sqrt{z^2 - m_u^2}) f_{n_p}(z) , \qquad (1.141)$$

with n_u defined in accordance with Eq. (1.3). This shows that one has the relation $f_{n_p}^{(u)}(z)|_{s=-1} = f_{n_p}^{(u)*}(z)|_{s=1}$ and, hence, the same for the functions $\Omega_{un_p}(ax, bx)$ and $G_{n_p,\mu}^{(u)}(ux, rx)$. Here the

star stands for the complex conjugate. Assuming that the masses m for the fields with s = +1and s = -1 are the same, we can see that the boundary-induced contributions from these fields to the charge density cancel each other. By taking into account that the same is the case for the boundary-free part (see Eq. (1.88)), we conclude that the VEV of the total charge density vanishes. For the VEV of the current density, the contributions from the fields $\psi'_{(+1)}$ and $\psi'_{(-1)}$ coincide for both the boundary-free and boundary-induced parts. The corresponding expressions for the total current (1.139) are obtained from those in the previous section for the case s = 1 with an additional factor 2. Now, for the field $\psi'_{(s)}$ the analog of the relation (1.131) between the charge and current densities on the edges has the form

$$\langle j'_{\phi} \rangle_{r=u} = -n_u \langle j^{0\prime} \rangle_{r=u}, \qquad (1.142)$$

where $\langle j^{\mu'} \rangle = \langle \bar{\psi}'_{(-1)} \gamma^{\mu} \psi'_{(-1)} \rangle$, u = a, b, and n_u is the same as in Eq. (1.131).

Note that we could consider another class of P- and T-invariant models with two-components fields $\psi_{(s)}$ obeying the boundary conditions in which the sign of the term in Eq. (1.134), containing the normal vector, is reversed. For these models, the same reversion should be made in the boundary condition (1.136) for the primed fields. Now we see that the charge and current densities for the field $\psi'_{(-1)}$ coincide with those in the previous sections for s = -1, whereas the results for the field $\psi'_{(+1)}$ are obtained from those before in the case s = 1 by the replacement (1.140). The total charge density, combined from the fields with s = 1 and s = -1 vanishes as in the previous case, and the current density is obtained from the expressions in section 1.4 in the case s = -1 with the additional factor 2. As it has been seen, in this case the dependence of the VEV on the field mass is more interesting. Naturally, there is another possibility when the boundary condition (1.134) with the additional factor s in the term involving the normal vector. In this case the primed fields will obey the same boundary condition (Eq. (1.136) with s = 1) and the corresponding VEVs exactly coincide with those in sections 1.3 and 1.4. In this variant, there is no cancellation and the total charge density does not vanish.

The field theoretical models we have considered can be realized by various graphene made structures. For example, the geometry with a single boundary at r = a corresponds to a circular graphene dot if the region r < a is considered and to a single circular nanohorn (or nanopore) for the region r > a. The influence of boundaries on the electronic properties of a circular graphene quantum dot in a magnetic field has been discussed in [141, 142]. Comparing the analytical results obtained within the continuum model to those obtained from the tightbinding model, the authors conclude that the Dirac model with the infinite-mass boundary condition describes rather well its tight-binding analog and is in good qualitative agreement with experiments. Considering different boundary conditions in the Dirac model for graphene devices, a similar conclusion is made in Ref. [138]. The Aharonov-Bohm effect and persistent currents in graphene nanorings have been recently investigated in [126],[143]-[149]. The effect of impurity on persistent currents in strictly one-dimensional Dirac systems is discussed in [150].

The results obtained above can be applied for the investigation of the ground state charge and current densities in graphene rings. Graphene is a monolayer of carbon atoms with honeycomb lattice containing two triangular sublattices A and B related by inversion symmetry. The electronic subsystem in a graphene sheet is among the most popular realizations of the Dirac physics in two spatial dimensions (for other planar condensed-matter systems with the lowenergy excitations described by the Dirac model see Ref. [151, 152]). For a given value of spin $S = \pm 1$, the corresponding long wavelength excitations are described in terms of 4-component spinors $\Psi_S = (\psi_{+,AS}, \psi_{+,BS}, \psi_{-,AS}, \psi_{-,BS})^T$ with the Lagrangian density (in the standard units)

$$L = \sum_{S=\pm 1} \bar{\Psi}_S (i\hbar\gamma^0 \partial_t + i\hbar v_F \gamma^l D_l - \Delta) \Psi_S.$$
(1.143)

Here, $D_l = (\nabla - ie\mathbf{A}/\hbar c)_l$, l = 1, 2, is the spatial part of the gauge extended covariant derivative and e = -|e| for electrons. The Fermi velocity v_F plays the role of the speed of light. It is expressed in terms of the microscopic parameters as $v_F = \sqrt{3}a_0\gamma_0/(2\hbar) \approx 7.9 \times 10^7$ cm/s, where $a_0 \approx 1.42$ Å is the inter-atomic spacing of graphene honeycomb lattice and $\gamma_0 \approx 2.9$ eV is the transfer integral between first-neighbor π orbitals. The components $\psi_{\pm,AS}$ and $\psi_{\pm,BS}$ of the spinor Ψ_S give the amplitude of the electron wave function on sublattices A and B. The indices + and - of these components correspond to inequivalent points, \mathbf{K}_+ and \mathbf{K}_- , at the corners of the two-dimensional Brillouin zone (see Ref. [15]-[17]). The energy gap Δ in Eq. (1.143) is related to the corresponding Dirac mass as $\Delta = mv_F^2$. It plays an important role in many physical applications (for the mechanisms of the gap generation in the energy spectrum of graphene see, for example, Ref. [15]-[17] and references therein). The values of the energy gap may vary in the range $1 \text{ meV} \lesssim \Delta \lesssim 1 \text{ eV}$, depending on the physical mechanism for the generation.

Comparing with the discussion above, we see that the values of the parameter $s = \pm 1$ and s = -1 correspond to the \mathbf{K}_+ and \mathbf{K}_- points of the graphene Brillouin zone and the Lagrangian density (1.143) is the analog of (1.137). From here it can be concluded that, for a given value of the spin S, the expressions for the VEVs of the charge and current densities for separate contributions coming from the points \mathbf{K}_+ and \mathbf{K}_- are obtained from the formulas in previous sections by the replacement $m \to a_0^{-1} \Delta/\gamma_F$, where $\gamma_F = \hbar v_F/a_0 \approx 2.51$ eV determines the energy scale in the model. In the expressions for the current density, an additional factor v_F should be added, because now the operator of the spatial components of the current density is defined as $j^{\mu} = ev_F \bar{\psi}(x) \gamma^{\mu} \psi(x)$, $\mu = 1, 2$. For a given spin S, the contributions from two valleys are combined in accordance with Eq. (1.139). In the problem at hand, the spins $S = \pm 1$ give the same contributions to the total VEVs. As it has been mentioned before, the charge density vanishes as a result of cancellation of the contributions from the \mathbf{K}_+ and \mathbf{K}_- points. The effective charge density may appear if the gap generation mechanism breaks the valley symmetry and the mass gap is different for $s = \pm 1$ and s = -1. Note that this will break P- and T-invariances of the model.

1.6 Summary

From the point of view of both the field theory and condensed matter, among the most interesting topics in quantum field theory is the investigation of the effects induced by gauge field fluxes on the properties of the ground state. In this chapter we have discussed the combined effects from the magnetic flux and boundaries on the VEVs of the fermionic charge and current densities in a two-dimensional circular ring. The examples of graphene nanoriboons and rings have already shown that the edge effects have important consequences on the physical properties of planar systems. In the present problem, for the field operator on the ring edges we have imposed the bag boundary conditions. The distribution of the magnetic flux inside the inner edge can be arbitrary. The boundary separating the ring from the region of the location for the gauge field strength is impenetrable for the fermionic field and the effect of the gauge field is purely topological. It depends on the total flux only. The latter gives rise to the Aharonov-Bohm effect for physical characteristics of the ground state. We have considered the problem for both irreducible representations of the Clifford algebra in (2+1) dimensions. In these representations the mass term in the Dirac equation breaks the parity and time-reversal invariances. For the evaluation of the VEVs we have employed the method based on the direct summation over a complete set of fermionic modes in the ring. The corresponding positiveand negative-energy wavefunctions are given by Eq. (1.25) with the radial functions defined by Eq. (1.26). The eigenvalues of the radial quantum number are quantized by the boundary conditions and are roots of the equation (1.29). The eigenvalue equations for the positiveand negative-energy modes differ by the sign of the energy. Alternatively, we can take the negative-energy modes in the form (1.39). With this representation, the eigenvalue equation for the negative-energy modes is obtained from the positiveenergy one by inverting the sign of the parameter α , the latter being the ratio of the magnetic flux to the flux quantum.

The mode-sums for VEVs of the charge and current densities, Eqs. (1.41) and (1.104), contain series over the roots of Eq. (1.29). The latter are given implicitly and these representations are not well adapted for the investigation of the VEVs. More convenient expressions are obtained by using the generalized Abel-Plana formula (1.43) for the summation of the series. There are two advantages in the formulas abtained in this manner: the explicit knowledge of the eigenvalues is not required and the boundary-induced contributions to the VEVs are explicitly extracted. In addition, instead of series with highly oscillatory terms for large values of quantum numbers, in the new representation one has exponentially convergent integrals for points away from the edges. This is an important point from the perspective of numerical evaluations.

The VEVs for both the charge and current densities are decomposed into boundary-free, single boundary-induced and the second boundary-induced contributions. These all are odd periodic functions of the magnetic flux with the period equal to the flux quantum. For the geometry with two boundaries we have provided two representations, given by Eqs. (1.57) and (1.92) for the charge and by Eqs. (1.123) and (1.127) for the azimuthal current. In these representations the contributions for the exterior or interior geometries with a single boundary are explicitly extracted. The last terms in all the representations are induced by the introduction of the second boundary to the geometry with a single boundary. The single boundary parts in the VEVs are given by the expressions (1.67) and (1.124) in the exterior region and by Eqs. (1.94) and (1.129) for the interior region.

In contrast to the case of the boundary-free geometry the charge and current densities in the ring are continuous at half-odd integer values for the ratio of the magnetic flux to the flux quantum, and both of them vanish at these points. We have shown that the behaviour of the VEVs as functions of the field mass (energy gap in field theoretical models of planar condensed matter system) is essentially different for the cases s = 1 and s = -1. With the initial increase of the mass from the zero value, the modulus for the charge and current densities decreases for the irreducible representation with s = 1 and increases for the one with s = -1. With further increase of the mass the vacuum densities are suppressed in both cases. An important feature that distinguishes the VEVs of the charge and current densities from those for the energy-momentum tensor is their finiteness on the boundaries. On the outer edge the current density is equal to the charge density whereas on the inner edge they have opposite signs. For a fixed values of the other parameters, both the charge and current densities decrease by the modulus with decreasing outer radius.

The boundary condition (1.2) we have considered contains no additional parameters and is a special case in a general class of boundary conditions for the Dirac equation confining the fermionic field in a finite volume. It is the most popular boundary condition in the investigations of the fermionic Casimir effect for various types of the bulk and boundary geometries. On the base of the analysis given above we can consider another boundary condition that differs from (1.2) by the sign of the term containing the normal to the boundary. The corresponding expressions for the VEVs of the charge and current densities are obtained from those in sections 1.3 and 1.4 by making the replacement (1.140) for n_u defined by Eq. (1.3). All the final formulas (for instance, Eqs. (1.57), (1.127)) remain the same with the only difference in the definition of the notation (1.59), where now n_u should be replaced by $-n_u$. Equivalently, the results for the field with a given s and with the modified boundary condition are obtained from the corresponding expressions for the field with -s and obeying the condition (1.2) replacing $f_{n_p}^{(u)}(z)$ by its complex conjugate, $f_{n_p}^{(u)*}(z)$. With the modified boundary condition, the current density is equal to the charge density on the inner edge and has the opposite sign on the outer edge.

The charge and current densities in parity and time-reversal models are obtained combining the results for the separate cases with s = 1 and s = -1. These models can be formulated in terms of four-component spinors constructed from the 2-component spinors realizing the two different irreducible representations. Assuming that both these spinors obey the boundary condition (1.2) and have the same mass, the resulting charge density vanishes, whereas the current density is obtained from the expressions given in section 1.4 with the additional factor 2. For the graphene circular rings, an additional factor 2 comes from the degree of freedom, related to spin.

Chapter 2: FERMIONIC CONDENSATE AND THE CASIMIR EFFECT IN A CONICAL DEFECT SPACETIME

The chapter is devoted to the investigation of combined effects of nontrivial topology, induced by a conical defect, and boundaries on the fermionic condensate and the VEV of the energymomentum tensor for a massive fermionic field. As geometry of boundaries we consider two plates perpendicular to the axis of the defect on which the field is constrained by the MIT bag boundary condition. By using the Abel-Plana type summation formula, the VEVs in the region between the plates are decomposed into the boundary-free and boundary-induced contributions for general case of the planar angle deficit. The boundary-induced parts in both the fermionic condensate and the energy-momentum tensor vanish on the defect. Fermionic condensate is positive near the defect and negative al large distances, whereas the vacuum energy density is negative everywhere. The radial stress is equal to the energy density. For a massless field, the boundary-induced contribution in the VEV of the energy-momentum tensor is different from zero in the region between the plates only and it does not depend on the coordinate along the string axis. In the region between the plates and at large distances from the defect, the decay of the topological part is exponential for both massive and massless fields. This behavior is in contrast to that for the VEV of the energy-momentum tensor in the boundaryfree geometry with the power law decay for a massless field. The vacuum pressure on the plates is inhomogeneous and vanishes at the location of the defect. The corresponding Casimir forces are attractive.

2.1 Geometry of the problem and the mode functions

We start with the description of the bulk and boundary geometries for the present problem. The background geometry is generated by an infinitely long straight string-like conical defects along the z axis. In the cylindrical coordinates $x^{\mu} = (t, r, \phi, z)$ the corresponding metric tensor reads $g_{\mu\nu} = \text{diag}(1, -1, -r^2, -1)$. For the line element one has

$$ds^{2} = dt^{2} - dr^{2} - r^{2}d\phi^{2} - dz^{2}.$$
(2.1)

The difference from the Minkowskian spacetime comes from the planar angle deficit $2\pi - \phi_0$ for the azimuthal angle ϕ , $0 \leq \phi \leq \phi_0$. In the discussion below we will also use the parameter $q = 2\pi/\phi_0$. Among the examples of the condensed matter realization of the geometry in (2+1)dimensional context (in (2.1) the part with the coordinate z is absent) are graphitic cones. They are obtained from the graphene sheet if one or more sectors are removed. The opening angle of the cone is related to the number of sectors removed, N_c , by the formula $2\pi(1 - N_c/6)$, where $N_c = 1, 2, ..., 5$ (for the electronic properties of graphitic cones see, e.g., [153]-[159]). All these angles have been observed in experiments [158, 159].

For points outside the defect's core, the local geometry generated by a conical defect is flat and it coincides with that for the Minkowski spacetime. However, these two spacetimes differ in their global properties. The nontrivial topology coming from the conical defect induces shifts in the VEVs of physical observables for quantum fields. Here we are interested in the influence of topology on the Casimir effect for a quantum fermionic field $\psi(x)$ in the geometry of two parallel plates perpendicular to the defect axis. The field operator obeys the Dirac equation

$$(i\gamma^{\mu}\nabla_{\mu} - m)\psi(x) = 0, \qquad (2.2)$$

with the covariant derivative $\nabla_{\mu} = \partial_{\mu} + \Gamma_{\mu}$ and with Γ_{μ} being the spin connection. In cylindrical coordinates with a planar angle deficit, for the Dirac 4 × 4 matrices γ^{μ} we will take the representation

$$\gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma^{l} = \begin{pmatrix} 0 & \beta^{l} \\ -\beta^{l} & 0 \end{pmatrix},$$
(2.3)

where β^l , l = 1, 2, 3, are the Pauli matrices in the same coordinates. The latter are presented

as

$$\beta^{l} = (i/r)^{l-1} \begin{pmatrix} 0 & (-1)^{l-1} e^{-iq\phi} \\ e^{iq\phi} & 0 \end{pmatrix}, \ \beta^{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
(2.4)

with l = 1, 2.

We assume the presence of two boundaries located at z = 0 and z = a on which the field operator is constrained by the MIT bag boundary conditions:

$$(1 + in_{\mu}\gamma^{\mu})\psi(x) = 0 , \quad z = 0, a, \tag{2.5}$$

where n_{μ} is the outward-pointing normal to the boundary with respect to the region under consideration. In the discussion below we will consider the region $0 \leq z \leq a$ with $n_{\mu} = -\delta_{\mu 1}$ and $n_{\mu} = \delta_{\mu 1}$ for the boundaries at z = 0 and z = a respectively. In the regions $z \leq 0$ and $z \geq a$, the VEVs are obtained by the limiting transitions and they are the same as in the corresponding problem with a single boundary, discussed in [160]. Note that the boundaries z = 0, a are twodimensional cones with the angle deficit $2\pi - \phi_0$. The models with boundary conditions induced by the compactification of the conical defect along its axis have been considered in [161, 162].

We are interested in the investigation of the combined effects of topology and boundaries on the fermion condensate and on the VEV of the energy-momentum tensor. These quantities are expressed in the form of the sums over complete set of fermionic modes obeying the boundary conditions (2.5). Therefore, our first step will be the determination of the complete set of the fermionic wavefunctions in the region between the boundaries. In [160] it has been shown that, in the geometry of a single plate at z = 0, the positive-energy mode functions obeying the boundary condition (2.5) on that plate are presented as

$$\psi_{\sigma}^{(+)} = C_{\sigma}^{(+)} e^{i(qj\phi-\omega t)} \begin{pmatrix} f_{+}(z)J_{\beta}(\lambda r)e^{-iq\phi/2} \\ i\frac{\epsilon_{js}}{k}g_{+}(z)J_{\beta+\epsilon_{j}}(\lambda r)e^{iq\phi/2} \\ \frac{1}{k}g_{-}(z)J_{\beta}(\lambda r)e^{-iq\phi/2} \\ -i\epsilon_{j}sf_{-}(z)J_{\beta+\epsilon_{j}}(\lambda r)e^{iq\phi/2} \end{pmatrix}, \qquad (2.6)$$

where $J_{\beta}(x)$ is the Bessel function, $0 \leq \lambda < 0$, $\epsilon_j = 1$ for $j \ge 0$ and $\epsilon_j = -1$ for j < 0 with $j = \pm 1/2, \pm 3/2, \ldots$, and

$$\beta = q|j| - \epsilon_j/2, \ \omega = \sqrt{\lambda^2 + k^2 + m^2}.$$
(2.7)

In (2.6) we have defined the functions

$$f_{\pm}(z) = e^{ikz} \pm i\kappa_s e^{-ikz},$$

$$g_{\pm}(z) = (\omega \pm m) f_{\pm}(z) + s\lambda f_{\mp}(z),$$
(2.8)

with k > 0 and

$$\kappa_s = \frac{\omega + s\lambda}{k - im}, \ s = -1, 1. \tag{2.9}$$

In (2.6), σ stands for the set of quantum numbers specifying the mode functions (see below).

Now we should impose on the modes (2.6) the boundary condition (2.5) at z = a. For the bispinor $\psi = (\varphi_1, \varphi_2, \chi_1, \chi_2)$ one has

$$(1+i\gamma^{1})\psi = \begin{pmatrix} \varphi_{1}+i\chi_{1} \\ \varphi_{2}-i\chi_{2} \\ \chi_{1}-i\varphi_{1} \\ \chi_{2}+i\varphi_{2} \end{pmatrix} = 0.$$
(2.10)

From here we get two separate equations for the components

$$\varphi_1 + i\chi_1 = 0, \ \varphi_2 - i\chi_2 = 0.$$
 (2.11)

For the bispinor (2.6) these equations are reduced to

$$-\frac{1}{k}g_{-}(a)J_{\beta}(\lambda r)e^{-iq\phi/2} + if_{+}(a)J_{\beta}(\lambda r)e^{-iq\phi/2} = 0,$$

$$i\epsilon_{j}sf_{-}(a)J_{\beta+\epsilon_{j}}(\lambda r)e^{iq\phi/2} + \frac{\epsilon_{j}s}{k}g_{+}(a)J_{\beta+\epsilon_{j}}(\lambda r)e^{iq\phi/2} = 0,$$
(2.12)

or, after the further simplification,

$$f_{+}(a) + \frac{i}{k}g_{-}(a) = 0,$$

$$f_{-}(a) - \frac{i}{k}g_{+}(a) = 0.$$
(2.13)

By taking into account the expressions for the functions f_{\pm} and g_{\pm} , these equations are reduced to a single equation with respect to the quantum number k:

$$e^{2ika} = \frac{m - ik}{m + ik}.\tag{2.14}$$

This equation can also be alternatively written as

$$ma\sin(ka) + ka\cos(ka) = 0. \tag{2.15}$$

This equation has an infinite number of positive solutions with respect to ka. We will denote them by $x_n = ka$, $n = 1, 2, ..., x_{n+1} > x_n$. For the eigenvalues of k one has $k = k_n = x_n/a$.

For the negative-energy mode functions, in a similar way, one has the expression

$$\psi_{\sigma}^{(-)} = C_{\sigma}^{(-)} e^{i(qj\phi+\omega t)} \begin{pmatrix} -\frac{1}{k}g_{-}(z)J_{\beta}(\lambda r)e^{-iq\phi/2} \\ i\epsilon_{j}sf_{-}(z)J_{\beta+\epsilon_{j}}(\lambda r)e^{iq\phi/2} \\ f_{+}(z)J_{\beta}(\lambda r)e^{-iq\phi/2} \\ i\frac{\epsilon_{j}s}{k}g_{+}(z)J_{\beta+\epsilon_{j}}(\lambda r)e^{iq\phi/2} \end{pmatrix}, \qquad (2.16)$$

with the same notations as in (2.6) and with the same equation (2.15) for the eigenvalues of ka. It can be explicitly checked that the positive- and negative-energy modes (2.6) and (2.16) are orthogonal. The four-spinors $\psi_{\sigma}^{(\pm)}$ are eigenfunctions of the projection of the total angular momentum along the conical defect:

$$\hat{J}_3\psi_{\sigma}^{(\pm)} = \left(-i\partial_{\phi} + i\frac{q}{2}r\gamma^1\gamma^2\right)\psi_{\sigma}^{(\pm)} = qj\psi_{\sigma}^{(\pm)}.$$
(2.17)

Now, the set of quantum numbers for the modes are specified by $\sigma = (\lambda, j, n, s)$.

The coefficients $C_{\sigma}^{(\pm)}$ in (2.6) and (2.16) are found from the orthonormality conditions

$$\int d^3x \sqrt{|g|} \psi_{\sigma}^{(\pm)\dagger} \psi_{\sigma'}^{(\pm)} = \delta(\lambda - \lambda') \delta_{nn'} \delta_{j'j} \delta_{ss'} , \qquad (2.18)$$

By taking into account that

$$\int_0^\infty dr r J_\beta(\lambda r) J_\beta(\lambda' r) = \frac{1}{\lambda} \delta(\lambda - \lambda'), \qquad (2.19)$$

we find

$$|C_{\chi 1}^{(1)}|^{-2} = \frac{\phi_0}{\lambda} \int_0^a \left[dz f_+(z) f_+^*(z) + \frac{1}{k^2} g_-(z) g_-^*(z) + f_-(z) f_-^*(z) + \frac{1}{k^2} g_+(z) g_+^*(z) \right] .$$

$$(2.20)$$

For the product of the functions $g_{\pm}(z)$ one has

$$g_{+}(z)g_{+}^{*}(z) + g_{-}(z)g_{-}^{*}(z)$$

$$= \left[(\omega + m)^{2} + \lambda^{2} \right] f_{+}(z)f_{+}^{*}(z) + \left[(\omega - m)^{2} + \lambda^{2} \right] f_{-}(z)f_{-}^{*}(z)$$

$$+ 2s\lambda\omega \left[f_{+}(z)f_{-}^{*}(z) + f_{-}(z)f_{+}^{*}(z) \right]. \qquad (2.21)$$

For the products of the functions $f_{\pm}(z)$ we find

$$f_{+}(z)f_{-}^{*}(z) + f_{-}(z)f_{+}^{*}(z) = 2(1 - \kappa_{s}\kappa_{s}^{*}),$$

$$f_{+}(z)f_{+}^{*}(z) + f_{-}(z)f_{-}^{*}(z) = 2(1 + \kappa_{s}\kappa_{s}^{*}),$$

$$f_{+}(z)f_{+}^{*}(z) - f_{-}(z)f_{-}^{*}(z) = 2i\left(-\kappa_{s}^{*}e^{2ikz} + \kappa_{s}e^{-2ikz}\right).$$
(2.22)

And therefore one can write

$$f_{+}(z)f_{+}^{*}(z) + \frac{1}{k^{2}}g_{-}(z)g_{-}^{*}(z) + f_{-}(z)f_{-}^{*}(z) + \frac{1}{k^{2}}g_{+}(z)g_{+}^{*}(z) = \frac{4\omega}{k^{2}}\left[\omega\left(1 + \kappa_{s}\kappa_{s}^{*}\right) + s\lambda\left(1 - \kappa_{s}\kappa_{s}^{*}\right) - im\kappa_{s}^{*}e^{2ikz} + im\kappa_{s}e^{-2ikz}\right].$$
(2.23)

Inserting this into the expression for the normalization integral one gets

$$\int_{0}^{a} dz \left[f_{+}(z) f_{+}^{*}(z) + \frac{1}{k^{2}} g_{-}(z) g_{-}^{*}(z) + f_{-}(z) f_{-}^{*}(z) + \frac{1}{k^{2}} g_{+}(z) g_{+}^{*}(z) \right]$$

$$= \frac{8\omega \left(\omega + s\lambda\right) a}{k^{2}} \left[1 + \frac{m}{a \left(k^{2} + m^{2}\right)} \right]$$

$$= \frac{8\omega \left(\omega + s\lambda\right) a}{k^{2}} \left[1 - \frac{\sin(2ka)}{2ka} \right].$$
(2.24)

At the last step we have used the equation for the eigenvalues of k. Hence, for the normalization coefficient we find

$$\left|C_{\sigma}^{(\pm)}\right|^{-2} = \frac{\phi_0}{\lambda} \frac{8\omega \left(\omega + s\lambda\right) a}{k^2} \left[1 - \frac{\sin(2ka)}{2ka}\right],\tag{2.25}$$

with $k = x_n/a$.

2.2 Fermionic condensate

We start our investigation of the VEVs by the fermion condensate, defined as the expectation value $\langle 0|\bar{\psi}\psi|0\rangle \equiv \langle \bar{\psi}\psi \rangle$, with $|0\rangle$ being the vacuum state and $\bar{\psi} = \psi^{\dagger}\gamma^{0}$ is the Dirac adjoint. The fermionic condensate is among the most important quantities that characterize the properties of the quantum vacuum. It carries important information about both the local and global properties of the background geometry. In particular, the fermionic condensate plays an important role in the models of dynamical breaking of chiral symmetry.

Having the complete set of mode functions, the fermion condensate is evaluated by the mode sum

$$\langle \bar{\psi}\psi \rangle = \sum_{s=\pm 1} \sum_{j=\pm 1/2,\dots} \sum_{n=1}^{\infty} \int_0^\infty d\lambda \, \bar{\psi}_{\sigma}^{(-)} \psi_{\sigma}^{(-)}.$$
(2.26)

The expression in the right-hand side is divergent. We assume that some regularization scheme is used to make the expression in the right-hand side finite (for example, a cutoff function is introduced or the arguments of the operators in the product are shifted). The choice of the specific scheme is not essential in the further discussion and we will not display it explicitly.

By taking into account that expression for the negative-energy fernionic modes, the product appearing in (2.26) is presented in the form

$$\psi_{\sigma}^{(-)+}\gamma^{0}\psi_{\sigma}^{(-)} = \frac{1}{k^{2}} \left| C_{\chi 1}^{(1)} \right|^{2} \left\{ \left[|g_{-}(z)|^{2} - k^{2}|f_{+}(z)|^{2} \right] J_{\beta}^{2}(\lambda r) + \left[k^{2}|f_{-}(z)|^{2} - |g_{+}(z)|^{2} \right] J_{\beta+\epsilon_{j}}^{2}(\lambda r) \right\}.$$
(2.27)

Let us consider the functions $|g_{\pm}(z)|^2$, $|f_{\pm}(z)|^2$ and $f_{\pm}^*(z)f_{\mp}(z)$ separately. By taking the expressions given above one gets

$$|g_{\pm}(z)|^{2} = \left[(\omega \pm m) f_{\pm}^{*}(z) + s\lambda f_{\mp}^{*}(z) \right] \left[(\omega \pm m) f_{\pm}(z) + s\lambda f_{\mp}(z) \right]$$

= $(\omega \pm m)^{2} |f_{\pm}(z)|^{2} + s\lambda (\omega \pm m) \left[f_{\pm}^{*}(z) f_{\mp}(z) + f_{\mp}^{*}(z) f_{\pm}(z) \right] + \lambda^{2} |f_{\mp}(z)|^{2}, \quad (2.28)$

and for the others we find

$$|f_{\pm}(z)|^{2} = \left[e^{-ikz} \mp i\kappa_{s}^{*}e^{ikz}\right] \left[e^{ikz} \pm i\kappa_{s}e^{-ikz}\right]$$

$$= 1 + \kappa_{s}^{*}\kappa_{s} \pm i\left(\kappa_{s}e^{-2ikz} - \kappa_{s}^{*}e^{2ikz}\right),$$

(2.29)

$$f_{\pm}^{*}(z)f_{\mp}(z) = \left[e^{-ikz} \mp i\kappa_{s}^{*}e^{ikz}\right] \left[e^{ikz} \mp i\kappa_{s}e^{-ikz}\right]$$

$$= 1 - \kappa_{s}^{*}\kappa_{s} \mp i\kappa_{s}e^{-2ikz} \mp i\kappa_{s}^{*}e^{2ikz}.$$
(2.30)

From the last relation it follows that

$$f_{\pm}^{*}(z)f_{\mp}(z) + f_{\mp}^{*}(z)f_{\pm}(z) = 2\left(1 - \kappa_{s}^{*}\kappa_{s}\right).$$
(2.31)

Putting all these together

$$|g_{\pm}(z)|^{2} = \left[\left(\omega \pm m\right)^{2} + \lambda^{2} \right] \left(1 + \kappa_{s}^{*} \kappa_{s}\right) \pm i \left[\left(\omega \pm m\right)^{2} - \lambda^{2} \right] \left(\kappa_{s} e^{-2ikz} - \kappa_{s}^{*} e^{2ikz}\right) + 2s\lambda \left(\omega \pm m\right) \left(1 - \kappa_{s}^{*} \kappa_{s}\right),$$

$$(2.32)$$

which can be further simplifed by taking into account that

$$\kappa_s^* \kappa_s = \frac{\omega + s\lambda}{\omega - s\lambda},\tag{2.33}$$

and

$$\kappa_s e^{-2ikz} - \kappa_s^* e^{2ikz} = \frac{1}{\omega - s\lambda} \left[(k + im) e^{-2ikz} - (k - im) e^{2ikz} \right].$$
(2.34)

On the base of these relations one finds

$$|g_{\pm}(z)|^2 - k^2 |f_{\mp}(z)|^2 = \pm 2 \frac{\omega^2 - \lambda^2 \pm m\omega}{\omega - s\lambda} \left[2m + i \left[(k + im) e^{-2ikz} - (k - im) e^{2ikz} \right] \right].$$
(2.35)

Inserting this into the expression (2.27) we get

$$\psi_{\sigma}^{(-)+}\gamma^{0}\psi_{\sigma}^{(-)} = -\frac{2}{k^{2}}\left|C_{\chi1}^{(1)}\right|^{2}\left[2m+i\left[\left(k+im\right)e^{-2ikz}-\left(k-im\right)e^{2ikz}\right]\right] \\ \times \left\{\frac{\omega^{2}-\lambda^{2}}{\omega-s\lambda}\left[J_{\beta+\epsilon_{j}}^{2}(\lambda r)+J_{\beta}^{2}(\lambda r)\right]+\frac{m\omega}{\omega-s\lambda}\left[J_{\beta+\epsilon_{j}}^{2}(\lambda r)-J_{\beta}^{2}(\lambda r)\right]\right\}.$$

$$(2.36)$$

By taking into account that

$$\sum_{j=\pm 1/2,\dots} J_{\beta}^2(\lambda r) = \sum_{j=\pm 1/2,\dots} J_{\beta+\epsilon_j}^2(\lambda r) = \sum_j \left[J_{qj-1/2}^2(\lambda r) + J_{qj+1/2}^2(\lambda r) \right],$$
(2.37)

we can write for the condensate

$$\langle 0|\bar{\psi}\psi|0\rangle =$$

$$= -\frac{1}{\phi_0 a} \sum_j \int_0^\infty d\lambda \sum_{n=1}^\infty \frac{\lambda \left[J_{qj-1/2}^2(\lambda r) + J_{qj+1/2}^2(\lambda r)\right]}{\omega \left[1 - \frac{\sin(2ka)}{2ka}\right]}$$

$$\times \left[2m + i \left[(k+im) e^{-2ikz} - (k-im) e^{2ikz}\right]\right],$$
(2.38)

or

$$\langle \bar{\psi}\psi \rangle = -\frac{qm}{2\pi a} \sum_{j} \int_{0}^{\infty} d\lambda \,\lambda \sum_{n=1}^{\infty} \frac{J_{qj-1/2}^{2}(\lambda r) + J_{qj+1/2}^{2}(\lambda r)}{\omega \left[1 - \sin(2x_{n})/(2x_{n})\right]} h(k_{n}, z).$$
(2.39)

Here, $\omega = \sqrt{k_n^2 + \lambda^2 + m^2}$,

$$h(k,z) = 2 - \sum_{\eta=\pm 1} \left(1 + \eta i \frac{k}{m} \right) e^{2\eta i k z},$$
 (2.40)

and \sum_{j} stands for the summation over $j = 1/2, 3/2, \cdots$.

By using the integral representation

$$\omega = \frac{2}{\sqrt{\pi}} \int_0^\infty ds \, e^{-\omega^2 s^2},\tag{2.41}$$

the integral over λ in (2.39) is evaluated with the help of the formula [127]

$$\int_0^\infty d\lambda \,\lambda J_{qj\pm 1/2}^2(\lambda r)e^{-\lambda^2 s^2} = \frac{I_{qj\pm 1/2}(r^2/2s^2)}{2s^2 e^{r^2/2s^2}},\tag{2.42}$$

where $I_{\nu}(x)$ is the modified Bessel function. As a result, the fermionic condensate is presented in the form

$$\langle \bar{\psi}\psi \rangle = -\frac{qm}{(2\pi)^{3/2}ar} \sum_{n=1}^{\infty} \frac{h(x_n/a, z)}{1 - \sin(2x_n)/(2x_n)} \times \int_0^\infty dy \, y^{-1/2} e^{-(x_n^2/a^2 + m^2)r^2/2y - y} \mathcal{I}(q, y),$$
(2.43)

with the notation

$$\mathcal{I}(q,y) = \sum_{j} \left[I_{qj-1/2}(y) + I_{qj+1/2}(y) \right].$$
(2.44)

As the next step in the evaluation of the fermionic condensate, for the series (2.44) we use the representation [109]-[115], [128]

$$\mathcal{I}(q,x) = \frac{2}{q} \sum_{l=0}^{p} (-1)^{l} \cos(\pi l/q) e^{x \cos(2\pi l/q)} + \frac{2}{\pi} \cos(q\pi/2) \int_{0}^{\infty} du \frac{\sinh(qu/2)\sinh(u/2)}{\cosh(qu) - \cos(q\pi)} e^{-x \cosh u},$$
(2.45)

where p is the integer part of q/2, p = [q/2]. In (2.45), the prime on the sign of the sum means that the term l = 0 and the term l = p for even values of q should be taken with the coefficients 1/2. In the absence of the conical defect one has q = 1 and we get $\mathcal{I}(1, y) = e^y$. From here it follows that the contribution of the term l = 0 in (2.45) to $\langle \bar{\psi}\psi \rangle$ coincides with the fermionic condensate in the region between two plates in the Minkowski spacetime. Substituting (2.45) into (2.43), the integral over y is explicitly evaluated and one finds the following representation

$$\langle \bar{\psi}\psi \rangle = -\frac{m}{2\pi ar} \sum_{n=1}^{\infty} \frac{h(x_n/a, z)}{1 - \sin(2x_n)/(2x_n)} \left[\sum_{l=0}^{p} (-1)^l \frac{c_l}{s_l} e^{-2rs_l} \sqrt{k_n^2 + m^2} + \frac{2q}{\pi} \int_0^\infty dx \frac{\cos\left(q\pi/2\right) \sinh\left(qx\right) \sinh\left(x\right)}{\cosh(2qx) - \cos(q\pi)} \frac{e^{-2r\sqrt{k_n^2 + m^2} \cosh x}}{\cosh x} \right],$$
(2.46)

with the notations

$$s_l = \sin(\pi l/q), \ c_l = \cos(\pi l/q).$$
 (2.47)

For the general case of a massive field, the eigenvalues k_n in (2.46) are given implicitly, as roots of (2.15), and this form for the fermionic condensate is not convenient for the further investigation of its properties. For the summation over these eigenvalues we use the formula

$$\sum_{n=1}^{\infty} \frac{\pi f(x_n)}{1 - \sin(2x_n)/(2x_n)} = -\frac{\pi maf(0)}{2(ma+1)} + \int_0^{\infty} dx \, f(x) - i \int_0^{\infty} dx \frac{f(ix) - f(-ix)}{\frac{x+ma}{x-ma}e^{2x} + 1}, \qquad (2.48)$$

that is a consequence of the generalized Abel-Plana summation formula [163]. For the series in (2.46) the corresponding function is given by the expression

$$f(x) = h(x/a, z)e^{-2br\sqrt{x^2/a^2 + m^2}}.$$
(2.49)

For this function f(0) = 0 and the first term in the right-hand side of (2.48) is absent. The first integral in (2.48) is evaluated by making use of the formula

$$\int_{m}^{\infty} dx \sin(2y\sqrt{x^2 - m^2})e^{-2zx} = 2ym^2 f_1(2m\sqrt{z^2 + y^2}), \qquad (2.50)$$

with the notation

$$f_{\nu}(x) = \frac{K_{\nu}(x)}{x^{\nu}},$$
(2.51)

and the relation

$$\int_0^\infty dx \, \sum_{\eta=\pm 1} \left(m+\eta ix\right) e^{2\eta ixz} e^{-2y\sqrt{x^2+m^2}} = (2m+\partial_z) \int_m^\infty dx \, \sin(2y\sqrt{x^2-m^2}) e^{-2xz}.$$
 (2.52)

In deriving (2.52) we have rotated the integration contour of the left-hand side in the complex *x*-plane by the angle $\pi/2$ for $\eta = 1$ and by $-\pi/2$ for $\eta = -1$. As a result one finds the representation

$$\frac{\pi}{a} \sum_{n=1}^{\infty} \frac{h(x_n/a, z)e^{-2y\sqrt{x_n^2/a^2 + m^2}}}{1 - \sin(2x_n)/(2x_n)} = 4m^2 y \left[f_1(2my) - F(2mz, 2my)\right] - \frac{y}{m} B(z, y), \qquad (2.53)$$

where we have defined the functions

$$F(x,y) = f_1(\sqrt{x^2 + y^2}) - xf_2(\sqrt{x^2 + y^2}), \qquad (2.54)$$

and

$$B(z,y) = \frac{2}{y} \int_{m}^{\infty} dx \frac{\sin(2y\sqrt{x^2 - m^2})}{\frac{x + m}{x - m}e^{2ax} + 1} \left[2m - \sum_{\eta = \pm 1} (m + \eta x)e^{2\eta zx} \right].$$
 (2.55)

Here, the relation $f'_{\nu}(x) = -x f_{\nu+1}(x)$ is used for the function (2.51).

Substituting (2.53) into (2.46), the fermionic condensate is decomposed into three contributions. The first one comes from the term with $f_1(2my)$ in (2.53). This contribution does not depend on z and on a and corresponds to the fermionic condensate in the boundary-free conical defect spacetime. For points away from the boundaries, the only divergence in the fermionic condensate is contained in the l = 0 term of this contribution. The latter coincides with the fermionic condensate in boundary-free Minkowski spacetime. The renormalization is reduced to omitting this term. As a result, the renormalized fermionic condensate in the boundary-free conical defect geometry is presented as

$$\langle \bar{\psi}\psi \rangle_{\rm s} = -\frac{2m^3}{\pi^2} \bigg[\sum_{l=1}^p (-1)^l c_l f_1(2mrs_l) + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \\ \times \int_0^\infty dx \frac{\sinh(qx)\sinh(x)f_1(2mr\cosh x)}{\cosh(2qx) - \cos(q\pi)} \bigg].$$
(2.56)

This expression has been previously derived in [160] (for the generalization in the geometry of conical defect with a magnetic flux see [164]). For a massless field the boundary-free part (2.56) vanishes, $\langle \bar{\psi}\psi \rangle_s = 0$. For massive fields, the condensate $\langle \bar{\psi}\psi \rangle_s$ is positive everywhere. It diverges on the string as $1/r^2$ and is exponentially suppressed at large distances, $mr \gg 1$.

The contribution of the term with F(2mz, 2my) in (2.53) to the fermionic condensate does not depend on a whereas the contribution of the last term in the right-hand side of (2.53) vanishes in the limit $a \to \infty$. From here it follows that the part

$$\langle \bar{\psi}\psi \rangle_{\rm b}^{(1)} = \frac{2m^3}{\pi^2} \bigg[\sum_{l=0}^{p} (-1)^l c_l F(2mz, 2mrs_l) + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \\ \times \int_0^\infty dx \frac{\cos\left(q\pi/2\right) \sinh\left(qx\right) \sinh x}{\cosh(2qx) - \cos(q\pi)} F(2mz, 2mr\cosh x) \bigg],$$
(2.57)

is induced by the presence of the boundary at z = 0 when the second boundary is absent. Hence, in the geometry of a single boundary at z = 0 the renormalized fermionic condensate is decomposed as

$$\langle \bar{\psi}\psi\rangle^{(1)} = \langle \bar{\psi}\psi\rangle_{\rm s} + \langle \bar{\psi}\psi\rangle^{(1)}_{\rm b}.$$
(2.58)

This result, with (2.56) and (2.57), coincides with that obtained in [160]. Hence, in the region between two boundaries the fermionic condensate is written in the form

$$\langle \bar{\psi}\psi \rangle = \langle \bar{\psi}\psi \rangle^{(1)} + \frac{1}{2\pi^2} \left[\sum_{l=0}^{p} (-1)^l c_l B(z, rs_l) + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \int_0^\infty dx \frac{\sinh\left(qx\right)\sinh(x)B(z, r\cosh x)}{\cosh(2qx) - \cos(q\pi)} \right],$$
(2.59)

where the last term is induced by the presence of the second boundary at z = a.

By taking into account that

$$F(2mz, 2my) = -\frac{m^{-3}}{2y} \int_{m}^{\infty} dx \, \frac{x-m}{e^{2zx}} \sin(2y\sqrt{x^2-m^2}), \qquad (2.60)$$

and combining the single plate-induced part $\langle \bar{\psi}\psi \rangle_{\rm b}^{(1)}$ with the last term in (2.59), we find the decomposition

$$\langle \bar{\psi}\psi\rangle = \langle \bar{\psi}\psi\rangle_{\rm s} + \langle \bar{\psi}\psi\rangle_{\rm b}. \tag{2.61}$$

Here, the boundary-induced contribution to the fermionic condensate in the region between the plates is given by

$$\langle \bar{\psi}\psi \rangle_{\rm b} = \frac{1}{2\pi^2} \left[\sum_{l=0}^{p} (-1)^l c_l C(z, rs_l) + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \times \int_0^\infty dx \frac{\sinh\left(qx\right) \sinh(x) C(z, r\cosh x)}{\cosh(2qx) - \cos(q\pi)} \right],$$
(2.62)

with the function

$$C(z,y) = \frac{2}{y} \int_{m}^{\infty} dx \frac{\sin(2y\sqrt{x^2 - m^2})}{\frac{x + m}{x - m}e^{2ax} + 1} \left[2m - (m + x)\left(e^{2zx} + e^{2(a - z)x}\right)\right].$$
 (2.63)

Taking the limit $a \to \infty$, $z \to \infty$, with fixed a - z, and making use of the relation (2.60), one can see that from the second term in the right-hand side of (2.59) the boundary-induced part is obtained for a single plate at z = a. The latter is given by the expression (2.57) with the replacement $z \to a - z$.

For a massless field, m = 0, the function (2.63) can be presented in the form

$$C(z,y) = \frac{1}{y} \partial_z \left[-\frac{1}{a} \int_0^\infty du \frac{\sin(uy/a)}{e^u + 1} \cosh(uz/a) + \frac{1}{2} \int_0^\infty dx \sin(xy) e^{-zx} \right].$$
 (2.64)

For the integrals in this expression one has

$$\int_0^\infty du \frac{\sin(uy/a)}{e^u + 1} \cosh(uz/a) = \frac{ay}{2(z^2 + y^2)} - \frac{\pi \sinh(\pi y/a) \cos(\pi z/a)}{\cosh(2\pi y/a) - \cos(2\pi z/a)},$$
(2.65)

and

$$\int_0^\infty dx \sin(xy) e^{-zx} = \frac{y}{y^2 + z^2}.$$
(2.66)

Combining these results for a massless field one gets

$$C(z,y) = \frac{\pi}{ay} \partial_z \frac{\sinh(\pi y/a)\cos(\pi z/a)}{\cosh(2\pi y/a) - \cos(2\pi z/a)}.$$
(2.67)

In this special case the boundary-free part vanishes and for the single plate contribution one has (see [160])

$$\langle \bar{\psi}\psi \rangle^{(1)} = -\frac{q}{4\pi^2 r z^2} \frac{u^q}{u^{2q} - 1} \left(1 + \frac{qz}{\sqrt{r^2 + z^2}} \frac{u^{2q} + 1}{u^{2q} - 1} \right),$$
(2.68)

with the notation $u = z/r + \sqrt{1 + z^2/r^2}$.

For q = 1, in (2.62) the l = 0 term survives only and we get the corresponding result for two parallel plates in Minkowski bulk:

$$\langle \bar{\psi}\psi \rangle_{\rm M} = \frac{1}{\pi^2} \int_m^\infty dx \frac{\sqrt{x^2 - m^2}}{\frac{x + m}{x - m}e^{2ax} + 1} \left[2m - (m + x) \left(e^{2zx} + e^{2(a - z)x} \right) \right].$$
(2.69)

In this case the fermionic condensate is negative. The remaining part in (2.62) is induced by the presence of the conical defect. In particular, for a massless field the Minkowskian part is simplified to

$$\langle \bar{\psi}\psi \rangle_{\rm M} = -\pi \frac{2 - \sin^2(\pi z/a)}{8a^3 \sin^3(\pi z/a)}.$$
 (2.70)

For points near the plate z = 0, to the leading order, we get

$$\langle \bar{\psi}\psi \rangle_{\rm M} \approx -\frac{1}{4\pi^2 z^3}.$$
 (2.71)

This expression gives the leading term in the asymptotic expansion for the fermionic condensate near the plate for massive fields as well. Moreover, to the leading order and under the condition $z \ll r$, the expression in the right-hand side of (2.71) gives the asymptotic in the presence of the conical defect: $\langle \bar{\psi}\psi \rangle \approx -1/(4\pi^2 z^3)$. The generalization of (2.69) for arbitrary number of spatial dimensions is given in [165]. The fermionic condensate for the geometry of two parallel on AdS bulk has been investigated in [166].

For points outside the plates, $z \neq 0, a$, the boundary-induced contribution (2.62) is finite on the defect. By taking into account that $\lim_{y\to 0} C(z,y) = 4\pi^2 \langle \bar{\psi}\psi \rangle_M$, we obtain

$$\langle \bar{\psi}\psi \rangle_{\mathbf{b},r=0} = 2\langle \bar{\psi}\psi \rangle_M \left[\sum_{l=0}^{p'} (-1)^l c_l + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \int_0^\infty dx \frac{\sinh\left(qx\right) \sinh x}{\cosh(2qx) - \cos(q\pi)} \right].$$
(2.72)

The expression in the square brackets vanishes and, hence, the boundary-induced part in the fermionic condensate vanishes on the string. From here it follows that for points near the string and for a massive field the total fermionic condensate is dominated by the boundary-free contribution and is positive.

Now let us consider the asymptotic behavior of the fermionic condensate at large distances from the defect, $r \gg a, m^{-1}$. Here we need the asymptotic of the function C(z, y) for large values of the second argument. It is obtained by taking into account that for large y the dominant contribution to the integral in (2.63) comes from the region near the lower limit of the integration and we can write

$$C(z,y) \approx -\frac{2m^2}{y} \left[e^{2zm} + e^{2(a-z)m} - 1 \right] \int_1^\infty du \sin(2ym\sqrt{u^2 - 1}) \left(u - 1\right) e^{-2amu}.$$
 (2.73)

The integral is expressed in terms of the Macdonald functions $K_{1,2}(2m\sqrt{a^2+y^2})$. By using the corresponding asymptotic expression for large arguments, we can see that

$$C(z,y) \approx \sqrt{\pi}m^3 \left[e^{2zm} + e^{2(a-z)m} - 1 \right] \frac{e^{-2my}}{(my)^{3/2}},$$
(2.74)

for $y \gg a, m^{-1}$. At large distances, the leading contribution in the boundary-induced part of the fermionic condensate comes from the term l = 0 which coincides with $\langle \bar{\psi}\psi \rangle_{\rm M}$. For q > 2, the next to the leading term comes from the term l = 1 in (2.62). Adding the asymptotic for the boundary-free contribution one gets

$$\langle \bar{\psi}\psi \rangle \approx \langle \bar{\psi}\psi \rangle_{\rm M} - \left[e^{2zm} + e^{2(a-z)m} - 2\right] \frac{m^3 \cos\left(\pi/q\right) e^{-2mr \sin(\pi/q)}}{2(\pi mr \sin(\pi/q))^{3/2}}.$$
 (2.75)

The topological part, given by the second term in the right-hand side, is negative. For a massless field the asymptotic expression is directly obtained by using (2.67):

$$\langle \bar{\psi}\psi \rangle \approx \langle \bar{\psi}\psi \rangle_{\mathrm{M}} + \frac{\cot(\pi/q)\sin(\pi z/a)}{2\pi a^2 r e^{\pi r \sin(\pi/q)/a}}.$$
 (2.76)

Note that in this case the topological part is positive. For $q \leq 2$, the decay of the topological part in the boundary-induced fermion condensate at large distances from the string is stronger, like e^{-2mr} and $e^{-\pi r/a}$ for the massive and massless cases, respectively. Hence, at large distances from the string, to the leading order, the fermionic condensate coincides with that for the plates in Minkowski bulk and is negative. As it has been shown before, the condensate is positive near the string. As a consequence, the fermionic condensate vanishes for some intermediate value of the radial coordinate r. Note that at large distances from the string the topological part in the region between the plates is suppressed exponentially for both massive and massless fields.

Let us denote the part in the fermionic condensate induced by the conical defect as $\langle \bar{\psi}\psi \rangle_t$. It is given by

$$\langle \bar{\psi}\psi \rangle_{\rm t} = \langle \bar{\psi}\psi \rangle - \langle \bar{\psi}\psi \rangle_{\rm M} \tag{2.77}$$

and can be termed as topological part. The corresponding expression is obtained from (2.61) omitting the l = 0 term in the boundary-induced contribution (2.62). For a massless field, by taking into account that for $y \neq 0$ one has C(0, y) = C(a, y) = 0 (see (2.67)), we conclude that the topological part in the fermionic condensate vanishes on the plates. The same is the case for a massive field. In order to show that, we firstly consider the topological part in the boundary-induced contribution (2.62). This part is obtained from (2.62) omitting the term l = 0. For the evaluation of the topological part we cannot directly put z = 0, a in the corresponding expression. The limit $z \to 0, a$ should be taken after the evaluation of the integral for $z \neq 0, a$. In fact, we need to evaluate the limits $\lim_{z\to 0,a} C(z, y)$ for $y \neq 0$. These limits are the same and we will consider the case $z \to 0$ only. The evaluation procedure is simplified if we introduce in the integrand the factor $e^{-\alpha\sqrt{x^2-m^2}}$ with $\alpha > 0$ and take the limit $\alpha \to 0$ after the evaluation of the integral. With this factor we can directly put z = 0 in the integrand. In this manner it can be seen that

$$\lim_{z \to 0} C(z, y) = 4m^3 f_1(2my), \tag{2.78}$$

for $y \neq 0$. Comparing with the boundary-free fermionic condensate (2.56), we see that the boundary-induced contribution in the topological part exactly cancels the boundary-free part and the topological part of the fermionic condensate vanishes on the plates. This feature is seen from Fig. 10, where, for a massless field and for the value of the parameter q = 3, we have plotted the topological part in the fermionic condensate, $a^3 \langle \bar{\psi} \psi \rangle_t$, as a function of the distance from the string and of the distance from the plate at z = 0.

An alternative representation for the fermionic condensate is obtained by the application of the summation formula (2.48) to the series over n in (2.39). The part with the first integral in the right-hand side of (2.48) will give the fermion condensate for the geometry of a single plate at z = 0. In the corresponding expression we first separate the boundary-free contribution, obtained from the first term in the right-hand side of (2.40), and in the remaining part we rotate the integration contour over k by the angle $\pi/2$ ($-\pi/2$) for the term with e^{2ikz} (e^{-2ikz}). Combining with the term coming from the second integral in (2.48), for the fermionic condensate



Figure 10: The topological part in the fermionic condensate, $a^3 \langle \bar{\psi}\psi \rangle_{cs}$, for a massless field as a function of the distances from the string and from the left plate. For the planar angle deficit we have taken the value corresponding to q = 3.

in the region between the plates we obtain the representation

$$\langle \bar{\psi}\psi \rangle = \langle \bar{\psi}\psi \rangle_{\rm s} - \frac{q}{\pi^2} \sum_j \int_0^\infty d\lambda \,\lambda \left[J_{qj-1/2}^2(\lambda r) + J_{qj+1/2}^2(\lambda r) \right] \\ \times \int_{\sqrt{\lambda^2 + m^2}}^\infty dx \frac{(x+m) \left[e^{2xz} + e^{2(a-z)x} \right] - 2m}{\sqrt{x^2 - \lambda^2 - m^2} \left(\frac{x+m}{x-m} e^{2ax} + 1 \right)}.$$

$$(2.79)$$

As is seen from this formula, the boundary-induced contribution to the fermionic condensate, given by the second term in the right-hand side of (2.79), is always negative. In the limit $r \rightarrow 0$, the dominant contribution to the boundary-induced part in (2.79) comes from the term j = 1/2and, by making use of the asymptotic for the Bessel function for small values of the argument, to the leading order we get

$$\begin{aligned} \langle \bar{\psi}\psi \rangle &\approx \langle \bar{\psi}\psi \rangle_{\rm s} - \frac{\pi^{-3/2}(r/2)^{q-1}}{\Gamma(q/2)\Gamma((q+1)/2)} \int_m^\infty dx \, \frac{(x^2 - m^2)^{q/2}}{\frac{x+m}{x-m}e^{2ax} + 1} \\ &\times \left[(x+m) \, (e^{2xz} + e^{2(a-z)x}) - 2m \right]. \end{aligned} \tag{2.80}$$

For q = 1 this asymptotic formula is reduced to the exact result (2.69).

In a number of field theoretical models, the Lagrangian, in addition to the fermionic mass term $m\bar{\psi}\psi$, contains other terms involving the product $\bar{\psi}\psi$. An example is the fermionc field nonminimally coupled to gravity with the interaction term $\xi R \bar{\psi} \psi$ (see, e.g., [167, 168]), where R is the scalar curvature for the background spacetime and ξ is a constant with the dimension of the inverse mass. The formation of nonzero fermionic condensate induces an additional term $\xi R \langle \bar{\psi} \psi \rangle$ in the Lagrangian for the gravitational field. This leads to the shift (in general, space-dependent) in the gravitational constant. Another example is the fermionic field interacting with a scalar field φ through the interaction Lagrangian proportional to $\varphi \bar{\psi} \psi$. The interaction terms of this form appear also in semibosonized versions of the four-fermion interaction models of the Gross-Neveu and Nambu-Jona-Lasinio type. The effects of the background geometry, topology and boundaries in these models have been discussed in [169], [170]-[173].

2.3 Energy-momentum tensor

The mode sum for the VEV of the energy-momentum tensor, $\langle 0 | T_{\mu\nu} | 0 \rangle \equiv \langle T_{\mu\nu} \rangle$, is given by the expression

$$\langle T_{\mu\nu} \rangle = \frac{i}{2} \sum_{s=\pm 1} \sum_{j=\pm 1/2,\dots} \sum_{n=1}^{\infty} \int_{0}^{\infty} d\lambda \left[\bar{\psi}_{\sigma}^{(-)} \gamma_{(\mu} \nabla_{\nu)} \psi_{\sigma}^{(-)} - (\nabla_{(\mu} \bar{\psi}_{\sigma}^{(-)}) \gamma_{\nu)} \psi_{\sigma}^{(-)} \right], \qquad (2.81)$$

where $\gamma_{\mu} = g_{\mu\nu}\gamma^{\nu}$ and the brackets enclosing the indices stand for the symmetrization. Similar to the case of the condensate, we assume that some regularization procedure is used for the right-hand side of (2.81) without explicitly writing it. Inserting the expression for the negativeenergy mode functions, one can see that all off-diagonal components vanish. The VEVs for the diagonal components are presented in the form (no summation over ν):

$$\langle T_{\nu}^{\nu} \rangle = \frac{q}{\pi a} \sum_{j} \sum_{n=1}^{\infty} \int_{0}^{\infty} d\lambda \frac{\lambda^{3}}{\omega} \frac{g_{qj-1/2}^{(\nu)}(\lambda r)h_{\nu}(k_{n}, z)}{1 - \sin(2x_{n})/(2x_{n})},$$
 (2.82)

with the notations

$$g_{\beta}^{(0)}(y) = -\frac{\omega^2}{k^2} g_{\beta}^{(3)}(y) = -\frac{\omega^2}{\lambda^2} [J_{\beta}^2(y) + J_{\beta+1}^2(y)],$$

$$g_{\beta}^{(1)}(y) = J_{\beta}^2(y) + J_{\beta+1}^2(y) - \frac{2qj}{y} J_{\beta}(y) J_{\beta+1}(y),$$

$$g_{\beta}^{(2)}(y) = \frac{2qj}{y} J_{\beta}(y) J_{\beta+1}(y),$$

(2.83)

and

$$h_{\nu}(k,z) = 1 - \frac{1 - \delta_{3}^{\nu}}{2} \sum_{\eta=\pm 1} \frac{m e^{2\eta i k z}}{m - \eta i k}.$$
(2.84)

Note that the axial stress does not depend on the coordinate z. We can see that the VEVs (2.82) obey the trace relation

$$\langle T_{\nu}^{\nu} \rangle = m \langle \bar{\psi} \psi \rangle. \tag{2.85}$$

We start the further transformation of the energy-momentum tensor with the energy density. The corresponding mode sum (2.81) is rewritten in the form

$$\langle T_0^0 \rangle = -\frac{q}{\pi a} \sum_{n=1}^{\infty} \frac{h_0(x_n/a, z)}{1 - \sin(2x_n)/(2x_n)} \\ \times \sum_j \int_0^\infty d\lambda \,\lambda\omega \left[J_{qj-1/2}^2(\lambda r) + J_{qj+1/2}^2(\lambda r) \right],$$
(2.86)

with the notation $h_0(k, z)$ defined in accordance with (2.84). For the further transformation we use the integral representation

$$\omega = -\frac{2}{\sqrt{\pi}} \int_0^\infty ds \partial_{s^2} e^{-\omega^2 s^2}.$$
(2.87)

Integrating over λ with the help of (2.42) and after the integration by parts in the integral over s, one gets

$$\langle T_0^0 \rangle = \frac{\pi^{-3/2} q}{\sqrt{2} r^3 a} \sum_{n=1}^{\infty} \frac{h_0(x_n/a, z)}{1 - \sin(2x_n)/(2x_n)} \\ \times \int_0^{\infty} dy \, y^{1/2} e^{-(x_n^2/a^2 + m^2)r^2/2y - y} \mathcal{I}(q, y).$$

$$(2.88)$$

As the next step the integral representation (2.45) is employed. After the integration over y, for the energy density this yields

$$\langle T_0^0 \rangle = -\frac{1}{4\pi ar} \partial_r \frac{1}{r} \sum_{n=1}^{\infty} \frac{h_0(x_n/a, z)}{1 - \sin(2x_n)/(2x_n)} \left[\sum_{l=0}^{p} (-1)^l \frac{c_l}{s_l^3} e^{-2rs_l} \sqrt{x_n^2/a^2 + m^2} + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \int_0^\infty du \frac{\sinh(qu)\sinh u}{\cosh(2qu) - \cos(q\pi)} \frac{e^{-2r} \sqrt{x_n^2/a^2 + m^2}\cosh u}{\cosh^3 u} \right].$$

$$(2.89)$$

Now let us consider the VEV of the azimuthal stress. By taking into account the relation

$$J_{\nu}(\lambda r)J_{\nu+1}(\lambda r) = \left(\frac{\nu}{r} - \frac{1}{2}\frac{d}{dr}\right)\frac{1}{\lambda}J_{\nu}^{2}(\lambda r), \qquad (2.90)$$

for the Bessel function, the corresponding expression is presented in the form

$$\langle T_2^2 \rangle = -\frac{2q^2}{\pi ar} \sum_{n=1}^{\infty} \frac{h_2(x_n/a, z)}{1 - \sin(2x_n)/(2x_n)} \\ \times \sum_j j \left(\frac{qj - 1/2}{r} - \frac{1}{2} \partial_r \right) \int_0^\infty d\lambda \, \frac{\lambda}{\omega} J_{qj-1/2}^2(\lambda r).$$
(2.91)
By using the integral representation (2.41), after the integration over λ , the VEV is rewritten as

$$\langle T_2^2 \rangle = \frac{4q^2 r^{-3}}{(2\pi)^{3/2} a} \sum_{n=1}^{\infty} \frac{h_2(x_n/a, z)}{1 - \sin(2x_n)/(2x_n)} \sum_j j \int_0^\infty dy$$

$$\times y^{-1/2} e^{-(k^2 + m^2)r^2/2y} \left(qj - 1/2 - y\partial_y\right) e^{-y} I_{qj-1/2}(y).$$

$$(2.92)$$

Now, with the help of the relation

$$(qj - 1/2 - y\partial_y) e^{-y} I_{qj-1/2}(y) = \frac{y}{qj} e^{-y} (y\partial_y - y + 1/2) \left[I_{qj-1/2}(y) + I_{qj+1/2}(y) \right], \quad (2.93)$$

the series over j is expressed in terms of the function $\mathcal{I}(q, y)$:

$$\langle T_2^2 \rangle = \frac{2qr^{-3}}{(2\pi)^{3/2} a} \sum_{n=1}^{\infty} \frac{h_2(x_n/a, z)}{1 - \sin(2x_n)/(2x_n)} \int_0^\infty dy$$

$$\times e^{-(k^2 + m^2)r^2/2y - y} y^{1/2} (y\partial_y - y + 1/2) \mathcal{I}(q, y).$$

$$(2.94)$$

Using the representation (2.45) one gets

$$\langle T_2^2 \rangle = \frac{-1}{2\pi a r^3} (1 - r\partial_r + \frac{1}{2} r^2 \partial_r^2) \sum_{n=1}^{\infty} \frac{h_2(x_n/a, z)}{1 - \sin(2x_n)/(2x_n)} \left[\sum_{l=0}^{p'} (-1)^l \frac{c_l}{s_l^3} e^{-2rs_l} \sqrt{x_n^2/a^2 + m^2} + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \int_0^\infty dx \frac{\sinh(qx) \sinh x}{\cosh(2qx) - \cos(q\pi)} \frac{e^{-2r\sqrt{x_n^2/a^2 + m^2} \cosh x}}{\cosh^3 x} \right].$$

$$(2.95)$$

For the axial stress the mode sum is reduced to

$$\langle T_3^3 \rangle = \frac{q}{\pi a^3} \sum_j \int_0^\infty d\lambda \,\lambda \sum_{n=1}^\infty \frac{x_n^2}{\omega_n} \frac{J_{qj-1/2}^2(\lambda r) + J_{qj+1/2}^2(\lambda r)}{1 - \sin(2x_n)/(2x_n)}.$$
 (2.96)

With the help of (2.41), after the integration over λ , we find

$$\left\langle T_3^3 \right\rangle = \frac{qa^{-3}}{\sqrt{2}\pi^{3/2}r} \sum_{n=1}^{\infty} \frac{x_n^2}{1 - \sin(2x_n)/(2x_n)} \int_0^\infty dy y^{-1/2} e^{-\left(m^2 + k^2\right)r^2/2y - y} \mathcal{I}(q, y).$$
(2.97)

Taking into account (2.45) and integrating over y one gets the representation

$$\langle T_3^3 \rangle = \frac{1}{\pi a^3 r} \sum_{n=1}^{\infty} \frac{x_n^2}{1 - \sin(2x_n)/(2x_n)} \left[\sum_{l=0}^{p} \frac{(-1)^l \cot(\pi l/q)}{e^{2rs_l}\sqrt{x_n^2/a^2 + m^2}} + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \int_0^\infty dx \frac{\sinh(qx) \tanh x}{\cosh(2qx) - \cos(q\pi)} e^{-2r\sqrt{x_n^2/a^2 + m^2} \cosh x} \right].$$

$$(2.98)$$

The expression for the radial stress is obtained from the trace relation (2.85).

Combining the expressions obtained above the vacuum energy-momentum tensor is transformed to the form

$$\langle T_{\nu}^{\nu} \rangle = -\frac{1}{4\pi a} \sum_{n=1}^{\infty} \frac{1}{1 - \sin(2x_n)/(2x_n)} \bigg[\sum_{l=0}^{p} (-1)^l c_l D_{\nu}(z, rs_l, k_n) + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \int_0^{\infty} du \, \frac{\sinh(qu) \sinh(u) D_{\nu}(z, r\cosh u, k_n)}{\cosh(2qu) - \cos(q\pi)} \bigg],$$

$$(2.99)$$

with the notation

$$D_{\nu}(z, y, k) = \hat{D}_{\nu} e^{-2y\sqrt{k^2 + m^2}} h_{\nu}(k, z), \qquad (2.100)$$

where we have defined the operators

$$\hat{D}_{0} = \hat{D}_{1} = y^{-3}(y\partial_{y} - 1),$$

$$\hat{D}_{2} = 2y^{-3}(1 - y\partial_{y} + \frac{1}{2}y^{2}\partial_{y}^{2}),$$

$$\hat{D}_{3} = y^{-1}\left(4m^{2} - \partial_{y}^{2}\right).$$
(2.101)

For these operators one has the relation

$$\sum_{\nu=0}^{3} \hat{D}_{\nu} = 4m^2/y.$$
 (2.102)

As is seen from (2.101), the radial stress is equal to the energy density.

Now we apply to the series over n in (2.99) the summation formula (2.48) with the function

$$f(x) = e^{-2y\sqrt{x^2/a^2 + m^2}} h_{\nu}(x/a, z).$$
(2.103)

As a result, the VEVs are decomposed as

$$\langle T_{\nu}^{\nu} \rangle = \langle T_{\nu}^{\nu} \rangle^{(1)} + \frac{1}{2\pi^2} \bigg[\sum_{l=0}^{p} (-1)^l c_l B_{\nu}(z, rs_l) + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \int_0^{\infty} du \, \frac{\sinh\left(qu\right) \sinh(u) B_{\nu}(z, r\cosh u)}{\cosh(2qu) - \cos(q\pi)} \bigg],$$

$$(2.104)$$

with the function

$$B_{\nu}(z,y) = \hat{D}_{\nu} \int_{m}^{\infty} dx \frac{\sin(2y\sqrt{x^{2}-m^{2}})}{\frac{x+m}{x-m}e^{2ax}+1} \left(1 - \frac{1-\delta_{3}^{\nu}}{2} \sum_{\eta=\pm 1} \frac{me^{2\eta xz}}{m-\eta x}\right).$$
(2.105)

The first term in the right-hand side of (2.104) comes from the first integral in (2.48) and is given by the expression

$$\langle T_{\nu}^{\nu} \rangle^{(1)} = -\frac{1}{4\pi^2} \bigg[\sum_{l=0}^{p} (-1)^l c_l B_{\nu}^{(0)}(z, rs_l) + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \\ \times \int_0^{\infty} du \, \frac{\sinh\left(qu\right) \sinh(u) B_{\nu}^{(0)}(z, r\cosh u)}{\cosh(2qu) - \cos(q\pi)} \bigg],$$

$$(2.106)$$

where

$$B_{\nu}^{(0)}(z,y) = \hat{D}_{\nu} \int_{0}^{\infty} dx \, e^{-2y\sqrt{x^{2}+m^{2}}} h_{\nu}(x,z).$$
(2.107)

The contribution (2.106) does not depend on a, whereas the second term in the right-hand side of (2.104) vanishes in the limit $a \to \infty$. From here it follows that the part $\langle T_{\nu}^{\nu} \rangle^{(1)}$ corresponds to the VEV in the geometry of a single plate at z = 0.

The further transformation of the contribution $\langle T_{\nu}^{\nu} \rangle^{(1)}$ is similar to that we have used for $\langle \bar{\psi}\psi \rangle^{(1)}$ in the previous section. We decompose $\langle T_{\nu}^{\nu} \rangle^{(1)}$ into two parts coming from the first and second terms in the right-hand side of (2.84). The part with $h_{\nu}(x, z) \rightarrow 1$ corresponds to the VEV in the boundary-free cosmic string spacetime. In this part the corresponding integral (2.107) is expressed in terms of the Macdonald function and the renormalization is reduced to omitting the l = 0 term. The corresponding renormalization VEV is presented as (no summation over ν)

$$\langle T_{\nu}^{\nu} \rangle_{\rm s} = \frac{2m^4}{\pi^2} \bigg[\sum_{l=1}^p (-1)^l c_l F_{\nu}^{(0)}(2mrs_l) + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \\ \times \int_0^\infty dx \frac{\sinh\left(qx\right) \sinh(x) F_{\nu}^{(0)}(2mr\cosh x)}{\cosh(2qx) - \cos(q\pi)} \bigg],$$
(2.108)

where we have defined the functions

$$F_0^{(0)}(y) = F_1^{(0)}(y) = F_3^{(0)}(y) = f_2(y),$$

$$F_2^{(0)}(y) = -3f_2(y) - f_1(y),$$
(2.109)

with the notation (2.51). The corresponding energy density is negative everywhere. An alternative integral representation is given in [108]. The fermionic VEVs $\langle T^{\nu}_{\nu} \rangle_{\rm s}$ in the special case q < 2 have been previously investigated in [107]. For a massless field the renormalized VEV of the energy-momentum tensor in the boundary-free conical defect spacetime was found in [92, 93] and is obtained from (2.108) taking the limit $m \to 0$:

$$\langle T_{\nu}^{\nu} \rangle_{\rm s} = -\frac{(q^2 - 1)(7q^2 + 17)}{2880\pi^2 r^4} {\rm diag}(1, 1, -3, 1).$$
 (2.110)

Note that in the boundary-free conical defect spacetime one has the relation $\langle T_0^0 \rangle_s = \langle T_3^3 \rangle_s$. The latter is a consequence of the boost invariance along the axis of the defect. For a massive field and for points near the string, $mr \ll 1$, the renormalized VEV diverges on the string as r^{-4} with the leading term given by (2.110). At large distances, $mr \gg 1$, for q < 2 the first term in the square brackets in (2.108) is absent and the dominant contribution in the second term comes from the region near the lower limit of the integration. In this case $\langle T_{\nu}^{\nu} \rangle_{\rm s}$ is suppressed by the factor e^{-2mr} . For q > 2 the dominant contribution at large distances comes from the l = 1 term in (2.108) and the suppression factor is $e^{-2mr\sin(\pi/q)}$. Note that the contribution of the first term in the square brackets of (2.108) is always negative for $\nu = 0, 1, 3$, and positive for $\nu = 2$. It can be easily checked that the boundary-free contribution (2.108) separately obeys the trace relation (2.85).

The contribution in (2.106), coming from the second term in the right-hand side of (2.84), is induced by the boundary at z = 0 when the second boundary is absent. After the rotation of the integration contours in the integral (2.107) one gets

$$\int_{0}^{\infty} dx \, e^{-2y\sqrt{x^2 + m^2}} \sum_{\eta = \pm 1} \frac{m e^{2\eta i k z}}{m - \eta i k} = 2m G(2mz, 2my), \tag{2.111}$$

where

$$G(2mz, 2my) = \int_{m}^{\infty} dx \, \frac{e^{-2xz}}{m+x} \sin(2y\sqrt{x^2 - m^2}).$$
(2.112)

An equivalent expression is obtained by making use of the representation

$$\frac{1}{m+x} = \int_0^\infty dt \, e^{-(m+x)t}.$$
(2.113)

The integral over x is expressed in terms of the Macdonald function and we find

$$G(u, v) = vG_1(u, v).$$
 (2.114)

Here and below we use the notation

$$G_{\nu}(u,v) = e^{u} \int_{u}^{\infty} dx \, e^{-x} f_{\nu}(\sqrt{x^{2} + v^{2}}).$$
(2.115)

As a result, the VEV in the geometry of the single plate at z = 0 is decomposed as (no summation over ν)

$$\left\langle T_{\nu}^{\nu}\right\rangle^{(1)} = \left\langle T_{\nu}^{\nu}\right\rangle_{\rm s} + \left\langle T_{\nu}^{\nu}\right\rangle_{\rm b}^{(1)},\tag{2.116}$$

where the boundary-induced contribution is given by

$$\langle T_{\nu}^{\nu} \rangle_{\rm b}^{(1)} = \frac{2m^4}{\pi^2} \bigg[\sum_{l=0}^{p} (-1)^l c_l F_{\nu}(2mz, 2mrs_l) + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \\ \times \int_0^{\infty} dx \frac{\sinh\left(qx\right) \sinh x}{\cosh(2qx) - \cos(q\pi)} F_{\nu}(2mz, 2mr\cosh x) \bigg],$$

$$(2.117)$$

with the functions

$$F_0(x,y) = F_1(x,y) = -G_2(x,y),$$

$$F_2(x,y) = 2G_2(x,y) + F(x,y),$$
(2.118)

and $F_3(x, y) = 0$. As we see, the boundary-induced part in the axial stress vanishes. Note that we have the relation

$$F_{\nu}(2mz, 2my) = \frac{1 - \delta_3^{\nu}}{8m^3} \hat{D}_{\nu} G(2mz, 2my).$$
(2.119)

By using this relation, we can see that in the limit $a \to \infty$, $z \to \infty$, with fixed a - z, from the second term in the right-hand side of (2.104) we obtain the boundary-induced part in the geometry of a single plate at z = a. The latter is given by (2.117) with the replacement $z \to a - z$.

Taking into account (2.119) and combining the last term in the right-hand side of (2.104) with the single plate-induced part (2.117), one gets the following representation

$$\langle T_{\nu}^{\nu} \rangle = \langle T_{\nu}^{\nu} \rangle_{\rm s} + \langle T_{\nu}^{\nu} \rangle_{\rm b} \,. \tag{2.120}$$

Here the boundary-free part is given by (2.108) and for the boundary-induced contribution in the region between the plates one has the expression

$$\langle T_{\nu}^{\nu} \rangle_{\rm b} = \frac{1}{2\pi^2} \bigg[\sum_{l=0}^{p} (-1)^l c_l C_{\nu}(z, rs_l) + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \\ \times \int_0^{\infty} du \, \frac{\sinh\left(qu\right) \sinh(u) C_{\nu}(z, r\cosh u)}{\cosh(2qu) - \cos(q\pi)} \bigg],$$
(2.121)

with the notation

$$C_{\nu}(z,y) = \hat{D}_{\nu} \int_{m}^{\infty} dx \frac{\sin(2y\sqrt{x^{2}-m^{2}})}{\frac{x+m}{x-m}e^{2ax}+1} \left[1 + \frac{m}{2}\left(1-\delta_{3}^{\nu}\right)\frac{e^{2xz}+e^{2(a-z)x}}{x-m}\right].$$
 (2.122)

As we could expect, in the region between the plates the VEVs are symmetric with respect to the plane z = a/2. For a massless field one obtains

$$C_{\nu}(z,y) = \frac{\pi}{4a} \hat{D}_{\nu} \left[\frac{1}{\pi y/a} - \frac{1}{\sinh(\pi y/a)} \right].$$
 (2.123)

Note that in the latter case the single plate part in the VEV of the energy-momentum tensor vanishes and in the region between the plates the VEV of the energy-momentum tensor does not depend on the coordinate z.

It can be checked that the boundary-induced contribution obeys the covariant conservation equation $\nabla_{\mu} \langle T^{\mu}_{\nu} \rangle_{\rm b} = 0$. For the geometry under consideration the latter is reduced to a single equation $\partial_r \left(r \langle T^1_1 \rangle_{\rm b} \right) = \langle T^2_2 \rangle_{\rm b}$. The latter equation directly follows from the relation

$$(y\partial_y + 1)\hat{D}_1\sin(by) = \hat{D}_2\sin(by).$$
 (2.124)

By taking into account the relation (2.102) we can see that $\sum_{\nu=0}^{3} C_{\nu}(z, y) = mC(z, y)$. From here it follows that the boundary induced contributions obey the trace relation (2.85).

For the further transformation of the VEV (2.121) in the case of a massless field, we first separate the Minkowskian part (the l = 0 term). It can be seen that the contribution of the first term in the square brackets of (2.123) to the remaining (topological) part of $\langle T_{\nu}^{\nu} \rangle_{\rm b}$ is equal to $-\langle T_{\nu}^{\nu} \rangle_{\rm s}$. Hence, in the total VEV, this contribution is cancelled by the boundary-free part and for the total VEV in the massless case we get

$$\langle T_{\nu}^{\nu} \rangle = \langle T_{\nu}^{\nu} \rangle_{\mathrm{M}} - \frac{\pi^{2}}{8a^{4}} \bigg[\sum_{l=1}^{p} (-1)^{l} c_{l} A_{\nu}(\pi s_{l} r/a) + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \\ \times \int_{0}^{\infty} du \, \frac{\sinh\left(qu\right) \sinh(u) A_{\nu}(\pi(r/a) \cosh u)}{\cosh(2qu) - \cos(q\pi)} \bigg],$$

$$(2.125)$$

with the functions

$$A_{\nu}(y) = \hat{D}_{\nu} \frac{1}{\sinh y}.$$
 (2.126)

For a massless field from (2.102) one has $\sum_{\nu=0}^{3} \hat{D}_{\nu} = 0$ and both the Minkowskian and topological parts in (2.125) are traceless.

In the absence of the conical defect, the only nonzero contribution to (2.121) comes from the term with l = 0 and we obtain the VEVs in the region between two plates in Minkowski bulk (no summation over ν):

$$\langle T_{\nu}^{\nu} \rangle_{\mathrm{M}} = -\frac{2}{3\pi^{2}} \int_{m}^{\infty} dx \frac{(x^{2} - m^{2})^{3/2}}{\frac{x + m}{x - m}e^{2ax} + 1} \left[1 + \frac{m}{2} \frac{e^{2xz} + e^{2(a-z)x}}{x - m} \right],$$

$$\langle T_{3}^{3} \rangle_{\mathrm{M}} = \frac{2}{\pi^{2}} \int_{m}^{\infty} dx \frac{x^{2} \sqrt{x^{2} - m^{2}}}{\frac{x + m}{x - m}e^{2ax} + 1},$$

$$(2.127)$$

where $\nu = 0, 1, 2$. In particular, the energy density is negative. For a massless field we have $\langle T_3^3 \rangle_{\rm M} = -3 \langle T_0^0 \rangle_{\rm M} = 7\pi^2/(960a^4)$. The fermionic Casimir densities for the geometry of two parallel plates in background of flat spacetime with an arbitrary number toroidally compactified spatial dimensions have been investigated in [165]. The expressions (2.127) are special cases

of the corresponding general formulas. The VEV of the energy-momentum tensor for parallel branes on AdS bulk is investigated in [166]. The fermionic Casimir energy for two parallel plates in 4-dimensional Minkowski spacetime has been studied in [118, 174]. The corresponding result for an arbitrary number of dimensions is generalized in [175, 176]. The influence of the compactification of spatial dimensions on the fermionic Casimir energy has been discussed in [68, 69].

Let us consider the boundary-induced contribution in the VEV of the energy-momentum tensor in the asymptotic regions. For points outside the plates, the boundary-induced part is finite on the string. Taking the limit $r \to 0$ it can be seen that $C_{\nu}(z,0) = 4\pi^2 \langle T^{\nu}_{\nu} \rangle_{\rm M}$. Hence, from (2.121) one gets

$$\langle T_{\nu}^{\nu} \rangle_{\mathbf{b},r=0} = 2 \langle T_{\nu}^{\nu} \rangle_{\mathbf{M}} \left[\sum_{l=0}^{p} (-1)^{l} c_{l} + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \int_{0}^{\infty} dx \, \frac{\sinh\left(qx\right) \sinh x}{\cosh(2qx) - \cos(q\pi)} \right].$$
(2.128)

Note that the expression in the square brackets is the same as that in the corresponding formula (2.72) for the fermionic condensate. This expression is zero and, hence, the boundary-induced contribution in the VEV of the energy-momentum tensor vanishes on the string. This means that in the region near the conical defect the VEV of the energy-momentum tensor is dominated by the boundary-free part and the corresponding energy density is negative.

In order to understand the behavior of the vacuum energy-momentum tensor at large distances from the string, assuming that $mr \gg 1$, we need to find the asymptotic estimate for the function $C_{\nu}(z, y)$ for large values of y. In this limit the dominant contribution to the integral in (2.122) comes from the region near the lower limit of the integration and we get

$$C_{\nu}(z,y) \approx \frac{e^{2mz} + e^{2m(a-z)}}{4m} \hat{D}_{\nu} \int_{m}^{\infty} dx \, x \sin(2y\sqrt{x^2 - m^2}) e^{-2ax},$$

$$C_{3}(z,y) \approx \frac{1}{4} \hat{D}_{3} \int_{m}^{\infty} dx \, \left(x^2/m^2 - 1\right) \sin(2y\sqrt{x^2 - m^2}) e^{-2ax},$$
(2.129)

where $\nu = 0, 1, 2$. The integrals are expressed in terms of the function $K_{\nu}(2m\sqrt{a^2 + y^2})$. By making use of the corresponding asymptotic expression for large values of the argument, we find

$$C_{\nu}(z,y) \approx -\sqrt{\pi} a m^{3/2} \left[e^{2mz} + e^{2m(a-z)} \right] \frac{e^{-2my}}{4y^{7/2}},$$

$$C_{3}(z,y) \approx -\sqrt{\pi} a m^{5/2} \frac{e^{-2my}}{2y^{5/2}},$$
(2.130)

for $\nu = 0, 1$ and $C_2(z, y) \approx -2myC_0(z, y)$. Substituting into (2.121), we see that for q > 2 the dominant contribution comes from the l = 1 term and for $mr \gg 1$ one gets:

$$\langle T_{\nu}^{\nu} \rangle_{\rm b} \approx \langle T_{\nu}^{\nu} \rangle_{\rm M} + \frac{am^5}{8\pi^{3/2}} \left[e^{2mz} + e^{2m(a-z)} \right] \frac{\cos\left(\pi/q\right) e^{-2mr\sin(\pi/q)}}{[mr\sin(\pi/q)]^{7/2}},$$

$$\langle T_2^2 \rangle_{\rm b} \approx \langle T_2^2 \rangle_{\rm M} - \frac{am^5}{4\pi^{3/2}} \left[e^{2mz} + e^{2m(a-z)} \right] \frac{\cos\left(\pi/q\right) e^{-2mr\sin(\pi/q)}}{[mr\sin(\pi/q)]^{5/2}},$$

$$\langle T_3^3 \rangle_{\rm b} \approx \langle T_3^3 \rangle_{\rm M} + \frac{am^5}{4\pi^{3/2}} \frac{\cos\left(\pi/q\right) e^{-2mr\sin(\pi/q)}}{[mr\sin(\pi/q)]^{5/2}}.$$

$$(2.131)$$

Note that for the boundary-free part (2.108) one has the large distance asymptotic

$$\langle T_{\nu}^{\nu} \rangle_{\rm s} \approx -\frac{m^4}{4\pi^{3/2}} \frac{\cos(\pi/q)e^{-2mr\sin(\pi/q)}}{[mr\sin(\pi/q)]^{5/2}},$$
(2.132)

for $\nu = 0, 1, 3$ and $\langle T_2^2 \rangle_s \approx -3 \langle T_0^0 \rangle_s$. In the case $1 \leq q \leq 2$ the suppression of the string-induced contribution is stronger, by the factor e^{-2mr} .

For a massless field, the topological part in the VEV of the energy-momentum tensor is given by the second term in the right-hand side of (2.125). At large distances from the string, $r \gg a$, and for q > 2 the dominant contribution to the topological part is due to the term with l = 1. To the leading order we find

$$\langle T_{\nu}^{\nu} \rangle \approx \langle T_{\nu}^{\nu} \rangle_{\mathrm{M}} + \frac{\pi A_{\nu}^{(0)}(\pi(r/a)\sin(\pi/q))}{4a^3 r e^{\pi(r/a)\sin(\pi/q)}} \cot(\pi/q),$$
 (2.133)

where

$$A_0^{(0)}(y) = A_1^{(0)}(y) = \frac{1}{y}, \ A_2^{(0)}(y) = -A_3^{(0)}(y) = -1.$$
(2.134)

For $q \leq 2$, in the topological part of (2.125) the integral term remains only and the topological part falls off as $e^{-\pi r/a}$. For a massless field the boundary-free part $\langle T^{\nu}_{\nu} \rangle_{\rm s}$ decays as $1/r^4$. Note that in the region between two plates the decay of the topological part at large distances from the string is exponential for both the massive and massless cases.

The VEVs $\langle T_{\nu}^{\nu} \rangle$ for $\nu = 0, 1, 2$ diverge on the plates. This type of surface divergences are well known in quantum field theory with boundaries and have been investigated for various bulk and boundary geometries. In the present setting of the problem, for points away from the coical defect, $r \neq 0$, the divergences are the same as those for the plates in Minkowski bulk. This means that the part in the VEV induced by the conial defect (the topological part), $\langle T_{\nu}^{\nu} \rangle_{\rm t} = \langle T_{\nu}^{\nu} \rangle - \langle T_{\nu}^{\nu} \rangle_{\rm M}$, is finite on the plates. This feature could be deduced from general arguments. Indeed, for points $r \neq 0$, both the local bulk and boundary geometries for the conical defect and for Minkowski bulks are the same. The divergences are determined by the local geometrical characteristics (curvature tensors for the bulk and boundaries) and, consequently, they are the same as well. In order to find the leading terms in the asymptotic expansion of $\langle T^{\nu}_{\nu} \rangle$, $\nu = 0, 1, 2$, for points near the plate at z = 0, we note that the dominant contribution to the integral in (2.127) comes from large values of x. One can see that to the leading order $\langle T^{\nu}_{\nu} \rangle \approx -m/(12\pi^2 z^3)$. The axial stress is finite on the plates.

In Fig. 11 we have plotted the ratio of the boundary-induced parts in the energy density $(\nu = 0, \text{ full curves})$ and the azimuthal stress $(\nu = 2, \text{ dashed curves})$ to the corresponding quantities for parallel plates in Minkowski spacetime versus the distance from the string-like defect. The graphs are plotted for a massless fermionic field and the numbers near the curves are the values of the parameter q.



Figure 11: The ratio of the plate-induced contributions in the energy density (full curves) and in the azimuthal stress (dashed curves) to the corresponding quantities in the Minkowski bulk as functions of the distance from the string for a massless fermionic field. The numbers near the curves are the values of the parameter q.

We could apply the summation formula (2.48) to the series over n in the formula (2.82). This gives the following equivalent representation for the diagonal components of the vacuum energy-momentum tensor (no summation over ν):

$$\langle T_{\nu}^{\nu} \rangle = \langle T_{\nu}^{\nu} \rangle_{\rm s} - \frac{q}{\pi^2} \sum_{j} \int_0^\infty d\lambda \, \lambda \bar{g}_{qj-1/2}^{(\nu)}(\lambda r) \times \int_{\sqrt{\lambda^2 + m^2}}^\infty dx \, \frac{f^{(\nu)}(x, z)}{\frac{x + m}{x - m} e^{2ax} + 1} \left(x^2 - \lambda^2 - m^2 \right)^{-1/2}.$$

$$(2.135)$$

Here, we have defined the functions

$$\bar{g}_{\beta}^{(0)}(y) = \bar{g}_{\beta}^{(3)}(y) = J_{\beta}^{2}(y) + J_{\beta+1}^{2}(y), \qquad (2.136)$$

 $\bar{g}_{\beta}^{(\nu)}(y) = g_{\beta}^{(\nu)}(y)$ for $\nu = 1, 2$, and

$$f^{(0)}(x,z) = (x^2 - \lambda^2 - m^2) f(x,z),$$

$$f^{(1)}(x,z) = f^{(2)}(x,z) = \lambda^2 f(x,z), \ f^{(3)}(x,z) = -2x^2,$$
(2.137)

with

$$f(x,z) = 2 + m \frac{e^{2xz} + e^{2x(a-z)}}{x-m}.$$
(2.138)

The second term in the right-hand side of (2.135) gives the contribution induced by the boundaries. As it has been shown above, the radial stress is equal to the energy density. From (2.135) it follows that for $r \neq 0$ the boundary-induced contribution is negative to the energy density and positive for the axial stress. By taking into account that the boundary-free part in the energy density is negative as well, we conclude that the total energy density is negative everywhere, $\langle T_0^0 \rangle < 0$.

We have already shown that the boundary-induced VEVs vanish on the conical defect. For points near the defect, the contribution of the term j = 1/2 into the boundary-induced part in (2.135) dominates and to the leading order we get (no summation over ν)

$$\langle T_{\nu}^{\nu} \rangle \approx \langle T_{\nu}^{\nu} \rangle_{\rm s} - \frac{g_{\nu} \pi^{-3/2} (r/2)^{q-1}}{(q+2)\Gamma(q/2)\Gamma((q+1)/2)} \int_{m}^{\infty} dx \, \frac{(x^2 - m^2)^{q/2+1}}{\frac{x+m}{x-m}e^{2ax} + 1} f(x,z), \tag{2.139}$$

for $\nu = 0, 1, 2$ with

$$g_0 = g_1 = 1, \ g_2 = q, \tag{2.140}$$

and

$$\langle T_3^3 \rangle \approx \langle T_3^3 \rangle_{\rm s} + \frac{2\pi^{-3/2}(r/2)^{q-1}}{\Gamma(q/2)\Gamma((q+1)/2)} \int_m^\infty dx \, \frac{x^2(x^2 - m^2)^{q/2}}{\frac{x+m}{x-m}e^{2ax} + 1}.$$
 (2.141)

In the special case q = 1 these asymptotics are reduced to the exact results (2.127).

2.4 Casimir force

The Casimir force acting per unit surface of the plate is determined by the normal stress $\langle T_3^3 \rangle$. For the corresponding effective pressure one has: $p = -\langle T_3^3 \rangle$. The boundary free part of the pressure is the same on both the sides of the plate and it does not contribute to the net force. Hence, the force per unit surface of the plate is determined by the boundary-induced part of the pressure along the z-direction. By taking into account that $\langle T_3^3 \rangle_{\rm b}^{(1)} = 0$, we find

$$p(r) = p_{\rm M} - \frac{2}{\pi^2} \bigg[\sum_{l=1}^{p} (-1)^l c_l F(rs_l) + \frac{2q}{\pi} \cos\left(\frac{q\pi}{2}\right) \\ \times \int_0^\infty dx \, \frac{\sinh(qx)\sinh(x)F(r\cosh x)}{\cosh(2qx) - \cos(q\pi)} \bigg],$$
(2.142)

with the function

$$F(y) = \frac{1}{y} \int_{m}^{\infty} dx \frac{x^2 \sin(2y\sqrt{x^2 - m^2})}{\frac{x + m}{x - m}e^{2ax} + 1}.$$
 (2.143)

Here $p_{\rm M} = -\langle T_3^3 \rangle_{\rm M}$, with $\langle T_3^3 \rangle_{\rm M}$ from (2.127), is the fermionic Casimir pressure for the plates in Minkowski bulk. From the results of the previous section it follows that for q > 1 the Casimir pressure vanishes on the string as r^{q-1} . An alternative representation for the Casimir pressure is obtained by using the formula (2.135) for the axial stress:

$$p(r) = -\frac{2q}{\pi^2} \sum_{j} \int_0^\infty d\lambda \,\lambda \left[J_{qj-1/2}^2(\lambda r) + J_{qj+1/2}^2(\lambda r) \right] \\ \times \int_{\sqrt{\lambda^2 + m^2}}^\infty dx \, x^2 \frac{\left(x^2 - \lambda^2 - m^2\right)^{-1/2}}{\frac{x + m}{x - m} e^{2ax} + 1}.$$
(2.144)

For $r \neq 0$ this pressure is always negative which means that the Casimir forces are attractive.

At large distances from the string and for a massive field, $mr \gg 1$, from (2.131) for q > 2we have

$$p(r) \approx p_{\rm M} - \frac{am^5}{4\pi^{3/2}} \frac{\cos\left(\pi/q\right) e^{-2mr\sin(\pi/q)}}{[mr\sin(\pi/q)]^{5/2}}.$$
 (2.145)

For q < 2 and if q is not too close to 2 we have

$$p \approx p_M + \frac{q^2 m^3 \cos\left(q\pi/2\right) e^{-2mr}}{8\pi^2 \sin^2(q\pi/2)(mr)^3},$$
(2.146)

with the exponential suppression of the topological part.

For a massless field, in (2.142) one has $p_{\rm M} = -7\pi^2/(960a^4)$ and

$$F(y) = -\frac{\pi}{16ay} \partial_y^2 \left[\frac{a}{\pi y} - \frac{1}{\sinh(y\pi/a)} \right] = -\frac{1}{8y^4} \left[1 - \left(\frac{y\pi}{a}\right)^3 \frac{1 + \sinh^2(y\pi/a)/2}{\sinh^3(y\pi/a)} \right].$$
 (2.147)

At large distances from the string, this leads to the asymptotic below:

$$p \approx -\frac{7\pi^2}{960a^4} \left[1 + \frac{(q^2 - 1)(q^2 + 17/7)}{3\pi^4 (r/a)^4} \right], \ r \gg a.$$
 (2.148)

In this case one has a power law decay for the topological part. As it has been shown before, for a massless field and at large distances from the string the topological part of the axial stress in the region between the plates is suppressed by the factor $e^{-\pi(r/a)\sin(\pi/q)}$ for q > 2 and by $e^{-\pi r/a}$ for $q \leq 2$. In (2.148), the leading term in the topological part (the second term in the square brackets) comes from the pressure in the exterior region ($z \leq 0$ for the plate at z = 0and $z \geq a$ for the plate at z = a). The latter is dominated by the boundary-free part and induces an attractive force.

Let us denote by ε_t the topological part in the vacuum energy per unit surface of the plates:

$$\varepsilon_{\rm t} = \frac{1}{\phi_0} \int_0^a dz \int_0^{\phi_0} d\phi \left\langle T_0^0 \right\rangle_{\rm t}.$$
(2.149)

where the topological contribution to the energy density is obtained from (2.120) omitting the l = 0 term (corresponding to $\langle T_0^0 \rangle_{\rm M}$) in the boundary-induced part (2.121). Note that $\varepsilon_{\rm t}$ depends on the radial coordinate r. Integrating by parts, one can see that

$$\partial_a \int_0^a dz \, C_0(z, y) = 4F(y).$$
 (2.150)

From here the following relation is obtained between the topological parts in the vacuum energy and the pressure on the plates:

$$p_{\rm t} = -\partial_a \varepsilon_{\rm t},\tag{2.151}$$

where p_t is given by the second term in the right-hand side of (2.142). This is the standard thermodynamical relation between the energy and pressure for adiabatic processes. Note that the Minkowkian part $\langle T_0^0 \rangle_M$ diverges on the plates and in order to obtain the finite vacuum energy an additional renormalization is required.



Figure 12: The ratio of the Casimir pressure to the corresponding quantity in the Minkowski bulk versus the distance from the string. The left panel is for a massless field and for the right panel ma = 0.5. The numbers near the curves are the values of the parameter q.

Fig. 12 presents the ratio of the Casimir force per unit surface of the plate to the corresponding quantity in the Minkowski spacetime as a function of the distance from the conical defect. The left panel is plotted for a massless field and for the right panel we have taken ma = 0.5. In the latter case for the Minkowskian pressure one has $p_{\rm M} \approx -0.0384/a^4$. The numbers near the curves are the values of the parameter q. As is seen from the right panel, the dependence, in general, is not monotonic.

2.5 Summary

We have investigated the influence of the conical geometry on the characteristics of the fermionic vacuum in the region between two plates perpendicular to the cone axis. This geometry describes an idealized conical string-like defect with zero thickness core. On the plates the field obeys the boundary condition used in bag models to confine the quarks inside the hadrons. Among the most important local characteristics of the fermionic vacuum are the fermionic condensate and the expectation value of the energy-momentum tensor. For the evaluation of the corresponding VEVs we have constructed the complete set of positive- and negative-energy fermionic modes, given by the expressions (2.6) and (2.16). The eigenvalues of the quantum number, corresponding to the direction along the conical defect axis, are quantized by the boundary conditions and are solutions of the transcendental equation (2.15). The mode sums of the VEVs contain series over these eigenvalues. The application of the Abel-Plana type formula (2.48) allows to extract the parts in the VEVs for the geometry of a single plate and to present the second plate induced contributions in the form for which the explicit knowledge of the eigenvalues of the axial quantum number is not required.

The fermionic condensate is decomposed into the boundary-free and boundary-induced contributions (see (2.61)). The boundary-free part is given by (2.56) and it vanishes for a massless field. For the boundary-induced part we have derived the expression (2.62) with the function C(z, y) defined as (2.63). The corresponding integral is further evaluated for a massless field with the expression (2.67). The l = 0 term in the boundary-induced contribution (2.62) corresponds to the fermion condensate for two parallel plates in Minkowski bulk. The remaining part is induced by the nontrivial topology of the conical defect. We have shown that, for points outside the string core, the topological part vanishes on the plates as a consequence of the cancellation between the boundary-free and boundary-induced parts. The Minkowskian part diverges on the plates with the leading term inversely proportional to the cube of the distance from the plate. For points away from the plates, the boundary-induced fermionic condensate vanishes on the string as r^{q-1} . For q > 2, at large distances from the defect the topological part in the boundary-induced fermionic condensate decays as $e^{-2mr\sin(\pi/q)}$ for a massive field and as $e^{-2r\sin(\pi/q)/a}$ for a massless field. In the case $q \leq 2$ the decay is stronger, like e^{-2mr} and $e^{-2r/a}$ for the massive and massless cases, respectively. An alternative representation for the boundary-induced contribution to the fermionic condensate is given by the second term in the right-hand side of (2.79). This contribution is negative for points outside the defect. For points near the defect the fermionic condensate is dominated by the boundary-free part and is positive. At large distances from the string, the Minkowskian part dominates and the condensate is negative. Consequently, at some intermediate distances the fermionic condensate vanishes.

The vacuum energy-momentum tensor is diagonal and the corresponding radial stress is equal to the energy density. The boundary-induced contributions to the separate components are given by (2.121), where the functions $C_{\nu}(z, y)$ are defined by the expressions (2.122) and

(2.123) for massive and massless fields respectively. In the latter case the single plate parts vanish and the VEV of the energy-momentum tensor in the region between the plates does not depend on the coordinate z. For the axial stress, that is the case for a massive field as well. Similar to the fermionic condensate, for points away the plates, the boundary-induced contributions to the vacuum energy-momentum tensor vanish on the string. At distances from the conical defect larger than the Compton wavelength, $mr \gg 1$, the asymptotic for the topological parts is given by (2.131) and they are suppressed by the factor $e^{-2mr\sin(\pi/q)}$ for q > 2 and by the factor e^{-2mr} for $q \leq 2$. Compared with the azimuthal and axial stresses, the topological contributions to the energy density and the radial stress contain an additional suppression factor 1/(mr). For a massless field and at distances $r \gg a$, one has the asymptotic behavior (2.133) with the suppression factor $e^{-2r\sin(\pi/q)/a}$ for q > 2 and $e^{-2r/a}$ for $q \leq 2$. This behavior for the topological part is in contrast to that for the boundary-free geometry. In the latter case the decay of the vacuum energy-momentum tensor is power law, as $1/r^4$. Another representation for the VEV of the energy-momentum tensor in the region between the boundaries is given by (2.135) with the last term been the boundary-induced contribution. Both the boundary-free and boundary-induced contributions to the energy density are negative.

We have also considered the Casimir force per unit surface of the plates (the vacuum effective pressure), given by (2.142). It is decomposed into the Minkowskian and topological parts, the latter being induced by the conical defect. The vacuum pressure on the plates is not homogeneous. It vanishes at the point where the conical defect crosses the plates and is negative at other points. This corresponds to the attractive force between the plates. The Casimir pressure on the plates, in general, is not a monotonic function of the distance from the string. We have shown that the topological contributions in the vacuum energy and the pressure obey the standard thermodynamical relation (2.151).

Chapter 3: CASIMIR EFFECT FOR PARALLEL ME-TALLIC PLATES IN CONICAL DEFECT SPACETIME

We evaluate the renormalized VEVs of electric and magnetic fields squared and the energymomentum tensor for the electromagnetic field in the geometry of two parallel conducting plates on background of a conical defect. On the base of these results, the Casimir-Polder force on a polarizable particle and the Casimir forces acting on the plates are investigated. The VEVs are decomposed into the pure defect and plate-induced parts. The VEV of the electric field squared is negative for points with radial distance to the defect smaller than the distance to the plates and positive for the opposite situation. On the other hand the VEV for the magnetic field squared is negative everywhere. The boundary induced part in the VEV of the energymomentum tensor is different from zero in the region between the plates only. Moreover, this part only depends on the distance from the string. The boundary-induced part in the vacuum energy density is positive for points with distance to the defect smaller than the distance to the plates and negative in opposite situation. The Casimir stresses on the plates depend nonmonotonically on the distance from the defect. We show that the Casimir forces acting on the plates are always attractive.

3.1 Mode functions for electromagnetic field

In this chapter we consider the effect of two parallel metallic plates on the quantum fluctuations of electromagnetic field in background of a conical defect. For an idealized defect the corresponding line element in cylindrical coordinates with the defect along the z-axis is given by the expression

$$ds^{2} = dt^{2} - dr^{2} - r^{2}d\phi^{2} - dz^{2}, \qquad (3.1)$$

where $0 \leq \phi \leq \phi_0$ and the spatial points (r, ϕ, z) and $(r, \phi + \phi_0, z)$ are to be identified. We are interested in the change of the VEVs of the electric and magnetic fields squared and the energy-momentum tensor of the electromagnetic field, induced by the plates.

The VEV for a physical quantity $f \{F_i, F_k\}$, bilinear in the electric (F = E) or magnetic (F = B) fields, can be written as the mode-sum

$$\langle 0|f\{F_i, F_k\}|0\rangle = \sum_{\alpha} f\{F_{\alpha i}, F_{\alpha k}^*\}, \qquad (3.2)$$

where $\{F_{\alpha i}, F_{\alpha k}^*\}$ represents a complete set of normalized mode functions, specified by a set of quantum numbers α , and obeying the boundary conditions

$$\mathbf{n} \times \mathbf{E}_{\alpha} = 0, \quad \mathbf{n} \cdot \mathbf{B}_{\alpha} = 0, \quad z = 0, a, \tag{3.3}$$

where **n** is the normal vector to the plates (directed along the z axis). We will consider the VEVs in the region between the plates, $0 \le z \le a$. The VEVs for the regions $z \le 0$ and $z \ge a$ are obtained as limiting cases.

We have two classes of mode functions corresponding to the waves of the transverse magnetic (TM) and transverse electric (TE) types. In the case of TM waves the corresponding mode functions for the electric field, obeying the boundary conditions (3.3) on the plate at z = 0, are given by the expressions

$$E_{\alpha 1}^{(0)} = -\beta_{\alpha} k \gamma J_{q|m|}'(\gamma r) \sin(kz) e^{i(qm\phi - \omega t)},$$

$$E_{\alpha 2}^{(0)} = -i\beta_{\alpha} k \frac{qm}{r} J_{q|m|}(\gamma r) \sin(kz) e^{i(qm\phi - \omega t)},$$

$$E_{\alpha 3}^{(0)} = \beta_{\alpha} \gamma^{2} J_{q|m|}(\gamma r) \cos(kz) e^{i(qm\phi - \omega t)},$$
(3.4)

where $J_{\nu}(x)$ is the Bessel function, $0 \leq \gamma < \infty$, and

$$q = 2\pi/\phi_0, \ \omega^2 = \gamma^2 + k^2, \ m = 0, \pm 1, \pm 2, \dots$$
 (3.5)

In (3.4), $E_{\alpha l}^{(0)}$ is the *l*-th physical component of the electric field vector in cylindrical coordinates and the values l = 1, 2, 3 correspond to the r, ϕ, z coordinates, respectively. For the mode functions corresponding to the magnetic field we find

$$B_{\alpha 1}^{(0)} = \beta_{\alpha} \omega \frac{qm}{r} J_{q|m|}(\gamma r) \cos(kz) e^{i(qm\phi - \omega t)},$$

$$B_{\alpha 2}^{(0)} = i\beta_{\alpha} \omega \gamma J'_{q|m|}(\gamma r) \cos(kz) e^{i(qm\phi - \omega t)},$$
(3.6)

and $B_{\alpha 3}^{(0)} = 0$. The eigenvalues for k are quantized by the boundary conditions (3.3) on the plate at z = a:

$$k = k_n = \frac{\pi n}{a}, \quad n = 0, 1, 2, \dots$$
 (3.7)

In the case of the TE waves, the mode functions have the form

$$E_{\alpha 1}^{(1)} = -\beta_{\alpha} \omega \frac{qm}{r} J_{q|m|}(\gamma r) \sin(kz) e^{i(qm\phi - \omega t)},$$

$$E_{\alpha 2}^{(1)} = -i\beta_{\alpha} \omega \gamma J'_{q|m|}(\gamma r) \sin(kz) e^{i(qm\phi - \omega t)},$$

$$E_{\alpha 3}^{(1)} = 0,$$
(3.8)

for the electric field and

$$B_{\alpha 1}^{(1)} = \beta_{\alpha} k \gamma J_{q|m|}'(\gamma r) \cos(kz) e^{i(qm\phi - \omega t)},$$

$$B_{\alpha 2}^{(1)} = i\beta_{\alpha} k \frac{qm}{r} J_{q|m|}(\gamma r) \cos(kz) e^{i(qm\phi - \omega t)},$$

$$B_{\alpha 3}^{(1)} = \beta_{\alpha} \gamma^{2} J_{q|m|}(\gamma r) \sin(kz) e^{i(qm\phi - \omega t)}.$$
(3.9)

for the magnetic field, with the same notations as in (3.4). Now for the eigenvalues of k we have $k = k_n = \pi n/a$, with n = 1, 2, ... As it is seen from the formulas for the mode functions, they are specified by the set $\alpha = (\lambda, \gamma, m, n)$, where $\lambda = 0, 1$ corresponds to the TM and TE waves, respectively.

The mode functions for the vector potential are normalized by the condition

$$\int_{0}^{\infty} drr \int_{0}^{\phi_{0}} d\phi \int_{0}^{a} dz \, \mathbf{A}_{\alpha}^{(\lambda)} \cdot \mathbf{A}_{\alpha'}^{(\lambda)*} = \frac{2\pi}{\omega} \delta_{\alpha\alpha'}, \qquad (3.10)$$

where $\delta_{\alpha\alpha'}$ is understood as the Dirac delta function for continuous components of the collective index α and as the Kronecker delta for discrete ones. By taking into account that $\mathbf{E}_{\alpha} = -\partial_t \mathbf{A}_{\alpha} = i\omega \mathbf{A}_{\alpha}$, the normalization condition is written in terms of the electric field modes as

$$\int_{0}^{\infty} drr \int_{0}^{\phi_{0}} d\phi \int_{0}^{a} dz \, \mathbf{E}_{\alpha}^{(\lambda)} \cdot \mathbf{E}_{\alpha'}^{(\lambda)*} = 2\pi\omega\delta_{\alpha\alpha'}.$$
(3.11)

For TM waves and for $n \neq 0$ one gets

$$\frac{a}{2}\phi_0\beta_\alpha^2 \int_0^\infty drr \left\{ \left[k^2 \left(\frac{qm}{r}\right)^2 + \gamma^2 \gamma'^2 \right] J_{q|m|}(\gamma r) J_{q|m|}(\gamma' r) + k^2 \gamma \gamma' J_{q|m|}'(\gamma r) J_{q|m|}'(\gamma' r) \right\}$$

$$= 2\pi\omega\delta(\gamma - \gamma'). \tag{3.12}$$

For $\gamma' = \gamma$ the integral is divergent and the dominant contribution comes from large values of r. By using the asymptotic expression for the Bessel function for large values of the argument. The integral is expressed in terms of the delta function and we find $\beta_{\alpha}^2 = 4\pi/(\gamma \omega a \phi_0)$. The normalization of the n = 0 TM mode should be considered separately. For this mode one has $E_{\alpha 1}^{(0)} = 0, E_{\alpha 2}^{(0)} = 0, E_{\alpha 3}^{(0)} = \beta_{\alpha} \gamma^2 J_{q|m|}(\gamma r) e^{i(qm\phi - \omega t)}$ and, hence,

$$a\phi_0\gamma^2\gamma'^2\beta_\alpha^2\int_0^\infty drr\,J_{q|m|}(\gamma r)J_{q|m|}(\gamma' r) = 2\pi\omega\delta(\gamma-\gamma'). \tag{3.13}$$

By using the result (2.19) for the integral one finds $\beta_{\alpha}^2 = 2\pi/(a\phi_0\gamma^2)$, where we have taken into account that for the mode under consideration $\omega = \gamma$. Now let us consider the TE waves. For the corresponding mode functions one has

$$\frac{a}{2}\phi_0\beta_\alpha^2 \int_0^\infty drr \left[\omega\omega'\gamma\gamma' J_{q|m|}'(\gamma r) J_{q|m|}'(\gamma' r) + \omega\omega' \left(\frac{qm}{r}\right)^2 J_{q|m|}(\gamma r) J_{q|m|}(\gamma' r)\right]$$

$$= 2\pi\omega\delta(\gamma - \gamma'). \tag{3.14}$$

The integral is evaluated in a way similar to that for TM waves and we get $\beta_{\alpha}^2 = 4\pi/(\gamma \omega a \phi_0)$. Combining all these results, we see that the normalization coefficient is given by the expression

$$\beta_{\alpha}^2 = \frac{2q(1 - \delta_{n0}/2)}{\omega\gamma a},\tag{3.15}$$

for both TM and TE modes.

3.2 VEVs for the electric and magnetic field squared

In this section we consider the VEVs of the electric and magnetic fields squared for the physical situation specified in the previous section. These VEVs are evaluated by using the mode sum

$$\langle 0|F^2|0\rangle = \sum_{\alpha} \mathbf{F}_{\alpha}^{(\lambda)} \cdot \mathbf{F}_{\alpha}^{(\lambda)*}, \qquad (3.16)$$

where F = E and F = B for the electric and magnetic fields respectively. First we consider the region between the plates, $0 \leq z \leq a$. Substituting the mode functions into the corresponding mode-sum formula (3.16), we find

$$\langle 0|F^{2}|0\rangle = \frac{4q}{a} \sum_{m=0}^{\infty} \int_{0}^{\infty} d\gamma \frac{\gamma}{\omega} \left[G_{qm}(\gamma r) \sum_{n=1}^{\infty} \left(2k_{n}^{2} + \gamma^{2} \right) f_{1}^{(F)}(k_{n}z) \right. \\ \left. + \gamma^{2} J_{qm}^{2}(\gamma r) \sum_{n=0}^{\infty} f_{2}^{(F)}(k_{n}z) \right],$$

$$(3.17)$$

where

$$G_{qm}(x) = J_{qm}^{2}(x) + (qm/x)^{2} J_{qm}^{2}(x), \qquad (3.18)$$

and we have defined the functions

$$f_1^{(E)}(x) = f_2^{(B)}(x) = \sin^2 x,$$

$$f_2^{(E)}(x) = f_1^{(B)}(x) = \cos^2 x.$$
(3.19)

The expression in the right-hand side of (3.17) is divergent. We assume that a cutoff function is introduced to make them convergent without explicitly writing it. The special form of this function will not be important in the following discussion.

For the evaluation of the series over n in (3.17) we apply the Abel-Plana summation formula, written in the form (see, for example, [163]):

$$\frac{\pi}{a} \sum_{n=0}^{\infty} f(\pi n/a) = \int_0^\infty dx \, f(x) + i \int_0^\infty dx \, \frac{f(ix) - f(-ix)}{e^{2ax} - 1},\tag{3.20}$$

where the prime on the summation sign means that the term with n = 0 should be taken with the coefficient 1/2. This leads to the following representation of the VEVs:

$$\langle 0|F^2|0\rangle = \langle F^2\rangle_1 + \langle F^2\rangle_2, \qquad (3.21)$$

where the first and second terms in the right-hand side correspond to the first and second integrals in (3.20), respectively. For these separate terms we get the expressions

$$\langle F^2 \rangle_1 = \frac{4q}{\pi} \sum_{m=0}^{\infty} \int_0^\infty dk \int_0^\infty d\gamma \frac{\gamma}{\omega} [G_{qm}(\gamma r) \\ \times (2k^2 + \gamma^2) f_1^{(F)}(kz) + \gamma^2 J_{qm}^2(\gamma r) f_2^{(F)}(kz)],$$
 (3.22)

and

$$\langle F^2 \rangle_2 = \frac{8q}{\pi} \sum_{m=0}^{\infty} \int_0^\infty d\gamma \gamma \int_{\gamma}^\infty dx \frac{(x^2 - \gamma^2)^{-1/2}}{e^{2ax} - 1} \\ \times \left[G_{qm}(\gamma r) \left(2x^2 - \gamma^2 \right) g_1^{(F)}(xz) + \gamma^2 J_{qm}^2(\gamma r) g_2^{(F)}(xz) \right],$$
 (3.23)

with the functions

$$g_1^{(E)}(x) = -g_2^{(B)}(x) = \sinh^2 x,$$

$$g_2^{(E)}(x) = -g_1^{(B)}(x) = \cosh^2 x.$$
(3.24)

The term $\langle F^2 \rangle_1$ does not depend on the distance between the plates, whereas the term $\langle F^2 \rangle_2$ vanishes in the limit $a \to \infty$. From here it follows that the part $\langle F^2 \rangle_1$ corresponds to the VEV in the geometry of a single plate located at z = 0 when the second plate is absent. The part $\langle F^2 \rangle_2$ is induced by the second plate at z = a.

The single plate parts can be further decomposed as

$$\langle F^2 \rangle_1 = \langle F^2 \rangle^{(s)} + \langle F^2 \rangle_1^{(b)}, \qquad (3.25)$$

where

$$\langle E^2 \rangle^{(s)} = \langle B^2 \rangle^{(s)} = \frac{2q}{\pi} \sum_{m=0}^{\infty} \int_0^\infty dk \int_0^\infty d\gamma \times \frac{\gamma}{\omega} \left[G_{qm}(\gamma r) \left(2k^2 + \gamma^2 \right) + \gamma^2 J_{qm}^2(\gamma r) \right],$$
 (3.26)

is the corresponding VEV for a boundary-free string geometry. The terms

$$\langle E^2 \rangle_1^{(b)} = -\langle B^2 \rangle_1^{(b)} = -\frac{2q}{\pi} \sum_{m=0}^{\infty} \int_0^\infty dk \cos(2kz) \\ \times \int_0^\infty d\gamma \frac{\gamma}{\omega} \left[G_{qm}(\gamma r) \left(2k^2 + \gamma^2 \right) - \gamma^2 J_{qm}^2(\gamma r) \right],$$
 (3.27)

are the parts induced by the presence of a single plate at z = 0. Note that the latter is finite for points away from the plate and the renormalization is necessary for the boundary-free part only. The renormalized boundary-free part is given by the following simple expression [111]:

$$\langle E^2 \rangle_{\rm ren}^{(s)} = \langle B^2 \rangle_{\rm ren}^{(s)} = -\frac{(q^2 - 1)(q^2 + 11)}{180\pi r^4}.$$
 (3.28)

The corresponding VEV is negative.

In the plate-induced parts (3.27) the integrals over k are evaluated by using the integration formulas

$$\int_{0}^{\infty} dk \frac{\cos(2kz)}{\sqrt{k^{2} + \gamma^{2}}} = K_{0}(2z\gamma),$$

$$\int_{0}^{\infty} dk \frac{k^{2}\cos(2kz)}{\sqrt{k^{2} + \gamma^{2}}} = -\frac{1}{4}\partial_{z}^{2}\int_{0}^{\infty} dk \frac{\cos(2kz)}{\sqrt{k^{2} + \gamma^{2}}} = -\gamma^{2}K_{0}''(2z\gamma).$$
(3.29)

By taking into account that $K_0''(y) = K_0(y) + K_1(y)/y$, one gets

$$\langle E^2 \rangle_1^{(b)} = -\langle B^2 \rangle_1^{(b)} = \frac{2q}{\pi z^4} \sum_{m=0}^{\infty} \int_0^\infty du \, u^3 \\ \times \left[J_{qm}^2 (ur/z) K_0(2u) + G_{qm} (ur/z) Q(2u) \right],$$
 (3.30)

with the notation

$$Q(x) = K_0(x) + 2K_1(x)/x, \qquad (3.31)$$

As we see, the boundary-induced part is positive for the electric field and negative for the magnetic field.

For a metallic plate in background of Minkowski spacetime one has q = 1 and the summation over m in (3.30) can be explicitly done by using the formula $\sum_{m=0}^{\infty} J_{qm}^2(y) = 1/2$. In addition, differentiating the latter relation and using the equation for the Bessel function, it can be seen that $\sum_{m=0}^{\infty} G_{qm}(y) = 1/2$. The remaining integrals over u are evaluated by using the standard result for the integrals involving the Macdonald function. As as result we can see that

$$\langle E^2 \rangle_1^{(b)}|_{q=1} = -\langle B^2 \rangle_1^{(b)}|_{q=1} = \frac{3}{4\pi z^4}.$$
 (3.32)

For $r \ll z$, in the boundary-induced parts (3.30) we use the asymptotic expression for the Bessel function for small values of the argument [177]. In addition, assuming that $(r/z)^{q-1} \ll 1$ (q is not too close to 1) and 1 < q < 2, we find

$$\langle E^2 \rangle_1 \approx \frac{q}{\pi z^4} \int_0^\infty d\gamma \gamma^3 K_0(2\gamma) + \frac{4q}{2^{2q}\pi z^4} \\ \times \frac{(r/z)^{2q-2}}{\Gamma^2(q)} \int_0^\infty d\gamma \gamma^{2q+1} \left[K_0(2\gamma) + \frac{K_1(2\gamma)}{\gamma} \right] \\ = \frac{q}{4\pi z^4} + \frac{q^2(q+1)}{2^{2q}\pi z^4} (r/z)^{2q-2}.$$
 (3.33)

By taking into account (3.28), we see that in the region under consideration the total VEV of the electric field squared is dominated by the boundary-free part and it is negative. In the opposite limiting case, $r \gg z$, to the leading order the boundary-induced VEV coincides with the corresponding quantity in Minkowski bulk. In this limit the total VEV is dominated by the boundary-induced part and it is positive for the electric field. For the magnetic field the total VEV is negative everywhere.

For integer values of the parameter q, the summation over m in (3.30) can be done by using the formulas [94, 111, 127]

$$\sum_{m=0}^{\infty} J_{qm}^{2}(y) = \frac{1}{2q} \sum_{l=0}^{q-1} J_{0}(2ys_{l}),$$

$$\sum_{m=0}^{\infty} G_{qm}(y) = \frac{1}{2q} \sum_{l=0}^{q-1} \cos(2\pi l/q) J_{0}(2ys_{l}).$$
 (3.34)

with $s_l = \sin(\pi l/q)$. By using these formulas, for the single plate part we find:

$$\langle E^2 \rangle_1 = \frac{1}{16\pi} \sum_{l=0}^{q-1} \int_0^\infty d\gamma \gamma^3 \left[\left(\cos(2\pi l/q) + 1 \right) K_0(z\gamma) + 2\cos(2\pi l/q) \frac{K_1(z\gamma)}{z\gamma} \right] J_0(\gamma r s_l). \quad (3.35)$$

By taking into account that $K_1(y) = -K'_0(y)$, the VEV is expressed in terms of the integrals

$$\int_{0}^{\infty} d\gamma \gamma K_{0}(z\gamma) J_{0}(\gamma r s_{l}) = \frac{1}{z^{2} + r^{2} s_{l}^{2}},$$

$$\int_{0}^{\infty} d\gamma \gamma^{3} K_{0}(z\gamma) J_{0}(\gamma r s_{l}) = 4 \frac{z^{2} - r^{2} s_{l}^{2}}{[z^{2} + r^{2} s_{l}^{2}]^{3}}.$$
(3.36)

As a result, for the single plate-induced parts we find:

$$\langle E^2 \rangle_1^{(b)} = -\langle B^2 \rangle_1^{(b)} = \frac{1}{4\pi} \sum_{l=0}^{q-1} \frac{(3-4s_l^2) z^2 - r^2 s_l^2}{(z^2 + r^2 s_l^2)^3}.$$
(3.37)

The l = 0 term in this expression coincides with the corresponding VEV for a plate in background of Minkowski spacetime: $\langle E^2 \rangle_1^{(b)}|_{q=1} = 3/(4\pi z^4)$. For $r \ll z$, from (3.37) we find

$$\langle E^2 \rangle_1 \approx \frac{1}{4\pi z^4} \sum_{l=0}^{q-1} \left\{ 3 - 4s_l^2 + 4(r/z)^2 s_l^2 \left(2 - 3s_l^2\right) \right\}$$
(3.38)

By taking into account that

$$\sum_{l=0}^{q-1} s_l^2 = \frac{q}{2}, \quad \sum_{l=0}^{q-1} s_l^4 = \frac{3q}{8}, \tag{3.39}$$

we find

$$\langle E^2 \rangle_1 \approx \frac{q}{4\pi z^4} \left[1 - \frac{1}{2} (r/z)^2 \right].$$
 (3.40)

For $r \gg z$, keeping the next-to-leading term, from (3.37) one gets

$$\langle E^2 \rangle_1 \approx \frac{3}{4\pi z^4} - \frac{1}{4\pi r^4} \sum_{l=1}^{q-1} \frac{1}{s_l^4}.$$
 (3.41)

Note that the leading term in the correction induced by the defect does not depend on z.

In figure 1 we plot the VEVs for the electric (full curves) and magnetic (dashed curves) field squared in the geometry of a single conducting plate located at z = 0 as a function of the ratio r/z. The numbers near the curves are the corresponding values of the parameter q. The dot-dashed curves correspond to the boundary-induced part in the VEV of the field squared, $z^4 \langle E^2 \rangle_1^{(b)}$. Note that, in general, the boundary-induced part $\langle E^2 \rangle_1^{(b)}$ is a non-monotonic function of r. The corresponding VEVs for a plate in Minkowski spacetime would be presented by the horizontal lines $3/(4\pi)$ and $-3/(4\pi)$ for the electric and magnetic fields respectively.and therefore putting them together for the total VEVs we get

In Fig. 13 we plot the VEVs for the electric (full curves) and magnetic (dashed curves) field squared in the geometry of a single conducting plate located at z = 0 as a function of the ratio r/z. The numbers near the curves are the corresponding values of the parameter q. The dot-dashed curves correspond to the boundary-induced part in the VEV of the field squared, $z^4 \langle E^2 \rangle_1^{(b)}$. Note that, in general, the boundary induced part $\langle E^2 \rangle_1^{(b)}$ is a non-monotonic function of r. The corresponding VEVs for a plate in Minkowski spacetime would be presented by the horizontal lines $3/(4\pi)$ and $-3/(4\pi)$ for the electric and magnetic fields respectively.

Now let us turn to the second plate induced parts given by (3.23). These parts for the electric and magnetic fields are presented in the form

$$\langle E^2 \rangle_2 = \frac{1}{a^4} C(r/a) + \frac{1}{a^4} \sum_{j=\pm 1} D(r/a, jz/a), \langle B^2 \rangle_2 = \frac{1}{a^4} C(r/a) - \frac{1}{a^4} \sum_{j=\pm 1} D(r/a, jz/a),$$
 (3.42)



Figure 13: VEV of the field squared, $z^4 \langle F^2 \rangle_1$, for the electric (full curves) and magnetic (dashed curves) fields in the geometry of a single metallic plate. The numbers near curves are the values of the parameter q. The boundary-induced part in the VEV of the electric field squared is plotted separately by the dot-dashed curves

where

$$C(x) = -\frac{4q}{\pi} \sum_{m=0}^{\infty} \int_{0}^{\infty} d\gamma \gamma \int_{\gamma}^{\infty} du \frac{(u^{2} - \gamma^{2})^{-1/2}}{e^{2u} - 1} \left[G_{qm}(\gamma x) \left(2u^{2} - \gamma^{2} \right) - \gamma^{2} J_{qm}^{2}(\gamma x) \right],$$

$$D(x, y) = \frac{2q}{\pi} \sum_{m=0}^{\infty} \int_{0}^{\infty} d\gamma \gamma \int_{\gamma}^{\infty} du \frac{(u^{2} - \gamma^{2})^{-1/2}}{e^{2u} - 1} e^{2yu} \left[G_{qm}(\gamma x) \left(2u^{2} - \gamma^{2} \right) + \gamma^{2} J_{qm}^{2}(\gamma x) \right].$$

For the evaluation of the integrals over u we use the expansion

$$\frac{1}{e^{2u} - 1} = \sum_{n=1}^{\infty} e^{-2nu}.$$
(3.43)

This gives

$$\int_{\gamma}^{\infty} du \frac{e^{2yu}}{(e^{2u}-1)\sqrt{u^2-\gamma^2}} = \sum_{n=1}^{\infty} \int_{\gamma}^{\infty} dx (x^2-\gamma^2)^{-1/2} e^{-2(an-jz)x}$$
$$= \sum_{n=1}^{\infty} K_0(2(n-jy)\gamma), \qquad (3.44)$$

and

$$\int_{\gamma}^{\infty} dx \frac{u^2 e^{2yu}}{(e^{2u} - 1)\sqrt{u^2 - \gamma^2}} = \sum_{n=1}^{\infty} \gamma^2 K_0''(2(n - jy)\gamma).$$
(3.45)

The remaining integrals are obtained taking in the last two results y = 0. By taking into account these formulas we find the following representations

$$C(x) = -\frac{4q}{\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_{0}^{\infty} d\gamma \gamma^{3} \left[G_{qm}(\gamma x)Q(2n\gamma) - J_{qm}^{2}(\gamma x)K_{0}(2n\gamma) \right],$$

$$D(x,y) = \frac{2q}{\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_{0}^{\infty} d\gamma \gamma^{3} \left[G_{qm}(\gamma x)Q(2(n-y)\gamma) + J_{qm}^{2}(\gamma x)K_{0}(2(n-y)\gamma) \right] (3.46)$$

Note that the z-dependent parts for the VEVs of the electric and magnetic fields have opposite signs. In particular, they are cancelled in the corresponding energy density (see the next section). The quantity $\langle E^2 \rangle_2$ is finite on the plate at z = 0 and diverges on the plate at z = a. The divergence comes from the n = 1 term in the expression for the function D(x, y). Note that the contribution of this term to the VEV of the electric field squared coincides with (3.30) with the replacement $z \to a - z$. Hence, it presents the VEV induced by the plate at z = awhen the plate at z = 0 is absent.

Combining the results for the single plate and second plate-induced parts, the total VEV is presented in the form

$$\langle F^2 \rangle = \langle F^2 \rangle_{\rm ren}^{(s)} - \frac{4q}{\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty'} \int_0^\infty d\gamma \gamma^3 \left[G_{qm}(\gamma r) Q(2na\gamma) - J_{qm}^2(\gamma r) K_0(2na\gamma) \right]$$

+ $\delta_F \frac{2q}{\pi} \sum_{n=-\infty}^\infty \sum_{m=0}^{\infty'} \int_0^\infty d\gamma \gamma^3 \left[G_{qm}(\gamma r) Q(2|na-z|\gamma) + J_{qm}^2(\gamma r) K_0(2|na-z|\gamma) \right]$ (3.47)

where $\delta_E = 1$ and $\delta_B = -1$. The single plate parts in this expression are presented by the n = 0 and n = 1 terms in the last summation on the right-hand side. The VEV is not changed under the replacement $z \to a - z$, which is a direct consequence of the problem symmetry with respect to the plane z = a/2.

For integer values of the parameter q, by using formulas (3.34), one finds

$$C(x) = -\frac{1}{2\pi} \sum_{l=0}^{q-1} \sum_{n=1}^{\infty} \frac{(1-4s_l^2)n^2 + x^2s_l^2}{(n^2 + x^2s_l^2)^3},$$

$$D(x,y) = \frac{1}{4\pi} \sum_{l=0}^{q-1} \sum_{n=1}^{\infty} \frac{(3-4s_l^2)(n-y)^2 - x^2s_l^2}{[(n-y)^2 + x^2s_l^2]^3}.$$
(3.48)

After the summation over n, the function C(x) can also be presented in the form

$$C(x) = -\frac{1}{2\pi} \sum_{l=0}^{q-1} \left[\left(1 - 4s_l^2 \right) h_2(xs_l) + 4s_l^2 h_3(xs_l) \right], \qquad (3.49)$$

with the notations

$$h_{2}(b) = \sum_{n=1}^{\infty} \frac{1}{(n^{2}+b^{2})^{2}} = -\frac{1}{2b^{4}} + \frac{\pi}{4b^{3}} \left[\coth(\pi b) + \frac{\pi b}{\sinh^{2}(\pi b)} \right],$$

$$h_{3}(b) = \sum_{n=1}^{\infty} \frac{b^{2}}{(n^{2}+b^{2})^{3}}$$

$$= -\frac{1}{2b^{4}} + \frac{\pi}{16b^{3}} \left[3 \coth(\pi b) + \frac{3\pi b}{\sinh^{2}(\pi b)} + \frac{2\pi^{2}b^{2}}{\sinh^{2}(\pi b)} \coth(\pi b) \right].$$
 (3.50)

Note that $h_2(0) = \pi^4/90$. For integer q, the expression of the total VEV takes the form:

$$\langle F^2 \rangle = \langle F^2 \rangle_{\rm ren}^{(s)} - \frac{1}{2\pi} \sum_{l=0}^{q-1} \sum_{n=1}^{\infty} \frac{(1-4s_l^2) n^2 a^2 + r^2 s_l^2}{(n^2 a^2 + r^2 s_l^2)^3} + \frac{\delta_F}{4\pi} \sum_{l=0}^{q-1} \sum_{n=-\infty}^{\infty} \frac{(3-4s_l^2) (na-z)^2 - r^2 s_l^2}{[(na-z)^2 + r^2 s_l^2]^3}.$$
 (3.51)

In Fig. 14 we display the dependence on r/a and z/a for the VEVs of the electric (left plot) and magnetic (right plot) field squared, in the region between the plates for a cosmic string with q = 2. For the VEV for the electric field squared the boundary-induced part dominates near the plates and it is positive in this region. Near the string the VEV is dominated by the boundary-free part and it is negative. The VEV of the magnetic field squared is negative everywhere.



Figure 14: VEVs of the electric field squared (left plot), $a^4 \langle E^2 \rangle$, and the magnetic field squared (right plot), $a^4 \langle B^2 \rangle$, in the region between two plates for q = 2

In the region z < 0 the VEVs for the field squared are given by (3.25), where the plate induced part is still given by the expression (3.27). The same decomposition (3.25) is valid for the region z > a, with the difference that now the single plate part is given by the expression (3.27) with the replacement $z \to z - a$.

3.3 An alternative representation of the electric field squared and the Casimir-Polder potential

Due to the boundary conditions imposed on the vacuum fluctuations of the electric and magnetic fields by the presence of the plates, the spectrum of these fluctuations is changed. This gives rise the Casimir-Polder forces acting on a polarizable particle. Here we consider a simple case of an isotropic polarizability neglecting the dispersion effects. In this approximation, the Casimir-Polder interaction energy is expressed as

$$U(\mathbf{r}) = -\frac{1}{2}\alpha \langle 0|E^2|0\rangle, \qquad (3.52)$$

where α is the static polarizability of the microparticle. This static limit is a good approximation for separations $|z|, |a - z|, r \gg 1/\omega_j$, where ω_j is the eigenfrequency in the dispersion law for the polarizability. For example, in the oscillator model one has $\alpha(\omega) = \sum_j g_j/(\omega_j^2 - \omega^2)$, where g_j are the oscillator strengths.

In order to obtain an alternative representation for the VEV of the electric field squared we start with the formula (3.17) for the electric field:

$$\langle E^2 \rangle = \frac{4q}{a} \sum_{m,n=0}^{\infty} \int_0^\infty d\gamma \frac{\gamma}{\omega} \left[\gamma^2 \cos^2(k_n z) J_{qm}^2(\gamma r) + \left(2k_n^2 + \gamma^2\right) \sin^2(k_n z) G_{qm}(\gamma r) \right], \quad (3.53)$$

with the notation (3.18). For the further transformation we use the integral representation

$$\frac{1}{\omega} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{ds}{s^{1/2}} e^{-\omega^2 s}.$$
 (3.54)

Plugging this into (3.53) and introducing new integration variables $x = r^2/2s$ and $y = \gamma r$, the integrals over y are evaluated by making use of the formulae

$$\int_{0}^{\infty} dy y^{2p+1} e^{-y^{2}/2x} J_{qm}^{2}(y) = \left(2x^{2} \partial_{x}\right)^{p} \left[x e^{-x} I_{qm}(x)\right],$$

$$\int_{0}^{\infty} dy \ y^{2p+1} e^{-y^{2}/2x} G_{qm}(y) = \left(2x^{2} \partial_{x}\right)^{p} \left[x e^{-x} \partial_{x} I_{qm}(x)\right],$$
(3.55)

where p = 0, 1, and $I_{qm}(x)$ is the modified Bessel function of the first kind. The first integral in (3.55) is given, for example, in [127]. The remaining integrals are obtained from the formulas given in [178]. For the VEV one obtains

$$\langle E^2 \rangle = \frac{8q}{\sqrt{2\pi}ar^3} \sum_{m,n=0}^{\infty} \int_0^\infty dx \sqrt{x} e^{-k_n^2 r^2/2x} \left\{ \cos^2(k_n z) \partial_x \left(x e^{-x} \right) + \sin^2(k_n z) \left[(k_n^2 r^2/x) e^{-x} \partial_x + \partial_x \left(x e^{-x} \partial_x \right) \right] \right\} I_{qm}(x).$$

$$(3.56)$$

Next we write the functions $\cos^2(k_n z)$ and $\sin^2(k_n z)$ in terms of the function $\cos(2k_n z)$ and use the relation

$$\sum_{n=0}^{\infty} e^{-k_n^2 r^2/2x} \cos(2k_n z) = \frac{a\sqrt{2x}}{2\sqrt{\pi}r} \sum_{n=-\infty}^{+\infty} e^{-2x(z-an)^2/r^2},$$
(3.57)

and similar relation for z = 0. The formula (3.57) directly follows from the Poisson resummation formula (see also [178]). As a result, the VEV of the electric field squared is decomposed into two parts

$$\langle E^2 \rangle = \langle E^2 \rangle_1 + \langle E^2 \rangle_2, \tag{3.58}$$

where

$$\langle E^2 \rangle_1 = \frac{2}{\pi r^4} \sum_{n=-\infty}^{+\infty} \int_0^\infty dx x e^{-2x(an/r)^2} \left[-4x(an/r)^2 \left(\partial_x + 1\right) + x \partial_x^2 + 2\left(1+x\right) \partial_x + 3 \right] \mathcal{J}_q(x),$$
(3.59)

and

$$\langle E^2 \rangle_2 = \frac{2}{\pi r^4} \sum_{n=-\infty}^{+\infty} \int_0^\infty dx x e^{-2x(z-an)^2/r^2} \left[\left[4x(z-an)^2/r^2 - 1 \right] (\partial_x + 1) - x \partial_x^2 - \partial_x \right] \mathcal{J}_q(x).$$
(3.60)

Here we have introduced the function

$$\mathcal{J}_{q}(x) = q e^{-x} \sum_{m=0}^{\infty} I_{qm}(x).$$
(3.61)

For this function we have the following representation [179]

$$\mathcal{J}_{q}(x) = \sum_{l=0}^{\lfloor q/2 \rfloor} e^{-2xs_{l}^{2}} - \frac{q\sin(q\pi)}{\pi} \int_{0}^{\infty} dy \frac{e^{-2x\cosh^{2}y}}{\cosh(2qy) - \cos(q\pi)},$$
(3.62)

with the notation $s_l = \sin(\pi l/q)$ and [q/2] is the integer part of q/2. The prime on the summation sign means that the terms l = 0 and l = q/2 (if q is and even integer) should be taken with coefficients 1/2.

Substituting (3.62) into (3.59), after integrating over x, we find

$$\langle E^2 \rangle_1 = -\frac{1}{2\pi a^4} \left[\frac{\pi^4}{90} + \sum_{l=1}^{[q/2]'} f_1(s_l^2, (r/a)^2) - \frac{q\sin(q\pi)}{\pi} \int_0^\infty dy \frac{f_1(\cosh^2 y, (r/a)^2)}{\cosh(2qy) - \cos(q\pi)} \right], \quad (3.63)$$

where the first term in the square brackets come from the l = 0 term in (3.62) and

$$f_1(x,y) = \frac{1}{x^2 y^2} + 2\sum_{n=1}^{+\infty} \frac{n^2 (1-4x) + xy}{(xy+n^2)^3}.$$
(3.64)

The prime on the summation sign in (3.63) means that in the case of even integer values of qthe term with l = q/2 should be taken with an additional coefficient 1/2. The contribution of the first term in the right-hand side of (3.64) to $\langle E^2 \rangle_1$ is expressed in terms of the function

$$c_n(q) = \sum_{l=1}^{[q/2]} s_l^{-n} - \frac{q}{\pi} \sin(q\pi) \int_0^\infty dy \frac{\cosh^{-n} y}{\cosh(2qy) - \cos(q\pi)},$$
(3.65)

with n = 4 that is given by $c_4(q) = (q^2 - 1)(q^2 + 11)/90$. This contribution coincides with the renormalized VEV of the field squared in the boundary-free cosmic string spacetime [111] (for the VEV of the energy-momentum tensor see [92, 93],[180]), given by (3.28). For the radial component of the corresponding Casimir-Polder force one finds (more general results for an anisotropic polarizability tensor with dispersion are given in [181])

$$F_r^{(s)} = \alpha \frac{(q^2 - 1)(q^2 + 11)}{90\pi r^5}.$$
(3.66)

This force is repulsive.

In a similar way, by using the result

1

$$\sum_{n=-\infty}^{+\infty} \frac{1}{(n-z/a)^4} = \frac{\pi^4}{3} \frac{1+2\cos^2(\pi z/a)}{\sin^4(\pi z/a)},$$
(3.67)

for the contribution (3.60) we find

$$\langle E^2 \rangle_2 = \frac{\pi^3}{4a^4} \frac{1 + 2\cos^2(\pi z/a)}{\sin^4(\pi z/a)} + \frac{1}{2\pi a^4} \left[\sum_{l=1}^{[q/2]'} f_2(s_l^2, (r/a)^2, z/a) - \frac{q\sin(q\pi)}{\pi} \int_0^\infty dy \frac{f_2(\cosh^2 y, (r/a)^2, z/a)}{\cosh(2qy) - \cos(q\pi)} \right],$$

$$(3.68)$$

with the function

$$f_2(x, y, u) = \sum_{n = -\infty}^{+\infty} \frac{(3 - 4x)(n - u)^2 - xy}{\left[(n - u)^2 + xy\right]^3}.$$
(3.69)

By taking into account (3.63) and (3.68), the Casimir-Polder potential is presented as

$$U(\mathbf{r}) = U_{\rm M}(\mathbf{r}) + \frac{\alpha}{4\pi a^4} \left\{ \sum_{l=1}^{[q/2]'} f(s_l^2, (r/a)^2, z/a) - \frac{q\sin(q\pi)}{\pi} \int_0^\infty dy \frac{f(\cosh^2 y, (r/a)^2, z/a)}{\cosh(2qy) - \cos(q\pi)} \right\},$$
(3.70)

with the function

$$f(x, y, u) = f_1(x, y) - f_2(x, y, u)$$

= $\sum_{n=-\infty}^{+\infty} \left\{ \frac{n^2 (1 - 4x) + xy}{(xy + n^2)^3} - \frac{(3 - 4x) (n - u)^2 - xy}{[(n - u)^2 + xy]^3} \right\}.$ (3.71)

In (3.70)

$$U_{\rm M}(\mathbf{r}) = -\frac{\pi^3 \alpha}{8a^4} \left[\frac{1 + 2\cos^2(\pi z/a)}{\sin^4(\pi z/a)} - \frac{1}{45} \right].$$
 (3.72)

In the absence of the conical defect one has q = 1 and this term survives only. Hence, it presents the potential in the region between the plates on the Minkowski bulk. The remaining contribution in (3.70) is induced by the conical defect. It is easily seen that f(x, y, 1 - u) =f(x, y, u). From here it follows that the Casimir-Polder potential is symmetric with respect to the plane z = a/2. For integer values of the parameter q, the integral term in (3.70) vanishes. In this special case, the expression we have derived above for the VEV of the electric field squared coincides with the result presented in the previous section [161, 162].

The Casimir-Polder potential in the geometry of a single plate at z = 0 is obtained from the result presented above taking the limit $a \to \infty$ for fixed values of r and z. In this limit, the nonzero contribution in (3.70) comes from the term n = 0 only and for the Casimir-Polder potential one finds

$$U^{(1)}(\mathbf{r}) = -\frac{3\alpha}{8\pi z^4} + \frac{\alpha}{4\pi r^4} \left\{ c_4(q) + \sum_{l=1}^{[q/2]} f^{(1)}(s_l^2, z^2/r^2) - \frac{q\sin(q\pi)}{\pi} \int_0^\infty dy \frac{f^{(1)}(\cosh^2 y, z/r)}{\cosh(2qy) - \cos(q\pi)} \right\},$$
(3.73)

with the notation

$$f^{(1)}(x,y) = \frac{(4x-3)y+x}{(x+y)^3}.$$
(3.74)

The first term in the right-hand side is the potential for a plate in Minkowski bulk. The part with the first term in the figure braces corresponds to the potential when the plate is absent. In the region z < 0 the Casimir-Polder potential is given by (3.73). The corresponding formula in the region z > a is obtained by the replacement $z \rightarrow z - a$. Therefore, we have found the Casimir-Polder potential in all the region for the problem at hand.

Let us describe some properties of the Casimir-Polder force. First of all we note that the problem is symmetric with respect to the plane z = a/2 and the z-component of the CasimirPolder force, $F_z = -\partial_z U(\mathbf{r})$, vanishes on that plane, $F_z|_{z=a/2} = 0$. Let us consider the behavior of the Casimir-Polder potential in the asymptotic regions of the parameters. For $r \ll z, a-z$, the leading contribution comes from the first term in the right-hand side of (3.64) and it coincides with the corresponding quantity in a conical space in the absence of boundaries, given by (3.28). For points close to the plate, $z \ll a, r$, the dominant contribution to (3.70) comes from the first term in the square brackets of (3.72). Expanding it for small z/a, to the leading order one gets

$$U(\mathbf{r}) \approx -\frac{3\alpha}{8\pi z^4}.\tag{3.75}$$

The leading term in the right-hand side coincides with the VEV of the field squared induced by a single conducting plate in Minkowski spacetime.

For large values of $r \gg a$ we need the asymptotic expression for the function f(x, y, u) for $y \gg 1$. In this range the dominant contribution to the series in (3.69) comes from large values of |y| and we can replace the summation by the integration. To the leading order one finds $f(x, y, u) \approx \pi(xy)^{-3/2}/2$. For the VEV of the field squared we get

$$U(\mathbf{r}) \approx U_{\rm M}(\mathbf{r}) + \frac{\alpha c_3(q)}{8ar^3}.$$
(3.76)

This shows that the radial component of the Casimir-Polder force behaves as

$$F_r = -\partial_r U(\mathbf{r}) \approx \frac{3\alpha c_3(q)}{8ar^4}.$$
(3.77)

The force is repulsive with respect to the string. Recall that near the string and for points not too close to the plates, the the radial force is given by (3.66) and behaves as $1/r^5$.

In Fig. 15 we have plotted the Casimir-Polder potential in the region between the plates, $a^4 U(\mathbf{r})/\alpha$, as a function of the rescaled coordinates r/a and z/a for a conical defect with planar angle deficit corresponding to q = 1.5.

3.4 Energy-momentum tensor and the Casimir force

In this section we consider the VEV of the energy-momentum tensor for the geometry of two parallel metallic plates. This VEV is obtained from the general formula (3.2) considering the standard expression for the energy-momentum tensor of the electromagnetic field:

$$\langle T_{ik} \rangle = \frac{1}{4\pi} \sum_{\alpha} \left[-F_{\alpha il}^{(\lambda)} F_{\alpha k}^{(\lambda)l*} + \frac{1}{4} g_{ik} F_{\alpha nl}^{(\lambda)} F_{\alpha}^{(\lambda)nl*} \right], \qquad (3.78)$$



Figure 15: Casimir-Polder potential in the region between the plates, as a function of the rescaled distances from the defect and from the plate at z = 0.

where $F_{\alpha nl}^{(\lambda)} = \partial_n A_{\alpha l}^{(\lambda)} - \partial_n A_{\alpha l}^{(\lambda)}$ is the field tensor and $A_{\alpha l}^{(\lambda)}$ are the mode functions for the vector potential. The energy density is obtained by using the results for the electric and magnetic field squared investigated above:

$$\langle T_0^0 \rangle = \frac{\langle E^2 \rangle + \langle B^2 \rangle}{8\pi}.$$
(3.79)

Hence, it is sufficient to investigate the VEV

$$P_{ij} = \langle E_i E_j + B_i B_j \rangle, \tag{3.80}$$

with i, j = 1, 2, 3. From the symmetry of the problem it follows that the only nonzero components are the diagonal ones and possibly the off diagonal component $P_{13} = P_{31}$. Let us consider these components for TM and TE waves separately.

By using (3.4) and (3.6), for the separate components in the case of TM waves we have

$$E_{\alpha 1}^{(0)} E_{\alpha 1}^{(0)} + B_{\alpha 1}^{(0)} B_{\alpha 1}^{(0)} = \beta_{\alpha}^{2} k^{2} \gamma^{2} J_{q|m|}^{\prime 2}(\gamma r) \sin^{2}(kz) + \beta_{\alpha}^{2} \omega^{2} \left(\frac{qm}{r}\right)^{2} J_{q|m|}^{2}(\gamma r) \cos^{2}(kz),$$

$$E_{\alpha 2}^{(0)} E_{\alpha 2}^{(0)} + B_{\alpha 2}^{(0)} B_{\alpha 2}^{(0)} = \beta_{\alpha}^{2} k^{2} \left(\frac{qm}{r}\right)^{2} J_{q|m|}^{2}(\gamma r) \sin^{2}(kz) + \beta_{\alpha}^{2} \omega^{2} \gamma^{2} J_{q|m|}^{\prime 2}(\gamma r) \cos^{2}(kz),$$

$$E_{\alpha 3}^{(0)} E_{\alpha 3}^{(0)} + B_{\alpha 3}^{(0)} B_{\alpha 3}^{(0)} = \beta_{\alpha}^{2} \gamma^{4} J_{q|m|}^{2}(\gamma r) \cos^{2}(kz)$$

$$E_{\alpha 1}^{(0)} E_{\alpha 3}^{(0)} + B_{\alpha 1}^{(0)} B_{\alpha 3}^{(0)} = -\beta_{\alpha}^{2} k \gamma^{3} J_{q|m|}(\gamma r) J_{q|m|}^{\prime}(\gamma r) \sin(kz) \cos(kz).$$
(3.81)

For TE waves we get

$$E_{\alpha 1}^{(1)} E_{\alpha 1}^{(1)} + B_{\alpha 1}^{(1)} B_{\alpha 1}^{(1)} = \beta_{\alpha}^{2} \omega^{2} \left(\frac{qm}{r}\right)^{2} J_{q|m|}^{2} (\gamma r) \sin^{2}(kz) + \beta_{\alpha}^{2} k^{2} \gamma^{2} J_{q|m|}^{\prime 2} (\gamma r) \cos^{2}(kz),$$

$$E_{\alpha 2}^{(1)} E_{\alpha 2}^{(1)} + B_{\alpha 2}^{(1)} B_{\alpha 2}^{(1)} = \beta_{\alpha}^{2} \omega^{2} \gamma^{2} J_{q|m|}^{\prime 2} (\gamma r) \sin^{2}(kz) + \beta_{\alpha}^{2} k^{2} \left(\frac{qm}{r}\right)^{2} J_{q|m|}^{2} (\gamma r) \cos^{2}(kz),$$

$$E_{\alpha 3}^{(1)} E_{\alpha 3}^{(1)} + B_{\alpha 3}^{(1)} B_{\alpha 3}^{(1)} = \beta_{\alpha}^{2} \gamma^{4} J_{q|m|}^{2} (\gamma r) \sin^{2}(kz)$$

$$E_{\alpha 1}^{(1)} E_{\alpha 3}^{(1)} + B_{\alpha 1}^{(1)} B_{\alpha 3}^{(1)} = \beta_{\alpha}^{2} k \gamma^{3} J_{q|m|} (\gamma r) J_{q|m|}^{\prime} (\gamma r) \sin(kz) \cos(kz).$$
(3.82)

Now we see that the contributions to the off-diagonal component coming from the TM and TE modes cancel each other and the vacuum energy-momentum tensor is diagonal. Summing the contribution from the TM and TE parts one finds

$$\sum_{\lambda} \left[E_{\alpha 1}^{(\lambda)} E_{\alpha 1}^{(\lambda)} + B_{\alpha 1}^{(\lambda)} B_{\alpha 1}^{(\lambda)} \right] = \beta_{\alpha}^{2} \gamma^{2} \left[k^{2} J_{q|m|}^{\prime 2} (\gamma r) + \omega^{2} \left(\frac{qm}{\gamma r} \right)^{2} J_{q|m|}^{2} (\gamma r) \right],$$

$$\sum_{\lambda} \left[E_{\alpha 2}^{(\lambda)} E_{\alpha 2}^{(\lambda)} + B_{\alpha 2}^{(\lambda)} B_{\alpha 2}^{(\lambda)} \right] = \beta_{\alpha}^{2} \gamma^{2} \left[\omega^{2} J_{q|m|}^{\prime 2} (\gamma r) + k^{2} \left(\frac{qm}{\gamma r} \right)^{2} J_{q|m|}^{2} (\gamma r) \right],$$

$$\sum_{\lambda} \left[E_{\alpha 2}^{(\lambda)} E_{\alpha 2}^{(\lambda)} + B_{\alpha 2}^{(\lambda)} B_{\alpha 2}^{(\lambda)} \right] = \beta_{\alpha}^{2} \gamma^{4} J_{q|m|}^{2} (\gamma r).$$
(3.83)

As a consequence, the corresponding VEVs are presented in the form

$$P_{11} = \frac{4q}{a} \sum_{m,n=0}^{\infty} \int_{0}^{\infty} d\gamma \frac{\gamma}{\omega} \left[k^{2} J_{qm}^{\prime 2}(\gamma r) + \omega^{2} \left(\frac{qm}{\gamma r}\right)^{2} J_{qm}^{2}(\gamma r) \right],$$

$$P_{22} = \frac{4q}{a} \sum_{m,n=0}^{\infty} \int_{0}^{\infty} d\gamma \frac{\gamma}{\omega} \left[\omega^{2} J_{qm}^{\prime 2}(\gamma r) + k^{2} \left(\frac{qm}{\gamma r}\right)^{2} J_{qm}^{2}(\gamma r) \right],$$

$$P_{33} = \frac{4q}{a} \sum_{m,n=0}^{\infty} \int_{0}^{\infty} d\gamma \frac{\gamma^{3}}{\omega} J_{qm}^{2}(\gamma r),$$
(3.84)

where $k = k_n = \pi n/a$ and $\omega = \sqrt{\gamma^2 + (n\pi/a)^2}$. As before, the expressions on the right-hand sides are divergent and the presence of a cutoff function is assumed.

By using the summation formula (3.20), for the series over n in (3.84) one obtains

$$\frac{\pi}{a} \sum_{n=0}^{\infty} \frac{1}{\omega} = \int_{0}^{\infty} \frac{dk}{\sqrt{\gamma^{2} + k^{2}}} + 2\int_{\gamma}^{\infty} dx \, \frac{(x^{2} - \gamma^{2})^{-1/2}}{e^{2ax} - 1},$$

$$\frac{\pi}{a} \sum_{n=0}^{\infty} \frac{k^{2}}{\omega} = \int_{0}^{\infty} \frac{dk \, k^{2}}{\sqrt{\gamma^{2} + k^{2}}} - 2\int_{\gamma}^{\infty} dx \, \frac{x^{2}(x^{2} - \gamma^{2})^{-1/2}}{e^{2ax} - 1}.$$
(3.85)

The first terms in the right hand sides of these formulas correspond to the part in the boundaryfree conical defect. In the geometry of a single plate the corresponding VEVs vanish. Hence, we can write

$$P_{ij} = P_{ij}^{(0)} + P_{ij}^{(b)}, (3.86)$$

where the boundary-free part is given by

$$P_{11}^{(0)} = \frac{q}{\pi^2} \sum_{m=0}^{\infty} \int_0^\infty dk \int_0^\infty d\gamma \frac{\gamma}{\omega} \left[k^2 J_{qm}^{\prime 2}(\gamma r) + \omega^2 \left(\frac{qm}{\gamma r}\right)^2 J_{qm}^2(\gamma r) \right],$$

$$P_{22}^{(0)} = \frac{q}{\pi^2} \sum_{m=0}^\infty \int_0^\infty dk \int_0^\infty d\gamma \frac{\gamma}{\omega} \left[\omega^2 J_{qm}^{\prime 2}(\gamma r) + k^2 \left(\frac{qm}{\gamma r}\right)^2 J_{qm}^2(\gamma r) \right],$$

$$P_{33}^{(0)} = \frac{q}{\pi^2} \sum_{m=0}^\infty \int_0^\infty dk \int_0^\infty d\gamma \frac{\gamma^3}{\omega} J_{qm}^2(\gamma r).$$
(3.87)

The boundary-induced parts come from the last terms in (3.85)

$$P_{11}^{(b)} = -\frac{2q}{\pi^2} \sum_{m=0}^{\infty} \int_0^\infty d\gamma \int_{\gamma}^\infty dx \gamma \frac{(x^2 - \gamma^2)^{-1/2}}{e^{2ax} - 1} \\ \times \left[x^2 J_{qm}^{\prime 2}(\gamma r) + (x^2 - \gamma^2) \left(\frac{qm}{\gamma r}\right)^2 J_{qm}^2(\gamma r) \right], \\ P_{22}^{(b)} = -\frac{2q}{\pi^2} \sum_{m=0}^{\infty} \int_0^\infty d\gamma \int_{\gamma}^\infty dx \gamma \frac{(x^2 - \gamma^2)^{-1/2}}{e^{2ax} - 1} \\ \times \left[(x^2 - \gamma^2) J_{qm}^{\prime 2}(\gamma r) + x^2 \left(\frac{qm}{\gamma r}\right)^2 J_{qm}^2(\gamma r) \right], \\ P_{33}^{(b)} = \frac{2q}{\pi^2} \sum_{m=0}^\infty \int_0^\infty d\gamma \int_{\gamma}^\infty dx \gamma^3 \frac{(x^2 - \gamma^2)^{-1/2}}{e^{2ax} - 1} J_{qm}^2(\gamma r).$$
(3.88)

The integrals in (3.88) are evaluated by using the formulas

$$\int_{\gamma}^{\infty} dx \frac{(x^2 - \gamma^2)^{-1/2}}{e^{2ax} - 1} = \sum_{n=1}^{\infty} K_0(2an\gamma),$$

$$\int_{\gamma}^{\infty} dx \frac{x^2(x^2 - \gamma^2)^{-1/2}}{e^{2ax} - 1} = \sum_{n=1}^{\infty} \gamma^2 K_0''(2an\gamma),$$
(3.89)

where we have used the expansion (3.43). This leads to the following expressions

$$P_{11}^{(b)} = -\frac{2q}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \int_0^\infty d\gamma \gamma^3 \left[J_{qm}^{(2)}(\gamma r) K_0(2an\gamma) + G_{qm}(\gamma r) \frac{K_1(2an\gamma)}{2an\gamma} \right],$$

$$P_{22}^{(b)} = -\frac{2q}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \int_0^\infty d\gamma \gamma^3 \left[\left(\frac{qm}{\gamma r} \right)^2 J_{qm}^2(\gamma r) K_0(2an\gamma) + G_{qm}(\gamma r) \frac{K_1(2an\gamma)}{2an\gamma} \right],$$

$$P_{33}^{(b)} = \frac{2q}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \int_0^\infty d\gamma \gamma^3 J_{qm}^2(\gamma r) K_0(2an\gamma).$$
(3.90)

For points away from the plates the boundary-induced parts $P_{ll}^{(b)}$ are finite and the cutoff function, implicitly assumed in the discussion above, can be safely removed.

As it has been shown the single plate parts in the VEV of the energy-momentum tensor vanish, $\langle T_{\mu\nu} \rangle_1^{(b)} = 0$. Hence, in the regions z < 0 and z > a one has $\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu} \rangle^{(s)}$. The

corresponding energy density is obtained by using the VEVs of the electric and magnetic field squared, given by (3.28):

$$\langle T_0^0 \rangle^{(s)} = \frac{\langle E^2 \rangle_{\text{ren}}^{(s)} + \langle B^2 \rangle_{\text{ren}}^{(s)}}{8\pi} = -\frac{(q^2 - 1)(q^2 + 11)}{720\pi^2 r^4}.$$
 (3.91)

From the Lorentz invariance of the boundary-free problem with respect to the boosts along the defect axis it follows that $\langle T_3^3 \rangle^{(s)} = \langle T_0^0 \rangle^{(s)}$. The remaining components are found by using the continuity equation $\nabla_{\nu} \langle T_{\mu}^{\nu} \rangle^{(s)} = 0$ and the zero trace condition $\langle T_{\mu}^{\mu} \rangle^{(s)} = 0$. In this way we can see that

$$\left\langle T_{\mu}^{\nu}\right\rangle^{(s)} = -\frac{\left(q^2 - 1\right)\left(q^2 + 11\right)}{720\pi^2 r^4} \operatorname{diag}\left(1, 1, -3, 1\right).$$

$$(3.92)$$

This result was obtained in [13, 14].

In the region between the plates the total VEV is presented in the decomposed form

$$\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu} \rangle^{(s)} + \langle T_{\mu\nu} \rangle^{(b)} .$$
(3.93)

The boundary-induced part in the energy density is obtained by using the corresponding parts in the electric and magnetic field squared. The remaining components are obtained combining the result for the energy density with the formulas (3.90). This leads to the following expressions

$$\langle T_{0}^{0} \rangle^{(b)} = -\frac{q}{\pi^{2}} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty}' \int_{0}^{\infty} d\gamma \gamma^{3} \left[G_{qm}(\gamma r) Q(2an\gamma) - J_{qm}^{2}(\gamma r) K_{0}(2an\gamma) \right],$$

$$\langle T_{1}^{1} \rangle^{(b)} = -\frac{q}{\pi^{2}} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty}' \int_{0}^{\infty} d\gamma \gamma^{3} \left\{ J_{qm}'^{2}(\gamma r) + \left[1 - \left(\frac{qm}{\gamma r} \right)^{2} \right] J_{qm}^{2}(\gamma r) \right\} K_{0}(2an\gamma),$$

$$\langle T_{2}^{2} \rangle^{(b)} = \frac{q}{\pi^{2}} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty}' \int_{0}^{\infty} d\gamma \gamma^{3} \left\{ J_{qm}'^{2}(\gamma r) - \left[1 + \left(\frac{qm}{\gamma r} \right)^{2} \right] J_{qm}^{2}(\gamma r) \right\} K_{0}(2an\gamma),$$

$$\langle T_{3}^{3} \rangle^{(b)} = \frac{q}{\pi^{2}} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty}' \int_{0}^{\infty} d\gamma \gamma^{3} \left[G_{qm}(\gamma r) Q(2an\gamma) + J_{qm}^{2}(\gamma r) K_{0}(2an\gamma) \right],$$

$$(3.94)$$

and the off-diagonal components vanish. It is easily checked that the boundary-induced part is traceless. Moreover, it does not depend on the z-coordinate.

Note that, by using the relation

$$J_{qm}^{\prime 2}(\gamma r) + \left[1 - \left(\frac{qm}{\gamma r}\right)^2\right] J_{qm}^2(\gamma r) = \frac{2}{r^2} \int_0^r dx \, x J_{qm}^2(\gamma x), \tag{3.95}$$

the expression for the radial stress may also be written in the form

$$\langle T_1^1 \rangle^{(b)} = -\frac{2q}{\pi^2 r^2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_0^\infty d\gamma \gamma^3 K_0(2an\gamma) \int_0^r dx \, x J_{qm}^2(\gamma x),$$
 (3.96)
from which it follows that the boundary-induced part in the vacuum pressure along the radial direction, $p_1^{(b)} = -\langle T_1^1 \rangle^{(b)}$, is always positive. The same is the case for the boundary-free part. Note that, from (3.94) we also have $\langle T_1^1 \rangle^{(b)} + \langle T_2^2 \rangle^{(b)} \leq 0$.

It can be explicitly checked that the boundary-induced parts obey the covariant continuity equation, $\nabla_{\mu} \langle T^{\mu}_{\nu} \rangle^{(b)} = 0$. By taking into account that the energy-momentum tensor is diagonal, this equation is reduced to the single relation $\partial_r (r \langle T^1_1 \rangle) = \langle T^2_2 \rangle$ between the radial and azimuthal stresses. Indeed, from (3.96) we have

$$\partial_{r}(r \langle T_{1}^{1} \rangle^{(b)}) = \frac{2q}{\pi^{2}} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_{0}^{\infty} d\gamma \gamma^{3} K_{0}(2an\gamma) \left[\frac{1}{r^{2}} \int_{0}^{r} dx \, x J_{qm}^{2}(\gamma x) - J_{qm}^{2}(\gamma r) \right]$$

$$= - \langle T_{1}^{1} \rangle - \frac{2q}{\pi^{2}} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_{0}^{\infty} d\gamma \gamma^{3} K_{0}(2an\gamma) J_{qm}^{2}(\gamma r), \qquad (3.97)$$

which concides with the expression (3.94) for $\langle T_2^2 \rangle^{(b)}$, written in the form

$$\langle T_2^2 \rangle^{(b)} = - \langle T_1^1 \rangle^{(b)} - \frac{2q}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty}' \int_0^\infty d\gamma \gamma^3 K_0(2an\gamma) J_{qm}^2(\gamma r).$$
 (3.98)

From here one gets the relation between the azimuthal and radial pressures:

$$p_2^{(b)} = -p_1^{(b)} + \frac{2q}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty}' \int_0^\infty d\gamma \gamma^3 K_0(2an\gamma) J_{qm}^2(\gamma r), \qquad (3.99)$$

or $p_2^{(b)} > -p_1^{(b)}$.

Let us consider the behavior of the boundary-induced part in the VEV of the energymomentum tensor near the string assuming that $r \ll a$. Assuming that q > 1 and by using the formula for the Bessel function for small arguments, to the leading order we find

$$\langle T_0^0 \rangle^{(b)} \approx \langle T_3^3 \rangle^{(b)} \approx - \langle T_1^1 \rangle^{(b)} \approx - \langle T_2^2 \rangle^{(b)} \approx \frac{q\pi^2}{720a^4}.$$
 (3.100)

Here, additionally it was assumed that $(r/a)^{q-1} \ll 1$. The latter means that the parameter q is not too close to 1. The next-to-leading term in the expansion over r/a behaves as $(r/a)^{2q-2}$ for 1 < q < 2 and as $(r/a)^2$ for $q \ge 2$. In particular, we see that near the string the boundaryinduced part in the energy density is positive. Note that in this region the total VEV is dominated by the boundary-free part. At large distances from the string, $r \gg a$, the effects induced by the nontrivial topology of the string are small and, to the leading order, the VEV coincides with the corresponding expression for metallic plates in Minkowski spacetime 3.92:

$$\langle T^{\mu}_{\nu} \rangle \approx \langle T^{\mu}_{\nu} \rangle^{(b)} \approx \langle T^{\mu}_{\nu} \rangle_{\mathrm{M}} = -\frac{\pi^2}{720a^4} \mathrm{diag}(1, 1, 1, -3).$$
 (3.101)

In this region the both boundary-free and boundary-induced parts in the vacuum energy density are negative.

For integer values of q for the energy density

$$\langle T_0^0 \rangle^{(b)} = \frac{C(r/a)}{4\pi} = -\frac{1}{8\pi^2} \sum_{l=0}^{q-1} \sum_{n=1}^{\infty} \frac{(1-4s_l^2)(an)^2 + r^2 s_l^2}{[(an)^2 + r^2 s_l^2]^3}.$$
 (3.102)

For the summation over m in the expression for the spatial components we can use the formulas (3.34). In addition, we need to sum the series

$$\sum_{m=0}^{\infty} \left[J_{qm}^{\prime 2}(y) - \left(\frac{qm}{y}\right)^2 J_{qm}^2(y) \right].$$
 (3.103)

In order to do that we use the relation

$$\int_{0}^{y} dx \, x J_{qm}^{2}(x) = \frac{y^{2}}{2} \left[J_{qm}^{\prime 2}(y) + \left(1 - \frac{q^{2}m^{2}}{y^{2}}\right) J_{qm}^{2}(y) \right], \qquad (3.104)$$

for the Bessel function. From here one has

$$J_{qm}^{\prime 2}(y) - \left(\frac{qm}{y}\right)^2 J_{qm}^2(y) = \frac{2}{y^2} \int_0^y dx \, x J_{qm}^2(x) - J_{qm}^2(y), \qquad (3.105)$$

and, hence,

$$\sum_{m=0}^{\infty} \left[J_{qm}^{\prime 2}(y) - \left(\frac{qm}{y}\right)^2 J_{qm}^2(y) \right]$$

= $\frac{2}{y^2} \int_0^y dx \, x \sum_{m=0}^{\infty} J_{qm}^2(x) - \sum_{m=0}^{\infty} J_{qm}^2(y)$
= $\frac{1}{2q} \sum_{l=0}^{q-1} \left[\frac{2}{y^2} \int_0^y dx \, x J_0(2xs_l) - J_0(2ys_l) \right].$ (3.106)

For the integral in this expression we have

$$\int_{0}^{y} dx \, x J_0(2xs_l) = \frac{y J_1(2ys_l)}{2s_l}.$$
(3.107)

As a consequence we get

$$\sum_{m=0}^{\infty} \left[J_{qm}^{\prime 2}(y) - \left(\frac{qm}{y}\right)^2 J_{qm}^2(y) \right] = \frac{1}{2q} \sum_{l=0}^{q-1} \left[\frac{J_1(2ys_l)}{ys_l} - J_0(2ys_l) \right].$$
(3.108)

Combining (3.108) with (3.34), in the case of integer q for the stresses one finds the following

expressions

$$\langle T_1^1 \rangle^{(b)} = -\frac{a^{-4}}{8\pi^2} \sum_{l=0}^{q-1} \sum_{n=1}^{\infty} \frac{1}{[n^2 + (rs_l/a)^2]^2},$$

$$\langle T_2^2 \rangle^{(b)} = -\frac{a^{-4}}{8\pi^2} \sum_{l=0}^{q-1} \sum_{n=1}^{\infty} \frac{n^2 - 3(rs_l/a)^2}{[n^2 + (rs_l/a)^2]^3},$$

$$\langle T_3^3 \rangle^{(b)} = \frac{a^{-4}}{8\pi^2} \sum_{l=0}^{q-1} \sum_{n=1}^{\infty} \frac{(3 - 4s_l^2) n^2 - (rs_l/a)^2}{[n^2 + (rs_l/a)^2]^3}.$$

$$(3.109)$$

By using the functions $h_2(b)$ and $h_3(b)$ from (3.50), we can write the obtained formulas in a single expression (no summation over μ)

$$\langle T^{\mu}_{\mu} \rangle^{(b)} = -\frac{a^{-4}}{8\pi^2} \sum_{l=0}^{q-1} \left[f_{\mu,2}(s_l) h_2(rs_l/a) + f_{\mu,3}(s_l) h_3(rs_l/a) \right],$$
 (3.110)

where

$$f_{0,2}(x) = 1 - 4x^{2}, \quad f_{0,3}(x) = 4x^{2},$$

$$f_{1,2}(x) = 1, \quad f_{1,3}(x) = 0,$$

$$f_{2,2}(x) = 1, \quad f_{2,3}(x) = -4,$$

$$f_{3,2}(x) = 4x^{2} - 3, \quad f_{3,3}(x) = 4 - 4x^{2}.$$
(3.111)

The l = 0 terms in (3.110) coincide with the corresponding quantities for parallel plates in Minkowski spacetime (see (3.101)). In Fig. 16 we present the ratio of the boundary-induced part of the energy density to the corresponding quantity for parallel plates in Minkowski spacetime as a function of the distance from the string. The numbers near the curves are the values of the parameter q. In the regions z < 0 and z > a the boundary-induced part in the VEV of the energy-momentum tensor vanishes and in these regions $\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu} \rangle^{(s)}$.

The Casimir force acting per unit surface of the plate is determined by the component $\langle T_3^3 \rangle$. For the corresponding effective pressure one has: $p_3 = -\langle T_3^3 \rangle$. The boundary free part of the pressure is the same for both sides of the plate and it does not contribute to the net force. Hence, the force per unit surface of the plate is determined by the boundary induced part of the pressure along the z-direction:

$$p_{3}^{(b)} = -\frac{q}{\pi^{2}a^{4}} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_{0}^{\infty} d\gamma \gamma^{3} \left\{ G_{qm}(\gamma r/a)Q(2n\gamma) + J_{qm}^{2}(\gamma r)K_{0}(2n\gamma) \right\}.$$
 (3.112)

In contrast to the case of Minkowski bulk, the Casimir stress on the plates is not uniform. The effective pressure (3.112) is always negative and, consequently, the corresponding Casimir force is always attractive. For integer values of the parameter q one has

$$p_{3}^{(b)} = -\frac{a^{-4}}{8\pi^{2}} \sum_{l=0}^{q-1} \sum_{n=1}^{\infty} \frac{(3-4s_{l}^{2})n^{2} - (rs_{l}/a)^{2}}{[n^{2} + (rs_{l}/a)^{2}]^{3}}$$

$$= -\frac{a^{-4}}{8\pi^{2}} \sum_{l=0}^{q-1} \left[\left(3 - 4s_{l}^{2} \right) h_{2}(rs_{l}/a) - 4 \left(1 - s_{l}^{2} \right) h_{3}(rs_{l}/a) \right], \qquad (3.113)$$



Figure 16: The ratio of the boundary-induced part in the energy density to the corresponding quantity for parallel plates in Minkowski spacetime. The numbers near the curves are the values of the parameter q.

As it has been shown before, the corresponding forces are attractive. The l = 0 term in this expression coincides with the corresponding quantity for plates in Minkowski spacetime: $p_{M,3}$ $= -\pi^2 2a^{-4}/240$. In Fig. 17 is plotted the ratio $p_3^{(b)}/p_{M,3}$ as a function of the distance from the string (in units of the separation between the plates) for separate values of the parameter q(numbers near the curves).



Figure 17: The ratio of the Casimir stress on the plate to the corresponding quantity in Minkowski spacetime as a function of the distance r/a from the string for separate values of the parameter q.

3.5 Summary

In this chapter we have investigated the combined effects from nontrivial topology and boundaries on properties of the electromagnetic vacuum. The nontrivial topology of the spacetime is induced by the presence of the conical string-like defects, and the for the boundaries we have considered the classical Casimir geometry with two parallel conducting plates. Among the most important characteristics of the ground state are the VEVs of the electric and magnetic fields squared and the energy-momentum tensor. To evaluate these VEVs we have employed the direct mode summation technique. Applying the Abel-Plana summation formula to the mode sum over the eigenvalues of the wave vector component along the conical defects, we have explicitly decomposed the VEVs into the pure string and boundary-induced parts. The pure string parts in the VEVs of the electric and magnetic fields squared coincide and they are negative everywhere. The presence of the conical defect increases the boundary-induced parts in the VEVs of the field squared when compared with the Minkowski spacetime results.

The boundary part in the VEVs of the field squared are further split into the single plate

and the second plate-induced parts. Single plate parts are given by expression (3.29) and they have opposite signs for the electric and magnetic fields. For points near the string, $r \ll z$, the VEV of the electric field squared is dominated by the pure string part and is negative. Near the plate, $z \ll r$, the plate induced part dominates and the VEV is positive. The VEV of the magnetic field squared is negative everywhere. The second plate-induced parts are presented in the form (3.42) with the functions defined in (3.46). The z-dependent parts in the expressions for the electric and magnetic fields squared have opposite signs and they are cancelled in the expression for the vacuum energy density. The general formulas are simplified in a special case of integer values of the parameter q. The corresponding expressions take the form (3.51) and (3.48).

We have investigated the Casimir-Polder interaction of a polarizable microparticle in this geometry. In the static limit, when the dispersion of the polarizability can be neglected, the Casimir-Polder potential is expressed in terms of the VEV of the electric field squared. For the evaluation of the latter we have used the method of direct summation over a complete set of the electromagnetic field modes in the problem under consideration. We have provided a closed expression for the Casimir-Polder potential for an arbitrary value of the planar angle deficit, given by (3.70) in the region between the plates. In the remaining regions, the potential describes the interaction between a polarizable microparticle and a single plate. The corresponding expression for the region z < 0 is presented as (3.73). Near the plates and not too close to the defect, the leading term in asymptotic expansion coincides with the Casimir-Polder potential for a single plate in Minkowski bulk and is given by (3.75). For points close to the defect, the leading contribution comes from the part corresponding to the geometry in the absence of the plates with the radial force given by (3.66). The latter decays as $1/r^5$. At large distances from the defect, the radial force is presented as (3.77) and the decay is slower, like $1/r^4$.

The boundary-induced part in the VEV of the energy-momentum tensor for the electromagnetic field is different from zero in the region between the plates only. The expressions for the separate components are given by formula (3.94). The VEVs are uniform with respect to the coordinate along the conical defect and depend on the distance from the string only. The boundary-induced part in the vacuum energy density is positive near the string and negative at large distances from the string. The radial effective pressure is always positive. The general formulas for the VEV of the energy-momentum tensor are simplified for integer values of the parameter q. The corresponding expressions are given by (3.110). In contrast to the geometry of two conducting plates in Minkowski spacetime, in the geometry with conical defect the Casimir stresses on the plates are non-uniform. They depend on the distance from the string and this dependence is not monotonic. The Casimir forces acting on the plates are always attractive. They are given by (3.112) for the general case and by (3.113) for integer values of the parameter q.

CONCLUSIONS

- 1. The magnetic flux threading a two-dimensional circular planar ring induces the charge and azimuthal current densities in the ground state of a quantum fermionic field. They are odd periodic functions of the magnetic flux with the period equal to the flux quantum. On the outer edge of the ring the current density is equal to the charge density whereas on the inner edge they have opposite signs. For a fixed values of the other parameters, both the charge and current densities decrease by the modulus with decreasing outer radius.
- 2. Unlike the case of the boundary-free geometry the charge and current densities in the ring are continuous at half-odd integer values for the ratio of the magnetic flux to the flux quantum, and both of them vanish at these points. The behaviour of the mean charge and current densities as functions of the field mass (energy gap in field theoretical models of planar condensed matter system) is essentially different for two inequivalent representations of the Clifford algebra. With the initial increase of the mass from the zero value, the modulus for the charge and current densities decreases for one of the irreducible representations and increases for the other. With further increase of the mass the vacuum densities are suppressed in both cases.
- 3. The charge and current densities in parity and time-reversal symmetric models are obtained combining the results for two inequivalent representations. Assuming that for both these representations the boundary condition and the masses are the same, the resulting charge density vanishes, whereas the current density is doubled. The effective charge density may appear if the energy gap generation mechanism breaks the symmetry with respect to the representations.
- 4. The fermionic condensate and the ground state energy-momentum tensor for a fermionic field in the geometry of two parallel plates perpendicular to the conical defect are decomposed into the boundary-free and boundary-induced contributions. Fermionic condensate is positive near the string and negative at large distances, whereas the vacuum energy density is negative everywhere. The radial stress is equal to the energy density. For a

massless field, the boundary induced contribution in the VEV of the energy-momentum tensor is different from zero in the region between the plates only and it does not depend on the coordinate along the defect axis.

- 5. The fermionic Casimr pressures on the plates are inhomogeneous and vanishe at the location of the conical defect. The corresponding Casimir forces are attractive.
- 6. Combined effects from a conical defect and parallel conducting plates on the zero-point fluctuations of the electromagnetic field are investigated. The expectation value of the electric field squared is negative for points with the radial distance to the string smaller than the distance to the plates, and positive for the opposite situation. The expectation value for the magnetic field squared is negative everywhere. The boundary-induced part in the VEV of the energy-momentum tensor is different from zero in the region between the plates only. This part only depends on the distance from the string. The boundaryinduced part in the vacuum energy density is positive for points with a distance to the string smaller than the distance to the plates and negative in the opposite situation.
- 7. The electromagnetic Casimir stresses on the plates depend non-monotonically on the distance from the defect. The corresponding forces are always attractive. The Casimir-Polder forces acting on a polarziable microparticle are repulsive with respect to the defect and attractive with respect to the closer plate.

LIST OF PAPERS PUBLISHED ON THE TOPIC OF THE THESIS

- E. R. Bezerra de Mello, A. A. Saharian, A. Kh. Grigoryan, Casimir effect for parallel metallic plates in cosmic string spacetime, J. Phys. A: Math. Theor. 45, 374011 (2012) (15 pages).
- S. Bellucci, A. A. Saharian, A. Kh. Grigoryan, Induced fermionic charge and current densities in two-dimensional rings, Phys. Rev. D 94, 105007 (2016) (21 pages).
- A. Kh. Grigoryan, A. R. Mkrtchyan, A. A. Saharian, Fermionic condensate and the Casimir effect in cosmic string spacetime, Int. J. Mod. Phys. D 26, 1750064 (2017) (26 pages).
- A.Kh. Grigoryan, Casimir-Polder potential for parallel metallic plates in background of a conical defect, Arm.J. Phy., 10, 81-91 (2017).

ACKNOWLEDGMENTS

This thesis would not have been possible without the constant help, advise and direction of my supervisor Prof. A. A. Saharian, to whom I am much obliged. I am thankful to the members of the Institute of Applied Problems of Physics (National Academy of Science, Republic of Armenia) for their valuable discussions and comments. And the genial support of Hayk Hakobyan, Torgom Yezekyan and Gurgen Arabajyan meant much to me.

References

- N. D. Birrell, P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
- [2] A. A. Grib, S. G. Mamayev, V. M. Mostepanenko, Vacuum Quantum Effects in Strong Fields (Friedmann Laboratory Publishing, St. Petersburg, 1994).
- [3] S. A. Fulling, Aspects of Quantum Field Theory in Curved Space-Time (Cambridge University Press, Cambridge, England, 1996).
- [4] L. Parker, D. Toms, Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity (Cambridge University Press, Cambridge, England, 2009).
- [5] A. A. Abrikosov, L. P. Gorkov, I. E. Dzyaloshinskii, Methods of quantum field theory in statistical physics (Dover, New York, 1963).
- [6] A. L. Fetter, J. D. Walecka, Quantum theory of many-particle systems (McGraw-Hill, New York, 1971).
- S. Doniach, E. H. Sondheimer, Green's functions for solid state physicists (Addison-Wesley, Redwood City, 1982).
- [8] G. Parisi, *Statistical field theory* (Addison Wesley, New York, 1988).
- [9] E. Fradkin, *Field theory of condensed matter systems* (Cambridge University Press, Cambridge, 2013).
- [10] A. M. Tsvelik, Quantum field theory in condensed matter physics (Cambridge University Press, Cambridge, 1995).
- [11] Y. Imry, Introduction to Mesoscopic Physics (Oxford University Press, Oxford, 1997).
- [12] G.V. Dunne, Topological Aspects of Low Dimensional Systems (Springer, Berlin, 1999).
- [13] A. O. Gogolin, A. A. Nersesyan, A. M. Tsvelik, Bosonization and strongly correlated systems (Cambridge University Press, Cambridge, 1999).

- [14] N. Nagaosa, Quantum field theory in condensed matter physics and quantum field theory in strongly correlated electronic systems (Springer, Berlin, 1999).
- [15] V.P. Gusynin, S.G. Sharapov, and J.P. Carbotte, AC conductivity of graphene: from tight-binding model to 2+1-dimensional quantum electrodynamics, Int. J. Mod. Phys. B 21, 4611 (2007).
- [16] A.H. Castro Neto, F. Guinea, N.M.R. Peres, K.S. Novoselov, A.K. Geim, The electronic properties of graphene, Rev. Mod. Phys. 81, 109 (2009).
- [17] M.A.H. Vozmediano, M.I. Katsnelson, F. Guinea, Gauge fields in graphene, Phys. Rep. 496, 109 (2010).
- [18] A. Altland, B. Simons, Condensed matter field theory (Cambridge University Press, Cambridge, 2010).
- [19] Xiao-Liang Qi and Shou-Cheng Zhang, Topological insulators and superconductors, Rev. Mod. Phys. 83, 1057 (2011).
- [20] O. Vafek, A. Vishwanath, Dirac Fermions in Solids: From High-T_c Cuprates and Graphene to Topological Insulators and Weyl Semimetals, Annu. Rev. Condens. Matter Phys. 5, 83 (2014).
- [21] R. Shankar, Quantum field theory and condensed matter (Cambridge University Press, Cambridge, 2017).
- [22] E. C. Marino, Quantum field theory approach to condensed matter physics (Cambridge University Press, Cambridge, 2017).
- [23] S. Deser, R. Jackiw, and S. Templeton, Topologically massive gauge theories, Ann. Phys. (N.Y.) 140, 372 (1982).
- [24] A.J. Niemi and G.W. Semenoff, Axial-Anomaly-Induced Fermion Fractionization and Effective Gauge-Theory Actions in Odd-Dimensional Space-Times, Phys. Rev. Lett. 51, 2077 (1983).

- [25] P. Cea, Variational approach to (2+1)-dimensional QED with topological mass term, Phys. Rev. D 34, 3229 (1986).
- [26] Y. Hosotani, Spontaneous breakdown of Lorentz invariance, Phys. Lett. B 319, 332 (1993).
- [27] Y. Hosotani, Variational approach to (2+1)-dimensional QED with topological mass term, Phys. Rev. D 51, 2022 (1995).
- [28] D. Cangemi, E. D'Hoker, G. Dunne, Derivative expansion of the effective action and vacuum instability for QED in 2+1 dimensions, Phys. Rev. D 51, 2513 (1995).
- [29] D. Wesolowski, Y. Hosotani, Spontaneous magnetization in Lorentz invariant theories, Phys. Lett. B 354, 396 (1995).
- [30] T. Itoh, T. Sato, Charge condensation in QED3 with a Chern-Simons term, Phys. Lett. B 367, 290 (1996).
- [31] P. Cea, Vacuum stability for Dirac fermions in three dimensions, Phys. Rev. D 55, 7985 (1997).
- [32] K.G. Klimenko, Three-dimensional Gross-Neveu model at nonzero temperature and in the presence of an external electromagnetic field, Z. Phys. C 54, 323 (1992).
- [33] K.G. Klimenko, Three-dimensional Gross-Neveu model at nonzero temperature and in an external magnetic field, Theor. Math. Phys. 90, 1 (1992).
- [34] V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Dynamical flavor symmetry breaking by a magnetic field in 2+1 dimensions, Phys. Rev. D 52, 4718 (1995).
- [35] R.R. Parwani, On chiral symmetry breaking by external magnetic fields in QED₃, Phys. Lett. B 358, 101 (1995).
- [36] G. Dunne, T. Hall, Inhomogeneous condensates in planar QED, Phys. Rev. D 53, 2220 (1996).

- [37] A. Das, M. Hott, Thermal effects on the catalysis by a magnetic field, Phys. Rev. D 53, 2252 (1996).
- [38] M. de J. Anguiano-Galicia, A. Bashir, A. Raya, $\bar{\psi} \psi$ condensate in constant magnetic fields, Phys. Rev. D **76**, 127702 (2007).
- [39] A. Raya and E. Reyes, Fermion condensate and vacuum current density induced by homogeneous and inhomogeneous magnetic fields in (2+1) dimensions, Phys. Rev. D 82, 016004 (2010).
- [40] A.N. Redlich, Parity violation and gauge noninvariance of the effective gauge field action in three dimensions, Phys. Rev. D 29, 2366 (1984).
- [41] C. L. Bilchak, R. W. Brown, and J. D. Stroughair, W and Z⁰ polarization in pair production: Dominant helicities, Phys. Rev. D 29, 375 (1984).
- [42] D. Boyanovsky, R. Blankenbecler, Axial and parity anomalies and vacuum charge: A direct approach, Phys. Rev. D 31, 3234 (1985).
- [43] R. Blankenbecler, D. Boyanovsky, Induced quantum numbers in (2+1)-dimensional QED, Phys. Rev. D 34, 612 (1986).
- [44] A.P. Polychronakos, Induced vacuum quantum numbers in 2 + 1 dimensions, Nucl. Phys. B 278, 207 (1986).
- [45] T. Jaroszewicz, Fermion-solenoid interactions: Vacuum charge and scattering theory, Phys. Rev. D 34, 3128 (1986).
- [46] P. Górnicki, Aharonov-bohm effect and vacuum polarization, Ann. Phys. (N.Y.) 202, 271 (1990).
- [47] E.G. Flekkøy, J.M. Leinaas, Vacuum Currents around a Magnetic Flux String, Int. J. Mod. Phys. A 6, 5327 (1991).
- [48] H.N. Li, D.A. Coker, and A.S. Goldhaber, selfconsistent solutions for vacuum currents around a magnetic flux string, Phys. Rev. D 47, 694 (1993).

- [49] V.B. Bezerra, E.R. Bezerra de Mello, Induced quantum numbers in the fermion-vortex system on a cone, Class. Qiantum Grav. 11, 457 (1994).
- [50] E.R. Bezerra de Mello, More about the fermion-vortex system on a cone, Class. Qiantum Grav. 11, 1415 (1994).
- [51] Yu.A. Sitenko, Induced vacuum condensates in the background of a singular magnetic vortex in (2+1)-dimensional space-time, Phys. Rev. D 60, 125017 (1999).
- [52] R. Jackiw, A.I. Milstein, S.-Y. Pi, I.S. Terekhov, Induced current and Aharonov-Bohm effect in graphene, Phys. Rev. B 80, 033413 (2009).
- [53] A.I. Milstein, I.S. Terekhov, Induced current in the presence of a magnetic flux tube of small radius, Phys. Rev. B 83, 075420 (2011).
- [54] M.F. Lin, D.S. Chuu, Persistent currents in toroidal carbon nanotubes, Phys. Rev. B 57, 6731 (1998).
- [55] S. Latil, S. Roche, A. Rubio, Persistent currents in carbon nanotube based rings, Phys. Rev. B 67, 165420 (2003).
- [56] K. Sasaki, Y. Kawazoe, Characteristic Behavior of Toroidal Carbon Nanotubes: Kinematics of Persistent Currents, Prog. Theor. Phys. 112, 369 (2004).
- [57] Z. Zhang, J. Yuan, M. Qiu, J. Peng, F. Xiao, Persistent currents in carbon nanotori: Effects of structure deformations and chirality, J. Appl. Phys. 99, 104311 (2006).
- [58] N. Xu, J.W. Ding, H.B. Chen, M.M. Ma, Curvature and external electric field effects on the persistent current in chiral toroidal carbon nanotubes, Eur. Phys. J. B 67, 71 (2009).
- [59] H. Bluhm et al., Persistent Currents in Normal Metal Rings, Phys. Rev. Lett. 102, 136802 (2009).
- [60] A.C. Bleszynski-Jayich et al., Persistent Currents in Normal Metal Rings, Science 326, 272 (2009).

- [61] C.W.J. Beenakker, Colloquium: Andreev reflection and Klein tunneling in graphene, Rev. Mod. Phys. 80, 1337 (2008).
- [62] K. Wakabayashi, S. Dutta, Nanoscale and edge effect on electronic properties of graphene, Solid State Commun. 152, 1420 (2012).
- [63] V.M. Mostepanenko, N.N. Trunov, The Casimir Effect and Its Applications (Clarendon, Oxford, 1997).
- [64] K.A. Milton, The Casimir Effect: Physical Manifestation of Zero-Point Energy (World Scientific, Singapore, 2002).
- [65] V.A. Parsegian, Van der Waals Forces (Cambridge University Press, Cambridge, England, 2005).
- [66] M. Bordag, G.L. Klimchitskaya, U. Mohideen, V.M. Mostepanenko, Advances in the Casimir Effect (Oxford University Press, Oxford, 2009).
- [67] D. Dalvit, P. Milonni, D. Roberts, F. da Rosa (ed.), Casimir Physics Lecture Notes in Physics, Vol. 834 (Springer-Verlag, Berlin, 2011).
- [68] S. Bellucci, A.A. Saharian, Fermionic Casimir densities in toroidally compactified spacetimes with applications to nanotubes, Phys. Rev. D 79, 085019 (2009).
- [69] S. Bellucci, A.A. Saharian, V.M. Bardeghyan, Induced fermionic current in toroidally compactified spacetimes with applications to cylindrical and toroidal nanotubes, Phys. Rev. D 82, 065011 (2010).
- [70] S. Bellucci, A.A. Saharian, H.A. Nersisyan, Scalar and fermionic vacuum currents in de Sitter spacetime with compact dimensions, Phys. Rev. D 88, 024028 (2013).
- [71] E.R. Bezerra de Mello, A.A. Saharian, and V. Vardanyan, Induced vacuum currents in anti-de Sitter space with toral dimensions, Phys. Lett. B 741, 155 (2015).
- [72] S. Bellucci, A.A. Saharian, Fermionic current from topology and boundaries with applications to higher-dimensional models and nanophysics, Phys. Rev. D 87, 025005 (2013).

- [73] S. Bellucci, A.A. Saharian, N.A. Saharyan, Casimir effect for scalar current densities in topologically nontrivial spaces, Eur. Phys. J. C 75, 378 (2015).
- [74] S. Bellucci, A.A. Saharian, V. Vardanyan, Vacuum currents in braneworlds on AdS bulk with compact dimensions, JHEP 11, 092 (2015).
- [75] S. Bellucci, A.A. Saharian, V. Vardanyan, Hadamard function and the vacuum currents in braneworlds with compact dimensions: Two-branes geometry, Phys. Rev. D 93, 084011 (2016).
- [76] S. Bellucci, E.R. Bezerra de Mello, A.A. Saharian, Finite temperature fermionic condensate and currents in topologically nontrivial spaces, Phys. Rev. D 89, 085002 (2014).
- [77] A. Vilenkin, E.P.S. Shellard, Cosmic Strings and Other Topological Defects (Cambridge University Press, Cambridge, England, 1994).
- [78] M. B. Hindmarsh, T. W. B. Kibble, Cosmic strings, Rep. Prog. Phys. 58, 477 (1995).
- [79] D. R. Nelson, Defects and Geometry in Condensed Matter Physics (Cambridge University Press, Cambridge, England, 2002).
- [80] G. E. Volovik, The Universe in a Helium Droplet (Clarendon Press, Oxford, 2003).
- [81] M. O. Katanaev and V. O. Volovich, Theory of defects in solids and three-dimensional gravity, Ann. Phys. 216, 1 (1992).
- [82] W. H. Zurek, Cosmological experiments in condensed matter physics, Phys. Rept. 276, 177 (1996).
- [83] F. Moraes, Condensed matter physics as a laboratory for gravitation and cosmology, Braz.J. Phys. **30**, 304 (2000).
- [84] Y. M. Bunkov, He³: cosmological and atomic physics experiments, Phil. Trans. R. Soc. A 366, 2821 (2008).
- [85] M. J. Bowick, L. Chander, E. A. Schiff, A. M. Srivastava, The cosmological Kibble mechanism in the laboratory: String formation in liquid crystals, Science 263, 943 (1994).

- [86] C. Bäuerle, Yu. M. Bunkov, S. N. Fisher, H. Godfrin, G. R. Pickett, Laboratory simulation of cosmic string formation in the early Universe using superfluid He³, Nature **382**, 332 (1996).
- [87] V. M. H. Ruutu, V. B. Eltsov, A. J. Gill, T. W. B. Kibble, M. Krusius, Yu. G. Makhlin, B. Praçais, G. E. Volovik, W. Xu, Vortex formation in neutron irradiated superfluid He³ as an analogue of cosmological defect formation, Nature **382**, 334 (1996).
- [88] D. I. Bradley, S. N. Fisher, A. M. Guenault, R. P. Haley, J. Kopu, H. Martin, G. R. Pickett, J. E. Roberts, V. Tsepelin, Relic topological defects from brane annihilation simulated in superfluid He³, Nature Phys. 4, 46 (2008).
- [89] T.M. Helliwell and D. A. Konkowski, Vacuum fluctuations outside cosmic strings, Phys. Rev. D 34, 1918 (1986).
- [90] W. A. Hiscock, Semiclassical gravitational effects near cosmic strings, Phys. Lett. B 188, 317 (1987).
- [91] B. Linet, Quantum field theory in the space-time of a cosmic string, Phys. Rev. D 35, 536 (1987).
- [92] V. P. Frolov and E. M. Serebriany, Vacuum polarization in the gravitational field of a cosmic string, Phys. Rev. D 35 3779 (1987).
- [93] J. S. Dowker, Vacuum averages for arbitrary spin around a cosmic string, Phys. Rev. D 36, 3742 (1987).
- [94] P. C. W. Davies, V. Sahni, Quantum gravitational effects near cosmic strings, Class. Quantum Grav. 5, 1 (1988).
- [95] G. E. A. Matsas, Vacuum fluctuations of twisted fields in the spacetime of a cosmic string, Phys. Rev. D 41, 3846 (1990).
- [96] M. Bordag, On the Vacuum-Interaction of Two Pairallel Cosmic Strings, Annalen der Physik 47, 93 (1990).

- [97] K. Shiraishi, S. Hirenzaki, Quantum aspects of self-interacting fields around cosmic strings, Class. Quantum Grav. 9, 2277 (1992).
- [98] G. Cognola, K. Kirsten, L. Vanzo, Free and self-interacting scalar fields in the presence of conical singularities, Phys. Rev. D 49, 1029 (1994).
- [99] M. E. X. Guimarães, B. Linet, Scalar Green's functions in an Euclidean space with a conical-type line singularity, Commun. Math. Phys. 165, 297 (1994).
- [100] E. S. Moreira Jnr, Massive quantum fields in a conical background, Nucl. Phys. B 451, 365 (1995).
- [101] M. Bordag, K. Kirsten, S. Dowker, Heat-kernels and functional determinants on the generalized cone, Commun. Math. Phys. 182, 371 (1996).
- [102] D. Iellici, Massive scalar field near a cosmic string, Class. Quantum Grav. 14, 3287 (1997).
- [103] N. R. Khusnutdinov, M. Bordag, Ground state energy of a massive scalar field in the background of a cosmic string of finite thickness, Phys. Rev. D 59, 064017 (1999).
- [104] J. Spinelly, E. R. Bezerra de Mello, Vacuum polarization of a charged massless scalar field on cosmic string space-time in the presence of a magnetic field, Class. Quantum Grav. 20, 873 (2003).
- [105] V. B. Bezerra, N. R. Khusnutdinov, The vacuum expectation value of the spinor massive field in the cosmic string space-time, Class. Quantum Grav. 23, 3449 (2006).
- [106] Yu. A. Sitenko and N. D. Vlasii, Induced vacuum energy-momentum tensor in the background of a cosmic string, arXiv:1110.0731.
- [107] E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, A. S. Tarloyan, Vacuum polarization induced by a cylindrical boundary in the cosmic string spacetime, Phys. Rev. D 74, 025017 (2006).
- [108] E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, A. S. Tarloyan, Fermionic vacuum polarization by a cylindrical boundary in the cosmic string spacetime, Phys. Rev. D 78, 105007 (2008).

- [109] E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, V. M. Bardeghyan, Fermionic current densities induced by magnetic flux in a conical space with a circular boundary, Phys. Rev. D 82, 085033 (2010).
- [110] I. Brevik, T. Toverud, Electromagnetic energy density around a superconducting cosmic string, Class. Quantum Grav. 12, 1229 (1995).
- [111] E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, Electromagnetic Casimir densities induced by a conducting cylindrical shell in the cosmic string spacetime, Phys. Lett. B 645, 245 (2007).
- [112] G. Fucci, K. Kirsten, The Casimir effect for conical pistons, JHEP 03, 016 (2011).
- [113] E. R. Bezerra de Mello, A. A. Saharian, Vacuum polarization by a flat boundary in cosmic string spacetime, Class. Quantum Grav. 28, 145008 (2011).
- [114] E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, H.H. Harutyunyan, Vacuum currents induced by a magnetic flux around a cosmic string with finite core, Phys. Rev. D 91, 064034 (2015).
- [115] A. Mohammadi, E.R. Bezerra de Mello, A.A. Saharian, Finite temperature fermionic charge and current densities induced by a cosmic string with magnetic flux, J. Phys. A Math. Theor. 48, 185401 (2015).
- [116] S. Bellucci1, E. R. Bezerra de Mello, E. Bragança, A. A. Saharian, Finite temperature fermion condensate, charge and current densities in a (2 + 1)-dimensional conical space. Eur. Phys. J. C 76, 350 (2016).
- [117] A. Chodos, R. Jaffe, K. Johnson, C. Thorn, and V. Weisskopf, New extended model of hadrons, Phys. Rev. D 9, 3471 (1974).
- [118] K. Johnson, The M.I.T. Bag Model, Acta Phys. Pol. B 6, 865 (1975).
- [119] M.V. Berry and R.J. Mondragon, Neutrino billiards: time-reversal symmetry-breaking without magnetic fields, Proc. R. Soc. London, Ser. A 412, 53 (1987).

- [120] E. McCann and V. I. Fal'ko, Symmetry of boundary conditions of the Dirac equation for electrons in carbon nanotubes, J. Phys.: Condens. Matter 16, 2371 (2004).
- [121] A.R. Akhmerov and C.W.J. Beenakker, Detection of Valley Polarization in Graphene by a Superconducting Contact, Phys. Rev. Lett. 98, 157003 (2007).
- [122] A.R. Akhmerov and C.W.J. Beenakker, Boundary conditions for Dirac fermions on a terminated honeycomb lattice, Phys. Rev. B 77, 085423 (2008).
- [123] J.A.M. van Ostaay, A.R. Akhmerov, C.W.J. Beenakker, and M. Wimmer, Dirac boundary condition at the reconstructed zigzag edge of graphene, Phys. Rev. B 84, 195434 (2011).
- [124] Yu.A. Sitenko, Casimir effect with quantized charged spinor matter in background magnetic field, Phys. Rev. D 91, 085012 (2015).
- [125] A.A. Saharian, The Generalized Abel-Plana Formula with Applications to Bessel Functions and Casimir Effect (Yerevan State University Publishing House, Yerevan, 2008); Preprint ICTP/2007/082; arXiv:0708.1187.
- [126] P. Recher, B. Trauzettel, A. Rycerz, Y.M. Blanter, C.W.J. Beenakker, and A.F. Morpurgo, Aharonov-Bohm effect and broken valley degeneracy in graphene rings, Phys. Rev. B 76, 235404 (2007).
- [127] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, *Integrals and Series* (Gordon and Breach, New York, 1986), Vol. 2.
- [128] E. R. Bezerra de Mello, F. Moraes, A. A. Saharian, Fermionic Casimir densities in a conical space with a circular boundary and magnetic flux, Phys. Rev. D 85, 045016 (2012).
- [129] P. de Sousa Gerbert and R. Jackiw, Classical and quantum scattering on a spinning cone, Commun. Math. Phys. 124, 229 (1989).
- [130] P. de Sousa Gerbert, Fermions in an Aharonov-Bohm field and cosmic strings, Phys. Rev. D 40, 1346 (1989).

- [131] Yu.A. Sitenko, Self-adjointness of the Two-Dimensional Massless Dirac Hamiltonian and Vacuum Polarization Effects in the Background of a Singular Magnetic Vortex, Ann. Phys. (N.Y.) 282, 167 (2000).
- [132] C.R. Hagen, Aharonov-Bohm scattering of particles with spin, Phys. Rev. Lett. 64, 503 (1990).
- [133] C.R. Hagen, Spin dependence of the Aharonov-Bohm effect: A Review, Int. J. Mod. Phys. A 6, 3119 (1991).
- [134] C.R. Hagen, Effects of nongauge potentials on the spin-¹/₂ Aharonov-Bohm problem, Phys. Rev. D 48, 5935 (1993).
- [135] M. Bordag and S. Voropaev, Bound states of an electron in the field of the magnetic string, Phys. Lett. B 333, 238 (1994).
- [136] C.R. Hagen and D.K. Park, Relativistic Aharonov-Bohm-Coulomb problem, Ann. Phys. (N.Y.) 251, 45 (1996).
- [137] F.M. Andrade, E.O. Silva, and M. Pereira, Physical regularization for the spin-1/2 Aharonov-Bohm problem in conical space, Phys. Rev. D 85, 041701(R) (2012).
- [138] C.G. Beneventano and E.M. Santangelo, Boundary conditions in the Dirac approach to graphene devices, Int. J. Mod. Phys.: Conf. Series 14, 240 (2012).
- [139] S. Bellucci, E.R. Bezerra de Mello, E. Bragança, and A.A. Saharian, Finite temperature fermion condensate, charge and current densities in a (2+1)-dimensional conical space, Eur. Phys. J. C 76, 359 (2016).
- [140] F.C. Khanna, A.P.C. Malbouisson, J.M.C. Malbouisson, and A.E. Santana, Quantum field theory on toroidal topology: Algebraic structure and applications, Phys. Rep. 539, 135 (2014).
- [141] S. Schnez, K. Ensslin, M. Sigrist, and T. Ihn, Analytic model of the energy spectrum of a graphene quantum dot in a perpendicular magnetic field, Phys. Rev. B 78, 195427 (2008).

- [142] M. Grujić, M. Zarenia, A. Chaves, M. Tadić, G.A. Farias, and F.M. Peeters, Electronic and optical properties of a circular graphene quantum dot in a magnetic field: Influence of the boundary conditions, Phys. Rev. B 84, 205441 (2011).
- [143] C.-H. Yan and L.-F. Wei, Size effects in Aharonov-Bohm graphene rings, J. Phys. Condens. Matter 22, 295503 (2010).
- [144] J. Wurm, M. Wimmer, H. U. Baranger, and K. Richter, Graphene rings in magnetic fields: Aharonov–Bohm effect and valley splitting, Semicond. Sci. Technol. 25, 034003 (2010).
- [145] M. M. Ma and J. W. Ding, Geometry dependence of persistent currents in diamond-like graphene rings, Solid State Commun. 150, 1196 (2010).
- [146] M. Zarenia, J. Milton Pereira, A. Chaves, F.M. Peeters, and G. A. Farias, Simplified model for the energy levels of quantum rings in single layer and bilayer graphene, Phys. Rev. B 81, 045431 (2010).
- [147] J. Schelter, P. Recher, and B. Trauzettel, The Aharonov–Bohm effect in graphene rings, Solid State Commun. 152, 1411 (2012).
- [148] I. Romanovsky, C. Yannouleas, and U. Landman, Patterns of the Aharonov-Bohm oscillations in graphene nanorings, Phys. Rev. B 85, 165434 (2012).
- [149] D. Faria, A. Latgé, S. E. Ulloa, and N. Sandler, Currents and pseudomagnetic fields in strained graphene rings, Phys. Rev. B 87, 241403(R) (2013).
- [150] D. Sticlet, B. Dóra, and J. Cayssol, Persistent currents in Dirac fermion rings, Phys. Rev. B 88, 205401 (2013).
- [151] V.P. Gusynin and S.G. Sharapov, Transport of Dirac quasiparticles in graphene: Hall and optical conductivities, Phys. Rev. B 73, 245411 (2006).
- [152] O. Vafek and A. Vishwanath, Dirac Fermions in Solids: From High-Tc Cuprates and Graphene to Topological Insulators and Weyl Semimetals, Annu. Rev. Condens. Matter Phys. 5, 83 (2014).

- [153] P. E. Lammert, V. H. Crespi, Topological Phases in Graphitic Cones, Phys. Rev. Lett. 85, 5190 (2000).
- [154] A. Cortijo, M. A. H. Vozmediano, Effects of topological defects and local curvature on the electronic properties of planar graphene, Nucl. Phys. B 763, 293 (2007).
- [155] Yu. A. Sitenko, N. D. Vlasii, Electronic properties of graphene with a topological defect, Nucl. Phys. B 787, 241 (2007).
- [156] C. Furtado, F. Moraes, A. M. M. Carvalho, Geometric phases in graphitic cones, Phys. Lett. A 372, 5368 (2008).
- [157] A. Jorio, G. Dresselhaus, M. S. Dresselhaus, Carbon Nanotubes: Advanced Topics in the Synthesis, Structure, Properties and Applications (Springer, Berlin, 2008).
- [158] A. Krishnan et al., Graphitic cones and the nucleation of curved carbon surfaces, Nature (London) 388, 451 (1997).
- [159] S.N. Naess, A. Elgsaeter, G. Helgesen, K. D. Knudsen, Carbon nanocones: wall structure and morphology, Sci. Tech. Adv. Mater. 10, 065002 (2009).
- [160] E.R.Bezerra de Mello, A.A. Saharian, and S.V. Abajyan, Fermionic vacuum polarization by a flat boundary in cosmic string spacetime, Class. Quantum Grav. 30, 015002 (2013).
- [161] E.R. Bezerra de Mello and A.A. Saharian, Topological Casimir effect in compactified cosmic string spacetime, Class. Quantum Grav. 29, 035006 (2012).
- [162] E.R. Bezerra de Mello and A.A. Saharian, Fermionic current induced by magnetic flux in compactified cosmic string spacetime, Eur. Phys. J. C 73, 2532 (2013).
- [163] A.A. Saharian, The Generalized Abel-Plana Formula with Applications to Bessel Functions and Casimir Effect (Yerevan State University Publishing House, Yerevan, 2008) (arXiv:0708.1187).
- [164] S. Bellucci, E.R. Bezerra de Mello, A. de Padua, and A.A. Saharian, Fermionic vacuum polarization in compactified cosmic string spacetime, Eur. Phys. J. C 74, 2688 (2014).

- [165] E. Elizalde, S.D. Odintsov, and A.A. Saharian, Fermionic condensate and Casimir densities in the presence of compact dimensions with applications to nanotubes, Phys. Rev. D 83, 105023 (2011).
- [166] E. Elizalde, S.D. Odintsov, and A.A. Saharian, Fermionic Casimir densities in anti-de Sitter spacetime, Phys. Rev. D 87, 084003 (2013).
- [167] M.O. Ribas, F.P. Devecchi, and G.M. Kremer, Cosmological model with non-minimally coupled fermionic field, Europhys. Lett. 81, 19001 (2008).
- [168] R.C. de Souza and G. M. Kremer, Noether symmetry for non-minimally coupled fermion fields, Class. Quant. Grav. 25, 225006 (2008).
- [169] T. Inagaki, T. Muta, and S.D. Odintsov, Dynamical symmetry breaking in curved spacetime, Prog. Theor. Phys. Suppl. 127, 93 (1997).
- [170] D. Ebert and K. G. Klimenko, Cooper pairing and finite-size effects in a Nambu–Jona-Lasinio-type four-fermion model, Phys. Rev. D 82, 025018 (2010).
- [171] A. Flachi, Interacting fermions, boundaries, and finite size effects, Phys. Rev. D 86, 104047 (2012).
- [172] A. Flachi, Strongly Interacting Fermions and Phases of the Casimir Effect, Phys. Rev. Lett. 110, 060401 (2013).
- [173] A. Flachi, Dual fermion condensates in curved space, Phys. Rev. D 88, 085011 (2013).
- [174] S.G. Mamaev and N.N. Trunov, Vacuum expectation values of the energy-momentum tensor of quantized fields on manifolds with different topologies and geometries. III, Sov. Phys. J. 23, 551 (1980).
- [175] R.D.M. De Paola, R.B. Rodrigues, and N.F. Svaiter, Casimir Energy of Massless Fermions in the Slab-bag, Mod. Phys. Lett. A 14, 2353 (1999).
- [176] E. Elizalde, F.C. Santos, and A.C. Tort, The Casimir energy of a massive fermionic field confined in a d+1 dimensional slab-bag, Int. J. Mod. Phys. A 18, 1761 (2003).

- [177] M. Abramowitz and I. A. Stegun (ed.), Handbook of Mathematical Functions, (Dover, New York, 1972).
- [178] A. A. Saharian, V. F. Manukyan, N. A. Saharyan, Electromagnetic vacuum fluctuations around a cosmic string in de Sitter spacetime, Eur. Phys. J. C 77, 478 (2017).
- [179] E. R. Bezerra de Mello, A. A. Saharian, Topological Casimir effect in compactified cosmic string spacetime, Class. Quantum Grav. 29, 035006 (2012).
- [180] J. S. Dowker, Vacuum averages for arbitrary spin around a cosmic string, Phys. Rev. D 36, 3742 (1987).
- [181] A. A. Saharian, A. S. Kotanjyan, Repulsive Casimir–Polder forces from cosmic strings, Eur. Phys. J. 71, 1765 (2011).