INSTITUTE OF MATHEMATICS,
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BOUNDARY VALUE PROBLEMS IN HARDY WEIGHTED SPACES

Dissertation
01.01.02 "Differential Equations, Mathematical Physics"
Specialty

For the degree of candidate of
Physical and Mathematical sciences

Scientific Supervisor
Doctor of phys.-math.sciences
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Yerevan, 2017

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INTRODUCTION

Theory of differential equations has a central role in the science of mathematics. Development of this theory simulated the development of associated fields, such as: elasticity theory, hydrodynamics, aerodynamics, theory of electromagnetic field. Nowadays, differential equations are mostly applied in biology, mathematical modeling of physical processes, engineering, finance, ecology, etc. Thus, theory of differential equations is one of the basic tools of contemporary research.

In this dissertation there are investigated boundary value problems for elliptic equations in the unit circle. Riemann, Riemann-Hilbert and Dirichlet problems are solved in the weighted spaces. This kind of problems arose during the investigation of boundary value problems of partial differential elliptic equations which have applications in physical processes. General solution of partial differential elliptic equations (system of equations) with constant coefficients is given by linear combination of analytic functions (Bitsadze A.V. [10], [11]).

Boundary value problems in the classes of analytic functions are investigated with boundary conditions in C^{α} Holder classes, $L^p, 1 < p < \infty$ spaces by Muskhelishvili N.I. [59], Gakhov F.D. [15], Vekua I.N. [83],[84], Khvedelidze B.V. [50], Simonenko I.B. [67]. Riemann boundary value problem is investigated in the weighted spaces $L^p(d\mu)$ by Khvedelidze B.V. [51]. By Hayrapetyan H.M., Asatryan A.S. Riemann boundary value problem is investigated in L^{∞} [41]. Poghosyan L.V. and Hayrapetyan H.M. considered Riemann boundary value problem in the sense of weak convergence [39], [63]. During investigation of boundary value problems of differential equations the need has arisen to expend theory of singular integral equations, also theory of boundary value problems of theory of functions. By Bikchantayev I.A. [8], [9], Soldatov A.P. [71], [72], [74], Tovmasyan N.E. [76], [77] and others [16], [58], [60] are suggested other boundary conditions which are correct for both improperly and properly elliptic equations. By Hayrapetyan H.M., Meliksetyan P.E. [25] and Hayrapetyan H.M., Oganisyan I.V. [26] are investigated boundary conditions for improperly elliptic equations of second and third order in the weighted spaces. The same kind of problem in multiply connected domain is investigated by Babayan A.H. [2]. In the half-space the same kind of problems for the class of functions of polynomial growth are investigated in the works by Tovmasyan N.E. [79], Asatryan V.V. [1]. Note the work of Tovmasyan N.E. and Babayan A.H. which is dedicated to investigation of Riemann problem in the half-plane for properly elliptic equations of second order in the case when boundary functions are continuous with the weight in numeric axis [80]. Also, by Tovmasyan are studied partial differential equations in regard to their important applications in electrodynamics [81], [82]. Boundary value problems for poly-harmonic and poly-analytic functions are investigated by Hayrapetyan H.M. in the half-plane [20], [21], [22], [23]. Schwarz type problems are investigated in the class of bi-analytic functions by Hayrapetyan A.R. [19] and in the class of poly-analytic functions by Begehr H., Schmersau D. [7]. Note the work of Begehr H., Kumar A. [6], where boundary value problems are studied for the inhomogeneous poly-analytic equation. In this work it is shown that for solvability of the Dirichlet problem for Bitsadze equation uncountable linear independent conditions must hold. Also, note monograph of Tanabe H. [75], where he studies Dirichlet problem for strongly elliptic equation in the classes of continuous functions. In the unit circle Dirichlet boundary value problem for biharmonic functions is investigated by Hayrapetyan H.M. [30]. Also, note the work by Hayrapetyan H.M., Hayrapetyan A.R. [40], where they studied some questions regarding uniqueness of harmonic functions. Babayan V.A. and Hayrapetyan H.M. studied Riemann-Hilbert and Dirichlet problems in the classes of continuous functions [3], [4], [5], [35].

Relevance of the topic

Let L is Lyapunov simple, closed curve, D^+ and D^- are respectively interior and exterior domains. Riemann boundary value problem or conjugation problem in classical setting has the following statement:

Determine analytic function Φ in $D^+ \cup D^-$, bounded or vanishing at infinity such that the following holds:

$$\Phi^+(t) - a(t)\Phi^-(t) = f(t), \qquad t \in T, \tag{1}$$

where function a and f are defined in T and belong to Holder classes $C^{\delta}, \delta \in (0, 1]$. Besides, $a(t) \neq 0$ for any point of T.

Function a is called coefficient of Riemann problem and f is called free member.

This problem was solved by Gakhov F.D. [15]. Then, Khvedelidze B.V. has studied the case when function f belongs to the space L^p, where $1 < p < \infty$ [50], [52]. In the works of Simonenko I.B. [68], [69], [70] the same problem for $L^p(1 < p < \infty)$ was investigated with essentialy extended coefficient. For all these cases solution of the problem is given with Cauchy type integral of function f. Furthermore, if f belongs to either classes $C^{\delta}, \delta \in (0, 1)$ or $L^p, 1 < p < \infty$, then solution of the problem also belongs respectively to the classes $C^{\delta}, \delta \in (0, 1)$ or $L^p, 1 < p < \infty$ [17], [54], [57].

Boundary value problems when $f \in L^1$ becomes complicated as Cauchy type integral of function from L^1 is not bounded operator belonging to Smirnov class E^1 [44], [55]. For solving Riemann boundary value problem in space L^1 Hayrapetyan H.M. suggested new setting of the problem [28], [34]. This statement helped H.M. Hayrapetyan and his students Meliksetyan P.E., Hayrapetyan M.S., Tsutsulyan A.V. and others [24], [27], [43] to investigate Riemann boundary value problem and elliptic differential equations associated with it in the weighted spaces. Particularly, for unit circle $D^+ = \{z: |z| < 1\}$ statement is as follows:

Determine analytic function Φ in $D^+ \cup D^-$, bounded or vanishing at infinity such that the following holds:

$$\lim_{r \to 1-0} \|\Phi^+(rt) - a(t)\Phi^-(r^{-1}t) - f(t)\|_{L^1} = 0, \tag{2}$$

where functions a and f are defined in T, a belongs to Holder classes $C^{\delta}, \delta \in (0, 1]$, $a(t) \neq 0$ at any point of T and f belongs to L^1 space.

Then, it was shown that this setting is correct. In other words, if function Φ is a solution of Riemann boundary value problem with this setting for the function f from Holder classes or $L^p, 1 < p < \infty$ spaces, then it would be also a solution with the classical statement. Thus, attained results are generalization of classical results of theory of boundary value problems.

Taking into consideration above mentioned reasons, we may conclude that studied problems in this dissertation are relevant in the field of boundary value problems of differential equations in the weighted spaces.

Object of study

Boundary value problems in the weighted spaces in unit circle with coefficients belonging to Holder classes

Goals

- Study Riemann boundary value problem with a coefficient from Holder classes in the weighted spaces $L^1(\rho)$, where weight function is concentrated on finite number of singular points. Give necessary and sufficient conditions for solvability of the problem and determine solutions in explicit form.
- Study Discontinuous Riemann boundary value problem in the weighted spaces $L^1(\rho)$, where weight function is concentrated on one singular point. Give necessary and sufficient conditions for solvability of the problem and determine solutions in explicit form.
- Study Riemann-Hilbert boundary value problem in the weighted spaces $L^1(\rho)$, where weight function is concentrated on finite number of singular points by transforming it into Riemann boundary value problem, give necessary and sufficient conditions for solvability of the problem and determine solutions in explicit form.
- Study Dirichlet problem for biharmonic functions in the weighted spaces, where weight function is concentrated on one point, give necessary and sufficient conditions for solvability of the problem and determine solutions in explicit form.

Research methods

In this work there are applied methods of theory of analytic functions and boundary value problems

Scientific novelty of work

All the results of the dissertation are novel.

Theoretical and practical value

Generally speaking all the results have theoretical value and can be applied in the study of boundary value problems of differential and integral equations.

Approbation of work

The results were reported during both republican and international conferences and seminars:

- International Conference "Education, science and economics at universities and schools. Integration to international educational area", March 2014, Tsaghkadzor, Armenia
- Annual Conference, National Polytechnic University of Armenia, 2015
- AMU Annual Session dedicated to the 100th anniversary of Proffesor Haik Badalyan, June 2015, Yerevan, Armenia
- International Conference, Harmonic Analysis and Approximations, VI, September 2015, Tsaghkadzor, Armenia
- VI Russian-Armenian Conference on Mathematical Analysis, Mathematical Physics and Analytical Mechanics, September 2016, Rostov-on-Don, Russia
- Seminar, Mathematical Analysis Sessions, Wroclaw University of Science and Technology, April 2017, Wroclaw, Poland

Author's Publications in the topic of dissertation

Results of the dissertation are published in 8 works (4 papers and 4 abstracts), which are stated at the end of references.

The volume of the dissertation

The dissertation consists of introduction, three chapters, conclusion and references.

The volume of dissertation is 82 pages. References contain 92 items.

Work content

In the first chapter the Riemann boundary value problem is solved in the weighted spaces with a coefficient belonging to Holder classes in unit circle. The solution separately is given for two different cases regarding the order of singularity of the weight function. For stating the results precisely it is necessary to do some notations.

Let T be a unit circle in the complex plane z, and let D^+ and D^- be the interior and exterior domains, respectively bounded by the curve T. Also, by $L^1(\rho)$ we define the following space:

$$L^1(\rho) := L^1(\rho, T) = \left\{ f : \|f\|_{L^1(\rho)} := \int_T |f(t)|\rho(t)|dt| < \infty \right\},$$

where

$$\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}, \qquad k = 1, 2, \dots, m,$$

$t_k \in T$, and α_k, $k = 1, 2, \dots, m$ are real numbers. To formulate the problem we first introduce some notation. We set

$$\rho_r(t) = \rho^*(t) \prod_{k=1}^{m} \left| r^{\delta_k} t - t_k \right|^{n_k},$$

where

$$\rho^*(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k - n_k},$$

$$\delta_k = \begin{cases} 1, & \text{if } \alpha_k \le -1, \\ 0, & \text{if } \alpha_k > -1, \end{cases} \qquad n_k = \begin{cases} [\alpha_k] + 1, & \text{if } \alpha_k \text{ is noninteger}, \\ \alpha_k, & \text{if } \alpha_k \text{ is integer}. \end{cases}$$

We consider the Riemann boundary value problem in the following setting:

Problem R

Let f be an arbitrary measurable on T function from the space $L^1(\rho)$. Determine an analytic in $D^+ \cup D^-$ function $\Phi(z), \Phi(\infty) = 0$ to satisfy the boundary condition:

$$\lim_{r \to 1-0} \|\Phi^+(rt) - a(t)\Phi^-(r^{-1}t) - f(t)\|_{L^1(\rho_r)} = 0, \tag{3}$$

where $a(t), a(t) \neq 0$ is an arbitrary function from the class $C^\delta(T)$, $\delta > 0$, $D^+ = \{z: |z| < 1\}$, $D^- = \{z: |z| > 1\}$ and $\Phi^\pm$ are the contractions of function Φ on $D^\pm$ respectively.

Let $\kappa = \operatorname{ind} a(t)$ and $t \in T$. It is well known that the function a admits the representation (see [15])

$$a(t) = \frac{S^+(t)}{S^-(t)},$$

where

$$S^+(z) = exp\left\{\frac{1}{2\pi i}\int_T \frac{ln\left(t^{-\kappa}a(t)\right)}{t-z}dt\right\},\ \ z \in D^+,$$
$$S^-(z) = z^{-\kappa}\, exp\left\{\frac{1}{2\pi i}\int_T \frac{ln\left(t^{-\kappa}a(t)\right)}{t-z}dt\right\},\ \ z \in D^-, \quad (4)$$

$S^{\pm} \in C^{\delta}(\overline{D^{\pm}})$ and $|S^-(z)| = O(|z|^{-\kappa})$ as $z \to \infty$.

Also, by N we denote the following:

$$N = \sum_{k=1}^{m} n_k.$$

Lemma 1

*Let $\alpha_k > -1,\ k = 1,2,\dots,m,\ f \in L^1(\rho)$, and $\Phi(z)$ be some solution of the Problem **R**. Then the following assertions hold.*

a) If $N + \kappa \geq 0$, then $\Phi(z)$ admits the representation

$$\Phi(z) = \frac{S(z)}{2\pi i\Pi(z)}\int_T \frac{f(t)\Pi(t)}{S^+(t)(t-z)}dt + \frac{S(z)P(z)}{\Pi(z)}, \quad (5)$$

where $z \in D^+ \cup D^-$, $P(z)$ is some polynomial of degree $N + \kappa - 1$, and

$$\Pi(t) = \prod_{k=1}^{m}(t_k - t)^{n_k}.$$

b) If $N + \kappa < 0$, then $\Phi(z)$ has the representation (5), where $P(z) \equiv 0$ and the function f satisfies the conditions:

$$\int_T \frac{f(t)\Pi(t)}{S^+(t)}t^k dt = 0, \quad k = 0,1,\dots,-(N+\kappa)-1. \quad (6)$$

Let

$$K(f,z) = \frac{S(z)}{2\pi i\Pi(z)}\int_T \frac{f(t)\Pi(t)}{S^+(t)(t-z)}dt, \ \ z \in D^+ \cup D^-. \quad (7)$$

Theorem 1

Let $f \in L^1(\rho)$. *Then*

$$\lim_{r \to 1-0} \|K^+(f, rt) - a(t)K^-(f, r^{-1}t) - f(t)\|_{L^1(\rho_r)} = 0,$$

where $K(f,z)$ *is defined as in (7). Thus,* $K(f,z)$ *is a solution of the inhomogeneous Problem* ***R****, when* $N + \kappa \geq 0$. *If* $N + \kappa < 0$, *then* $K(f,z)$ *is a solution of the inhomogeneous Problem* ***R*** *if and only if* f *satisfies conditions (6).*

Theorem 2

Let $\alpha_k > -1,\ k = 1,2,\dots,m,\ f \in L^1(\rho)$. *Then the following assertions hold.*

a) If $\kappa \geq 0$, *then the general solution of the homogeneous Problem* ***R*** *can be represented in the form:*

$$\Phi_0(z) = S(z)\left(\sum_{k=1}^{m}\sum_{j=1}^{n_k}\frac{A_j^{(k)}}{(t_k - z)^j} + P(z)\right),$$

where $P(z)$ *is a polynomial of degree* $\kappa - 1$ *for* $\kappa > 0$ *and* $P(z) \equiv 0$ *for* $\kappa \equiv 0$.

b) If $\kappa < 0$ *and* $N + \kappa > 0$, *then the general solution of the homogeneous Problem* ***R*** *can be represented in the form:*

$$\Phi_0(z) = \frac{S(z)P(z)}{\Pi(z)},$$

where $P(z)$ *is a polynomial of degree* $N + \kappa - 1$.

c) If $N + \kappa \leq 0$, *then the homogeneous problem has only trivial solution.*

Theorem 3

Let $\alpha_k > -1,\ k = 1,2,\dots,m,\ f \in L^1(\rho)$. *Then the following assertions hold.*

a) If $N + \kappa \geq 0$, *then the general solution of the inhomogeneous Problem* ***R*** *can be represented in the form:*

$$\Phi(z) = K(f,z) + \Phi_0(z), \tag{8}$$

where $K(f,z)$ *is defined as in (7), and* $\Phi_0(z)$ *is the general solution of the homogeneous Problem* ***R****.*

*b) If $N+\kappa<0$, then the Problem **R** is solvable if and only if f satisfies the conditions (6). And the solution can be represented in the form (7).*

In general case when $\alpha_k, k=1,2,\dots,m$ are arbitrary real numbers, we need to demand from function a to belong some specific class of functions. Let define this class as follows:

Definition

Given $\alpha=\{\alpha_1,\alpha_2,\dots,\alpha_m\}$, we say that a function $a(t)$ belongs to the class R^{α} if

$$\lim_{r\to 1-0}\|S^{+}(rt)-a(t)S^{-}(r^{-1}t)\|_{L^1(\rho_r)}=0 \qquad (9)$$

Theorem 4

Let α_k, $k=1,2,\dots,m$ are arbitrary real numbers, $f\in L^1(\rho)$ and $a\in R^{\alpha}$. Then the following assertions hold.

*a) If $N+\kappa\geq 0$, then the general solution of the inhomogeneous Problem **R** can be represented in the form:*

$$\Phi(z)=K(f,z)+S(z)\left(A_0+\frac{P(z)}{\Pi(z)}\right),$$

where $K(f,z)$ is defined as in (7), A_0 is an arbitrary complex number for $\kappa>0$ and $A_0=0$ for $\kappa\leq 0$, and $P(z)$ is a polynomial of degree $N+\kappa-1$.

b) If $N+\kappa<0$ and $\kappa>0$, then $\Phi(z)$ can be represented in the form (8), where $\Phi_0(z)\equiv 0$ and f satisfies conditions (6).

*c) If $N<0$ and $\kappa\leq 0$, then the inhomogeneous Problem **R** has a unique solution:*

$$\Phi(z)=K(f,z)+A_0S(z),$$

where

$$A_0=-\frac{1}{2\pi i}\int_T\frac{f(t)}{S^{+}(t)}\Pi(t)\,t^{-(N+1)}dt$$

and f satisfies conditions (6) if $\kappa\neq -N-1$.

*d) If $N+\kappa<0$ and $N\geq 0$, then the Problem **R** has a single solution: $\Phi(z)=K(f,z)$ and f satisfies conditions (6).*

Theorem 5

Let α_k, $k = 1,2,\dots,m$ are arbitrary real numbers, $f \in L^1(\rho)$ and $a(t) \in R^{\alpha}$. Then the following assertions hold.

*a) If $N + \kappa > 0$, then the general solution of the homogeneous Problem **R** can be represented in the form:*

$$\Phi(z) = K(f,z) + S(z)\left(A_0 + \frac{P(z)}{\Pi(z)}\right),$$

where A_0 is an arbitrary complex number for $\kappa > 0$ and $A_0 = 0$ for $\kappa \le 0$, and $P(z)$ is a polynomial of degree $N + \kappa - 1$.

b) If $N + \kappa \le 0$ and $\kappa > 0$, then $\Phi(z) = A_0 S(z)$, where A_0 is an arbitrary complex number.

*c) If $N + \kappa \le 0$ and $\kappa \le 0$, then the homogeneous Problem **R** has only trivial solution: $\Phi(z) \equiv 0$.*

In the second chapter two problems are studied which are connected with Riemann boundary value problem: Riemann-Hilbert boundary value problem and Discontinuous Riemann boundary value problem.

If not stated opposite we use the same notations as in Chapter 1.

Problem H

Let f be a real-valued, measurable on T function from the space $L^1(\rho)$. Determine an analytic in D^+ function $\Phi(z)$ to satisfy the boundary condition:

$$\lim_{r\to 1-0} \|\mathrm{Re}\{a(t)\Phi(rt)\} - f(t)\|_{L^1(\rho_r)} = 0, \qquad (10)$$

where $a(t), a(t) \neq 0$ is an arbitrary function from the class $C^{\delta}(T)$, $\delta > 0$.

By making the following notations

$$\frac{\overline{a(t)}}{a(t)} = D(t), \qquad \frac{2f(t)}{a(t)} = g(t)$$

we come to the following Riemann Problem.

Problem R

Let g is some measurable on T function from the space $L^1(\rho)$. Determine an analytic in $D^+ \cup D^-$ function $\Phi(z), \Phi(\infty) = C$ to satisfy the boundary condition:

$$\lim_{r\to 1-0} \|\Phi^+(rt) - D(t)\Phi^-(r^{-1}t) - g(t)\|_{L^1(\rho_r)} = 0, \qquad (11)$$

where $D(t), D(t) \neq 0$ is a function from the class $C^\delta(T)$, $\delta > 0$ and $\Phi^\pm$ are the contractions of function Φ on $D^\pm$ respectively.

Let

$$D = \frac{u - iv}{u + iv},$$

where u and v are real-valued functions, such that $a = u + iv$.

Also,

$$\Omega(f,z) = \frac{S(z)}{2\pi i \Pi(z)}\left(\int_T \frac{(z^{N+\kappa} + t^{N+\kappa})f(t)\Pi(t)dt}{S^+(t)(u+iv)t^{N+\kappa}(t-z)} - z^{N+\kappa}\int_T \frac{f(t)\Pi(t)dt}{S^+(t)(u+iv)t^{N+\kappa+1}}\right) \qquad (12)$$

Theorem 6

The following assertions hold.

*a) If $N + \kappa \geq -1$, then $\Omega(f,z)$ is a solution of the inhomogeneous Problem **H**, where $\Omega(f,z)$ is defined as in (12).*

*b) If $N + \kappa < -1$, then $\Omega(f,z)$ is a solution of the inhomogeneous Problem **H** if and only if f satisfies the following conditions:*

$$\int_T \frac{f(t)}{(u(t) + iv(t))S^+(t)}\Pi(t)\, t^k dt = 0, \quad k = 0,1,\dots,-(N+\kappa) - 2. \qquad (13)$$

Theorem 7

Let $\alpha_k > -1, k = 1,2,\dots,m$. Otherwise, $D(t) \in R^\alpha$. Then the following assertions hold.

*a) If $N + \kappa > -1$, then the general solution of the homogeneous Problem **H** can be represented in the form:*

$$\Phi_0(z) = \frac{S(z)}{\Pi(z)}(c_0 z^{N+\kappa} + c_1 z^{N+\kappa-1} + \dots + c_{N+\kappa}),$$

where numbers $\{c_l\}_{l=0}^{N+\kappa}$ satisfy condition:

$$(-1)^{N+1}\overline{c_l}\prod_{k=1}^{m} t_k{}^{n_k} = c_{N+\kappa-l}, \qquad l = 0,1, \dots, N+\kappa.$$

*b) If $N+\kappa \le -1$, then the homogeneous Problem **H** has only trivial solution.*

Theorem 8

Let $\alpha_k > -1, k = 1,2, \dots, m$. Otherwise, $D(t) \in R^{\alpha}$. Then the following assertions hold.

*a) If $N+\kappa \ge -1$, then the general solution of the inhomogeneous Problem **H** can be represented in the form:*

$$\Phi(z) = \Omega(f,z) + \Phi_0(z),$$

*where $\Omega(f,z)$ is defined as in (12), and $\Phi_0(z)$ is the general solution of the homogeneous Problem **H**.*

*b) If $N+\kappa < -1$, then the Problem **H** is solvable if and only if f satisfies the conditions (13). And the solution can be represented in the form (12).*

Discontinuous Riemann boundary value problem is investigated in the case when coefficient of the problem belongs to Holder class in unit circle except from finite number of points where it has jump discontinuity. To state the problem we have to make following notations.

Definition

We say $a(t) \in H_0(T; t_1, t_2, \dots, t_m)$, if function a belongs to Holder class in any interval from T not including $t_k, k = 1,2, \dots, m$ points and has jump discontinuity at those points.

Let $\rho(t) = |t - t_0|^{\alpha}$, where $t_0 \in T$ and $\alpha > -1$ is arbitrary real number. By n we denote the following:

$$n = \begin{cases} [\alpha] + 1, & \text{if } \alpha \text{ is not integer}, \\ \alpha, & \text{if } \alpha \text{ is integer}. \end{cases}$$

By introducing $\varphi(t) = \ln a(t)$ function it is easy to get the following:

$$\alpha_k + i\beta_k = \frac{1}{2\pi i}\left(\varphi(t_k - 0) - \varphi(t_k + 0)\right), \qquad k = 1,2, \dots, m \tag{14}$$

Problem R

Let $f \in L^1(\rho)$, $a(t) \in H_0(T; t_1, t_2, \dots, t_m)$ *and* $a(t) \neq 0, t \in T$. *Also,* $t_0 \neq t_k$, $k = 1,2,\dots,m$. *Determine in* D *analytic function* $\Phi(z), \Phi(\infty) = 0$, *such that the following condition holds:*

$$\lim_{r \to 1-0} \|\Phi^+(rt) - a(t)\Phi^-(r^{-1}t) - f(t)\|_{L^1(\rho)} = 0.$$

Theorem 9

The general solution of the inhomogeneous Problem ***R****, which has some finite degree at infinity, is given by the following formula:*

$$\Phi^+(z) = \frac{S^+(z)}{2\pi i (z - t_0)^n} \int_T \frac{g(t)dt}{t - z}, \qquad z \in D^+,$$
$$\Phi^-(z) = \frac{S^-(z)}{2\pi i (z - t_0)^n} \int_T \frac{g(t)dt}{t - z} + S^-(z)P(z), \qquad z \in D^-, \tag{15}$$

where P *is any polynomial and*

$$g(t) = \left(P(t) + \frac{f(t)}{S^+(t)}\right)(t - t_0)^n.$$

Let introduce the following functions:

$$\Phi_k^+(z) = \frac{1}{2\pi i} \int_T \frac{t^k (t - t_0)^n}{t - z} dt, \qquad z \in D^+,$$
$$\Phi_k^-(z) = \frac{1}{2\pi i} \int_T \frac{t^k (t - t_0)^n}{t - z} dt + z^{n+k}, \qquad z \in D^-.$$

Then,

$$\tilde{g}(t) = \frac{f(t)(t - t_0)^n}{S^+(t)}.$$

Hence, for any polynomial $(z) = c_0 + c_1 z + \dots + c_m z^m$, equation (15) can be represented as follows:

$$\Phi^+(z) = \frac{S^+(z)}{2\pi i (z - t_0)^n} \int_T \frac{\tilde{g}(t)dt}{t - z} + \frac{S^+(z)}{(z - t_0)^n} \sum_{k=0}^{m} c_k \Phi_k^+(z), \qquad z \in D^+,$$

$$\Phi^-(z) = \frac{S^-(z)}{2\pi i(z-t_0)^n}\int_T \frac{\tilde{g}(t)dt}{t-z} + \frac{S^-(z)}{(z-t_0)^n}\sum_{k=0}^{m} c_k\Phi_k^-(z), \qquad z \in D^-.$$

It is well known that $\kappa = -\sum_{k=1}^{m}\lambda_k$, where λ_k are integers such that

$$-1 < \lambda_k + \alpha_k \le 0, \qquad k = 0,1,\dots,m,$$

and κ is the index of function a.

Theorem 10

The following assertions hold.

a) If $n+\kappa \ge 0$, *then the general solution of the inhomogeneous Problem **R** has the following representation:*

$$\Phi^+(z) = \frac{S^+(z)}{2\pi i(z-t_0)^n}\int_T \frac{\tilde{g}(t)dt}{t-z} + \frac{S^+(z)}{(z-t_0)^n}\sum_{k=0}^{\kappa-1} c_k\Phi_k^+(z), \qquad z \in D^+,$$

$$\Phi^-(z) = \frac{S^-(z)}{2\pi i(z-t_0)^n}\int_T \frac{\tilde{g}(t)dt}{t-z} + \frac{S^-(z)}{(z-t_0)^n}\sum_{k=0}^{\kappa-1} c_k\Phi_k^-(z), \qquad z \in D^-, \tag{16}$$

where $c_0, c_1, \dots, c_{\kappa-1}$ are arbitrary complex numbers when $\kappa \ge 1$ and $c_0 = c_1 = \dots = c_{\kappa-1} = 0$ when $\kappa = 0$.

b) If $n+\kappa < 0$, *then the Problem **R** has a solution if and only if:*

$$\int_T \frac{\tilde{g}(t)}{t-z} t^k dt = 0, \quad k = 0,1,\dots,-(n+\kappa)-1 \tag{17}$$

And, the solution has the representation (16), where $c_0 = c_1 = \dots = c_{\kappa-1} = 0$.

In the third chapter the Dirichlet Problem is investigated for biharmonic functions in the weighted spaces in unit circle. The problem is investigated in the case when boundary conditions of a function and his derivative are considered in different weighted spaces. Firstly we introduce some notations.

Let $\rho_0(t)$ is a weight function, $\rho_0(t) = |1-t|^\alpha; \alpha$ is arbitrary real number, $\rho_1(t) = \rho_0(t)|1-t|$, $\|\cdot\|_{L^1(\rho_k)}$ is the norm of space, $f_k(t) \in L^1(\rho_k)$, $k = 0,1$ are real-valued functions defined in T. The problem is investigated in the following setting:

Problem D

Determine function $u(z), z \in D^+$ *to satisfy the equation*

$$\Delta^2 u = 0 \tag{18}$$

and boundary condition

$$\lim_{r\to 1-0} \left\| \frac{\partial^k u(rt)}{\partial r^k} - f_k(t) \right\|_{L^1(\rho_k)} = 0, \qquad k = 0,1. \tag{19}$$

With this statement we prove that Problem (18), (19) is normally solvable, besides, if $\alpha \le -1$, then the problem is investigated with the following boundary conditions:

$$\lim_{r\to 1-0} \left\| \frac{\partial^k u(rt)}{\partial r^k} - f_k(t) \right\|_{L^1(\rho_{kr})} = 0, \qquad k = 0, 1, \tag{20}$$

where $\rho_{kr}(t) = |1 - rt|^{n+k} |1 - t|^{\alpha - n}$, $k = 0,1$.

Theorem 11

Let $\alpha \le -1$. *Then the homogeneous problem (18), (20) has only trivial solution:* $u(z) \equiv 0$.

Definition

We say that, $\{C_k\}_0^n$ *numbers belong to class* $S_0(n)$, *if*

$$C_k = (-1)^k \overline{C_{n-k}}, \qquad k = 0,1, \dots, n.$$

Theorem 12

Let $\alpha > -1$. *Then the general solution of the homogeneous problem (18), (19) can be represented as follows:*

$$u(z) = \mathrm{Re}\big(\Phi_0(z) + (1 - |z|^2)\Phi_1(z)\big), \qquad z \in D^+$$

where

$$\begin{cases} \Phi_0(z) = \displaystyle\sum_{k=0}^{n} \frac{A_k}{(1-z)^k}, \\ \Phi_1(z) = \displaystyle\frac{z}{2}\frac{\partial \Phi_0(z)}{\partial z} - \frac{1}{2}\sum_{k=0}^{n+1} \frac{B_k}{(1-z)^k}. \end{cases}$$

Besides, $\{A_k\}_0^n$ numbers belong to the class $S_0(n)$, and $\{B_k\}_0^{n+1}$ numbers to the class $S_0(n+1)$.

Theorem 13

Let $\alpha \leq -1$. Then the general solution of the inhomogeneous problem (18), (20) can be represented as follows:

$$u(z) = \mathrm{Re}\big(\Phi_0(z) + (1-|z|^2)\Phi_1(z)\big),$$

where Φ_0, Φ_1 are analytic functions in D^+, determined with the following formulas:

$$\begin{cases} \Phi_0(z) = \dfrac{1}{2\pi i(1-z)^n}\displaystyle\int_T \dfrac{f_0(t)(1-t)^n}{t-z}dt, \\ \Phi_1(z) = \dfrac{z}{2}\dfrac{\partial\Phi_0(z)}{\partial z} + \dfrac{1}{2\pi i(1-z)^{n+1}}\displaystyle\int_T \dfrac{f_1(t)(1-t)^{n+1}}{t-z}dt. \end{cases}$$

Besides, there are necessary and sufficient conditions for solvability of the problem, which have the following form:

$$\int_T t^k f_0(t)dt = 0, \int_T t^k f_1(t)dt = 0, \qquad k = 0,1,\dots,n.$$

Theorem 14

Let $\alpha > -1$. Then the general solution of the problem (18), (19) can be represented in the form:

$$u(z) = \mathrm{Re}\big(\Phi_0(z) + (1-|z|^2)\Phi_1(z)\big),$$

where Φ_0, Φ_1 are analytic functions in D^+, and are given by the following formulas:

$$\begin{cases} \Phi_0(z) = \dfrac{1}{2\pi i(1-z)^n}\displaystyle\int_T \dfrac{f_0(t)(1-t)^n}{t-z}dt + \sum_{k=0}^{n}\dfrac{A_k}{(1-z)^k}, \\ \Phi_1(z) = \dfrac{1}{2\pi i(1-z)^{n+1}}\displaystyle\int_T \dfrac{f_1(t)(1-t)^{n+1}}{t-z}dt + z\sum_{k=0}^{n}\dfrac{A_k}{(1-z)^{k+1}} - \sum_{k=0}^{n+1}\dfrac{B_k}{(1-z)^k}, \end{cases}$$

and $\{A_k\}_0^n$ numbers belong to the class $S_0(n)$, and $\{B_k\}_0^{n+1}$ numbers to the class $S_0(n+1)$.

CHAPTER 1. RIEMANN PROBLEM IN THE WEIGHTED SPACES $L^1(\rho)$

Riemann boundary value problem[1] firstly was stated in Riemann's work on differential equations with algebraic coefficients [66]. However, Riemann surprisingly did not try to solve the problem. Solution of the homogeneous problem firstly was given by Hilbert. The basic idea which helped him to solve the problem was that any complex-valued function is boundary value of some analytic function. Then, he constructed Fredholm integral equation for which the solution of the homogeneous problem satisfies. Analyzing this equation he came to the conclusion that one of the two problems, one with coefficient $a(t)$, second with $\overline{a(t)}$, is solvable. Further investigation of this problem by Privalov I.I. [65] and Picard E. [61] was based on the same approach: use Cauchy type integral as a basic tool. The main difference from the works of Hilbert was the alternative problem. They used $\frac{1}{a(t)}$ coefficient instead of $\overline{a(t)}$. This method is still used in Riemann boundary value problems in a lot of cases of unknown functions.

It is worth mentioning two fundemantal works done by Plemelj I. [62] and Carleman T. [14]. First proves that for single-valued function $\ln a(t)$, solution of the homogeneous problem can be found explicitly with Cauchy type integral of this function. Second, during the process of investigation of singular integral equation with Cauchy kernel coincidently solved Riemann inhomogeneous problem with a constant coefficient, when the contour is the interval $[0, 1]$ of real axis. These works are very specific, they refer to only very trivial cases regarding coefficient of the problem, but as one of the first fundamental solutions, they stand very highly in the theory of Riemann boundary value problem. It was Gakhov F.D. who firstly investigated Riemann boundary value problem with a coefficient which can have index other than zero in simply connected domains [15]. Khvedelize B.V. expanded his results in multiply connected domains [53].

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[1] Different authors use other names for this problem, such as: "Hilbert problem", "problem of conjugation", "Hilbert-Privalov problem", "Riemann-Privalov problem".

Riemann boundary value problem with a displacement firstly was studied by Haseman C. [18]. He used the same method as Hilbert by transforming the problem into Fredholm integral equation and obtained the same alternative to the problem as Hilbert for Riemann boundary value problem. Complete solution of this problem was given by Kveselava D.A. in his work [56].

It would be helpful for future investigation to state some basic properties, which can be found in the works [15], [59].

1. Holder class functions

Let L is closed, smooth curve and $\varphi(t)$ is a function defined on this curve. We say that function $\varphi(t)$ satisfy Holder condition, if for any two points of this curve following holds:

$$|\varphi(t_2) - \varphi(t_1)| < C|t_2 - t_1|^\delta,$$

where C is constant and $\delta \in (0; 1]$. The class of functions which satisfy Holder condition called Holder class: $\varphi \in C^\delta(L)$. Note that if $\delta = 1$, then Holder class coincides with Lipshitz class.

Let consider Cauchy type integral from φ function, which belongs to Holder classes $C^\delta(L), \delta \in (0; 1]$.

By G^+ and G^- we understand interior and exterior domains of curve L respectively. The following is true:

$$\begin{cases} \int_L \frac{1}{t-z} dt = 2\pi i, & z \in G^+ \\ \int_L \frac{1}{t-z} dt = 0, & z \in G^- \\ \int_L \frac{1}{t-z} dt = \pi i, & z \in L \end{cases}$$

Suppose

$$\Psi(t) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau, \tag{1.1}$$

where $t \in L$, and $\varphi \in C^\delta(L)$. This integral is understood by Cauchy principal value. Sometimes for easy calculations it is used in the following form:

$$\Psi(t)=\frac{1}{2\pi i}\int_L \frac{\varphi(\tau)-\varphi(t)}{\tau-t}d\tau+\varphi(t)\cdot\pi i$$

The fundamental property of (1.1) is that $\Psi\in C^{\delta}(L)$ with the same number δ in the case $\delta\in(0,1)$ and $\Psi\in C^{\delta_0}(L)$, where $\delta_0=1-\varepsilon$, $(\varepsilon>0)$, is arbitrary real number in the case $\delta=1$. In other words, Cauchy type integral of Holder class function also is a Holder class function. This property plays key role in the theory of boundary value problems.

2. Sokhotski-Plemelj formulas

Let

$$\Psi(z)=\frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau-z}d\tau,$$

where $\varphi\in C^{\delta}(L)$.

By $\Psi^+(t)$ and $\Psi^-(t)$ we understand the following:

$$\begin{cases}\Psi^+(t)=\lim\limits_{z\to t^+}\Psi(z)\,,\\ \Psi^-(t)=\lim\limits_{z\to t^-}\Psi(z)\,.\end{cases}$$

The following formulas are true and called Sokhotski-Plemelj formulas:

$$\begin{cases}\Psi^+(t)=\frac{1}{2}\varphi(t)+\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}d\tau\\ \Psi^-(t)=-\frac{1}{2}\varphi(t)+\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}d\tau\end{cases}\qquad(1.2)$$

3. Index

Let $a(t)$ is a continuous function defined in L, $a(t)\neq 0$ at any point of L.

Definition

Augmentation of argument of function $a(t)$ when traversing a curve in positive direction devided by 2π is called index of function $a(t)$ by contour L.

$$\kappa=\mathrm{Ind}\,a(t)=\frac{1}{2\pi}[\arg a(t)]_L\qquad(1.3)$$

4. Riemann boundary value problem

Statement

Let L is a closed, smooth curve dividing complex plane into interior domain G^+ and exterior domain G^-. Functions $a(t)$ and $f(t)$ belong to Holder class of functions, also

$a(t) \neq 0$. *Determine analytic functions* $\Phi^+(z)$ *and* $\Phi^-(z)$ *in* G^+ *and* G^- *including* $z = \infty$ *point respectively such that the following holds:*

$$\Phi^+(t) - a(t)\Phi^-(t) = f(t). \tag{1.4}$$

Function $a(t)$ is called coefficient of Riemann boundary value problem, $f(t)$ –free member. Supposing $f \equiv 0$ in (1.4) we get the homogeneous problem.

Now let consider particular case, when $a(t) \equiv 1$. Taking into consideration Sokhotski-Plemelj formulas (1.2) solution of the problem is:

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z} d\tau + C.$$

By demanding additional condition $\Phi^-(\infty) = 0$ solution of the problem would be unique:

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z} d\tau.$$

Now let $\kappa = \operatorname{Ind} a(t)$. Also, let Φ^+ and Φ^- are solutions of the homogeneous problem. Denote by N^+ and N^- the quantity of their zeroes in G^+ and G^- respectively. Then $\kappa = N^+ + N^-$. Thus, $\kappa \geq 0$, otherwise homogeneous problem has only trivial solution.

a) Let $\kappa = 0$, then $\ln a(t)$ is a single valued function and $\ln \Phi(z)$ is analytic, so we have

$$\ln \Phi^+(t) - \ln \Phi^-(t) = \ln a(t).$$

From Sokhotski-Plemelj formulas we get

$$\ln \Phi(z) = \frac{1}{2\pi i}\int_L \frac{\ln a(\tau)}{\tau - z} d\tau,$$

hence

$$\Phi(z) = C\exp\left\{\frac{1}{2\pi i}\int_L \frac{\ln a(\tau)}{\tau - z} d\tau\right\}.$$

By demanding additional condition $\Phi^-(\infty) = 0$ we get $C = 0$. Thus, the homogeneous problem has only trivial solution.

b) Let $\kappa > 0$. Transform (1.4) into the following

$$\Phi^+(t) - t^{\kappa}\left(t^{-\kappa}a(t)\right)\Phi^-(t) = 0. \tag{1.5}$$

Obviously, $\mathrm{Ind}a_1(t) = \mathrm{Ind}t^{-\kappa}a(t) = 0$, so

$$a_1(t) = \frac{e^{\Gamma^+(t)}}{e^{\Gamma^-(t)}},$$

where

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\ln\left(\tau^{-\kappa}a(\tau)\right)}{\tau - z}d\tau.$$

Then we have

$$\frac{\Phi^+(t)}{e^{\Gamma^+(t)}} = t^{\kappa}\frac{\Phi^-(t)}{e^{\Gamma^-(t)}}.$$

As function $\frac{\Phi^+(t)}{e^{\Gamma^+(t)}}$ is analytic in G^+ and function $z^{\kappa}\frac{\Phi^-(z)}{e^{\Gamma^-(z)}}$ is analytic in D^- except from maybe at infinity where it can have a pole of order not higher than $\kappa - 1$, then from Liouvill's generalized theorem we obtain

$$\begin{cases} \Phi^+(z) = e^{\Gamma^+(z)}P_{\kappa-1}(z) \\ \Phi^-(z) = e^{\Gamma^-(z)}z^{-\kappa}P_{\kappa-1}(z) \end{cases} \tag{1.6}$$

Now let

$$\begin{cases} S^+(z) = e^{\Gamma^+(z)} \\ S^-(z) = z^{-\kappa}e^{\Gamma^-(z)} \end{cases} \tag{1.7}$$

Obviously,

$$a(t) = \frac{S^+(t)}{S^-(t)}$$

Then general solution of the homogeneous problem can be represented in the form

$$\Phi(z) = S(z)P_{\kappa-1}(z).$$

Taking into account the equation above, the inhomogeneous problem can be written in the form

$$\frac{\Phi^+(t)}{S^+(t)} - \frac{\Phi^-(t)}{S^-(t)} = \frac{f(t)}{S^+(t)}. \tag{1.8}$$

Let

$$\Psi(z) = \frac{S(z)}{2\pi i}\int_L \frac{f(\tau)}{S^+(z)(\tau - z)}d\tau. \tag{1.9}$$

Taking into consideration Sokhotski-Plemelj formulas $\Psi(z)$ satisfies (1.8) equation.

a) $\kappa \geq 0$, then $S^-(z) \approx z^{-\kappa}$ at infinity and $\Phi^-(\infty) = 0$. Then the general solution of the inhomogeneous problem has the following representation

$$\Phi(z) = \Psi(z) + S(z)P_{\kappa-1}(z).$$

b) $\kappa < 0$, then $S^-(z)$ has a pole of order $-\kappa$ at infinity, thus for obtaining $\Phi^-(\infty) = 0$, we should determine necessary and sufficient conditions.

Clearly,

$$\frac{1}{\tau - z} = -\frac{1}{z} - \frac{\tau}{z^2} - \cdots - \frac{\tau^{n-1}}{z^n} - \cdots,$$

thus

$$\Psi^-(z) = S^-(z)\sum_{k=0}^{\infty}\frac{c_k}{z^{k+1}},$$

where

$$c_k = -\frac{1}{2\pi i}\int_L \frac{f(\tau)\tau^k}{S^+(\tau)}d\tau.$$

Then we get the following conditions

$$\int_L \frac{f(\tau)\tau^k}{S^+(\tau)}d\tau = 0, k = 0,1, \ldots, -\kappa. \tag{1.10}$$

The problem has unique solution which has (1.9) form and f necessarily satisfies conditions (1.10).

1.1 Statement of the problem

Let T be a unit circle in the complex plane z, and let D^+ and D^- be the interior and exterior domains, respectively bounded by the curve T. Also, by $L^1(\rho)$ we define the following space:

$$L^1(\rho) := L^1(\rho, T) = \left\{ f : \|f\|_{L^1(\rho)} := \int_T |f(t)|\rho(t)|dt| < \infty \right\},$$

where

$$\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}, \qquad k = 1,2,\dots,m, \tag{1.1.1}$$

$t_k \in T$, and α_k, $k = 1,2,\dots,m$ are real numbers. To formulate the problem we first introduce some notation. We set

$$\rho_r(t) = \rho^*(t) \prod_{k=1}^{m} \left| r^{\delta_k} t - t_k \right|^{n_k}, \tag{1.1.2}$$

where

$$\rho^*(t) = \prod_{k=1}^{m} |t - t_k|^{\beta_k},$$

$$\delta_k = \begin{cases} 1, & if\ \alpha_k \leq -1, \\ 0, & if\ \alpha_k > -1, \end{cases} \qquad n_k = \begin{cases} [\alpha_k] + 1, & if\ \alpha_k\ is\ noninteger, \\ \alpha_k, & if\ \alpha_k\ is\ integer, \end{cases}$$

$\beta_k = \alpha_k - n_k$. It is clear that $\beta_k \in (-1,0]$ and $\rho^*(t) \in L^1(T)$.

We consider the Riemann boundary value problem in the following setting:

Problem R

Let f be an arbitrary measurable on T function from the class $L^1(\rho)$. Determine an analytic in $D^+ \cup D^-$ function $\Phi(z), \Phi(\infty) = 0$ to satisfy the boundary condition:

$$\lim_{r \to 1-0} \|\Phi^+(rt) - a(t)\Phi^-(r^{-1}t) - f(t)\|_{L^1(\rho_r)} = 0, \tag{1.1.3}$$

where $a(t)$, $a(t) \neq 0$ is an arbitrary function from the class $C^\delta(T)$, $\delta > 0$, $D^+ = \{z : |z| < 1\}$, $D^- = \{z : |z| > 1\}$ and $\Phi^\pm$ are the contractions of function Φ on $D^\pm$ respectively.

1.2 Preliminary results

1.2.1 In the case $\alpha_k > -1,\ k = 1, 2, \dots, m$

Lemma 1.2.1

Let $t_k \in T, t_i \neq t_j,\ i \neq j,$ and let $\lambda_k \in [0, 1),\ k = 1,2, \dots, m,\ \delta > 0$ be arbitrary real numbers. The following assertions hold:

$$a)\ \sup_{r\in(0,1)} \prod_{k=1}^{m} |t_k - \tau|^{\lambda_k} \int_T \frac{(1-r)^\delta}{\prod_{k=1}^{m}|t_k - t|^{\lambda_k}|\tau - rt|}|dt| < C,$$

$$b)\ \sup_{r\in(0,1)} \prod_{k=1}^{m} |t_k - \tau|^{\lambda_k} \int_T \frac{1-r^2}{\prod_{k=1}^{m}|t_k - t|^{\lambda_k}\,|\tau - rt|^2}|dt| < C,$$

$$c)\ \sup_{r\in(0,1)} |t_k - \tau|^{\lambda_k} \int_T \frac{1-r^2}{|t_k - rt|^{1+\lambda_k}|\tau - rt|}|dt| < C,$$

where $C < \infty$ is independent of τ.

Proof. We choose half-open intervals $T_k \in T$ such that

$$T = \bigcup_{k=1}^{m} T_{k,}\, t_k \in T_k, \qquad T_i \cap T_j = \emptyset\,, \qquad i = j.$$

Then for any $r \in (0,1)$ and $\tau \in T$ we have

$$\prod_{k=1}^{m} |t_k - \tau|^{\lambda_k} \int_T \frac{(1-r)^\delta}{\prod_{k=1}^{m}|t_k - t|^{\lambda_k}\,|\tau - rt|}|dt| <$$

$$< C\sum_k |t_k - \tau|^{\lambda_k} \int_{T_k} \frac{(1-r)^\delta}{|t_k - t|^{\lambda_k}|\tau - rt|}$$

It is enough to show that for any $t_0 \in T$

$$\sup_{r\in(0,1)} |t_0 - \tau|^{\lambda} \int_T \frac{(1-r)^\delta}{|t_0 - t|^{\lambda}|\tau - rt|}|dt| < \infty.$$

Taking into account that

$$\sup_{r\in(0,1)}\int_T \frac{(1-r)^\delta}{|\tau-rt|}|dt| = \sup_{r\in(0,1)} (1-r)^\delta \ln|1-r|^{-1} < \infty,$$

we show that

$$\sup_{r\in(0,1)}\int_T \frac{(1-r)^\delta \big||t_0-\tau|^\lambda - |t_0-t|^\lambda\big|}{|t_0-t|^\lambda|\tau-rt|}|dt| < \infty.$$

Since $\big||t_0-\tau|^\lambda - |t_0-t|^\lambda\big| < C|\tau-t|^\lambda$ (see [42]), it is enough to show that

$$\sup_{r\in(0,1)}\int_T \frac{(1-r)^\delta|\tau-t|^\lambda}{|t_0-t|^\lambda|\tau-rt|}|dt| < \infty.$$

Indeed, we have

$$\int_T \frac{(1-r)^\delta}{|t_0-t|^\lambda|\tau-rt|^{1-\lambda}}|dt| \leq \int_T \frac{(1-r)^{\delta-\delta_1}}{|t_0-t|^\lambda|\tau-rt|^{1-\lambda-\delta_1}}|dt| \leq$$

$$\leq \int_T \frac{(1-r)^{\delta-\delta_1}}{|t_0-t|^{1-\delta_1}}|dt| + \int_T \frac{(1-r)^{\delta-\delta_1}}{|\tau-rt|^{1-\delta_1}}|dt| < C,$$

where $\delta_1 \in (0,\delta)$. This completes the proof of assertion a) of the lemma.

To prove assertion b), similar to the case a), it is enough to show that

$$\sup_{r\in(0,1)} |t_0-\tau|^\lambda \int_T \frac{(1-r^2)^\delta}{|t_0-t|^\lambda|\tau-rt|^2}|dt| < \infty, \qquad \lambda\in(0,1).$$

Taking into consideration that

$$|t_0-\tau|^\lambda \int_T \frac{1-r^2}{|t_0-t|^\lambda|\tau-rt|^2}|dt| \leq C\left(\int_T \frac{1-r^2}{|t_0-t|^\lambda|\tau-rt|^{2-\lambda}} + 1\right),$$

we set

$$I_1(r,t) = \int_{|t_0-t|\geq 1-r} \frac{1-r^2}{|t_0-t|^\lambda|\tau-rt|^{2-\lambda}}|dt|,$$

$$I_2(r,t) = \int_{|t_0-t|<1-r} \frac{1-r^2}{|t_0-t|^\lambda|\tau-rt|^{2-\lambda}}|dt|.$$

Since $|t_0-t| > 2^{-1}|t_0-rt|$ for $|t_0-t|\geq 1-r$, we can write

$$I_1(r,t) \leq 2\int_T \frac{1-r^2}{|t_0-rt|^\lambda|\tau-rt|^{2-\lambda}}|dt| \leq$$

$$\leq 2\left(\int_T \frac{1-r^2}{|t_0 - rt|^2}|dt| + \int_T \frac{1-r^2}{|\tau - rt|^2}|dt|\right) = 8\pi.$$

To complete the proof of assertion b) of the lemma, it remains to observe that

$$I_2(r,t) \leq \int_{|\theta|<1-r} \frac{1}{|1-r|^{1-\lambda}|\theta|^{\lambda}}|d\theta| = C < \infty.$$

Next, to prove assertion c), we first observe that

$$|t_k - \tau|^{\lambda} \int_T \frac{1-r^2}{|t_k - rt|^{1+\lambda}|\tau - rt|}|dt| \leq$$

$$\leq \int_T \frac{(1-r^2)\big||t_k-\tau|^{\lambda} - |t_k - rt|^{\lambda}\big|}{|t_k - rt|^{1+\lambda}|\tau - rt|}|dt| + \int_T \frac{(1-r^2)\big||t_k-\tau|^{\lambda} - |t_k - rt|^{\lambda}\big|}{|t_k - rt||\tau - rt|}|dt|.$$

Hence taking into account that

$$\int_T \frac{(1-r^2)}{|t_k - rt||\tau - rt|}|dt| \leq \int_T \frac{(1-r^2)}{|t_k - rt|^2}|dt| + \int_T \frac{(1-r^2)}{|\tau - rt|^2}|dt| = 4\pi,$$

we can apply Holder inequality to obtain

$$\int_T \frac{(1-r^2)\big||t_k-\tau|^{\lambda} - |t_k - rt|^{\lambda}\big|}{|t_k - rt|^{1+\lambda}|\tau - rt|}|dt| \leq C\int_T \frac{(1-r^2)}{|t_k - rt|^{1+\lambda}|\tau - rt|^{1-\lambda}}|dt|$$

$$\leq C\left(\int_T \frac{(1-r^2)}{|t_k - rt|^2}|dt| + \int_T \frac{(1-r^2)}{|\tau - rt|^2}|dt|\right) < \infty,$$

and the assertion c) follows. Lemma 1.2.1 is proved. □

Let $\kappa = \operatorname{ind} a(t)$ and $t \in T$. It is well known that the function a admits the representation (see [15])

$$a(t) = \frac{S^+(t)}{S^-(t)},$$

where

$$S^+(z) = exp\left\{\frac{1}{2\pi i}\int_T \frac{ln\left(t^{-\kappa}a(t)\right)}{t-z}dt\right\}, \quad z \in D^+,$$

$$S^-(z) = z^{-\kappa}\, exp\left\{\frac{1}{2\pi i}\int_T \frac{ln\left(t^{-\kappa}a(t)\right)}{t-z}dt\right\}, \quad z \in D^-, \tag{1.2.1}$$

$S^{\pm} \in C^{\delta}(\overline{D^{\pm}})$ and $|S^{-}(z)| = O(|z|^{-\kappa})$ as $z \to \infty$.

Also, by N we denote the following:

$$N = \sum_{k=1}^{m} n_k. \tag{1.2.2}$$

Lemma 1.2.2

Let $\alpha_k > -1,\ k = 1,2,\dots,m,\ f \in L^1(\rho)$, and $\Phi(z)$ be some solution of the Problem ***R****. Then the following assertions hold.*

a) If $N + \kappa \geq 0$, then $\Phi(z)$ admits the representation

$$\Phi(z) = \frac{S(z)}{2\pi i \Pi(z)} \int_T \frac{f(t)\Pi(t)}{S^{+}(t)(t-z)} dt + \frac{S(z)P(z)}{\Pi(z)}, \tag{1.2.3}$$

where $z \in D^{+} \cup D^{-}, P(z)$ is some polynomial of degree $N + \kappa - 1$, and

$$\Pi(t) = \prod_{k=1}^{m} (t_k - t)^{n_k}.$$

b) If $N + \kappa < 0$, then $\Phi(z)$ has the representation (1.2.3), where $P(z) \equiv 0$ and the function f satisfies the conditions:

$$\int_T \frac{f(t)\Pi(t)}{S^{+}(t)} t^k dt = 0, \quad k = 0,1,\dots,-(N+\kappa)-1. \tag{1.2.4}$$

Proof. First let $N + \kappa \geq 0$. Since $-1 < \alpha_k - n_k \leq 0$, we have

$$\lim_{r \to 1-0} \int_T \left| \frac{\Phi^{+}(rt)\Pi(t)}{S^{+}(t)} - \frac{\Phi^{-}(r^{-1}t)\Pi(t)}{S^{-}(t)} - \frac{f(t)\Pi(t)}{S^{+}(t)} \right| |dt| = 0.$$

Denote

$$\Psi_r^{+}(z) = \frac{\Phi^{+}(rz)\Pi(z)}{S^{+}(z)},$$

$$\Psi_r^{-}(z) = \frac{\Phi^{-}(r^{-1}z)\Pi(z)}{S^{+}(z)},$$

$$f_r(t) = \frac{\Phi^{+}(rt)\Pi(t)}{S^{+}(t)} - \frac{\Phi^{-}(r^{-1}t)\Pi(t)}{S^{+}(t)}, \qquad |t| = 1.$$

Then we obtain the Hilbert boundary value problem $\Psi_r^+(t) - \Psi_r^-(t) = f_r(t)$ with respect to function Ψ_r. Taking into account that the function Ψ_r^- has a pole of order $N + \kappa - 1$ at infinity, we can write

$$\Psi_r^\pm(z) = \frac{1}{2\pi i}\int_T \frac{f_r(t)dt}{t - z} + P_r(z), \qquad z \in D^\pm,$$

where $P_r(z)$ is a polynomial of degree $N + \kappa - 1$. Since

$$f_r(t) \to \frac{f(t)\Pi(t)}{S^+(t)}, \qquad \Psi_r(z) \to \frac{\Phi(z)\Pi(z)}{S^+(z)}$$

as $r \to 1 - 0$, by passing to the limit we complete the proof of assertion a).

Now let $N + \kappa < 0$, then we have $P_r(z) \equiv 0$ and

$$\Phi(z) = \frac{S(z)}{2\pi i \Pi(z)}\int_T \frac{f(t)\Pi(t)}{S^+(t)(t - z)}dt.$$

Taking into account that the function $S(z)(\Pi(z))^{-1}$has a pole of order $-(N + \kappa)$ at infinity, the assertion b) of the lemma follows. Lemma 1.2.2 is proved.□

Lemma 1.2.3

Let $f \in L^1(\rho)$ *and*

$$K(f,z) = \frac{S(z)}{2\pi i \Pi(z)}\int_T \frac{f(t)\Pi(t)}{S^+(t)(t - z)}dt, \quad z \in D^+ \cup D^-, \tag{1.2.5}$$

then

$$\|K^+(f,rt) - a(t)K^-(f,r^{-1}t)\|_{L^1(\rho_r)} \leq C\|f\|_{L^1(\rho)}.$$

Proof. We set

$$K^+(f,rt) - a(t)K^-(f,r^{-1}t) = I_1(r,t) + I_2(r,t),$$

where

$$I_1(r,t) = \frac{S^+(rt) - a(t)S^-(r^{-1}t)}{2\pi i \Pi(rt)}\int_T \frac{f(\tau)\Pi(\tau)}{S^+(\tau)(\tau - rt)}d\tau,$$

and

$$I_2(r,t) = \frac{a(t)S^-(r^{-1}t)}{2\pi i}\left(\frac{1}{\Pi(rt)}\int_T \frac{f(\tau)\Pi(\tau)}{S^+(\tau)(\tau - rt)}d\tau - \right.$$

$$-\frac{1}{\Pi(r^{-1}t)}\int_T \frac{f(\tau)\Pi(\tau)}{S^+(\tau)(\tau-r^{-1}t)}d\tau\Big).$$

Since $|S^+(rt)-a(t)S^-(r^{-1}t)|<C(1-r)^\delta$ for $\delta>0$ (see [42]), we obtain

$$|I_1(r,t)|\le\frac{C(1-r)^\delta}{|\Pi(rt)|}\int_T\frac{|f(\tau)||\Pi(\tau)|}{S^+(\tau)|\tau-rt|}|d\tau|.$$

Next, we have

$$I_2(r,t)=I_2^{(1)}(r,t)+I_2^{(2)}(r,t),$$

where

$$I_2^{(1)}(r,t)=\frac{a(t)S^-(r^{-1}t)}{2\pi i\Pi(rt)}\int_T\frac{f(\tau)\Pi(\tau)}{S^+(\tau)(\tau-rt)^2}d\tau,$$

$$I_2^{(2)}(r,t)=\frac{a(t)S^-(r^{-1}t)}{2\pi i}\left(\frac{1}{\Pi(rt)}-\frac{1}{\Pi(r^{-1}t)}\right)\int_T\frac{f(\tau)\Pi(\tau)}{S^+(\tau)(\tau-rt)}d\tau.$$

Taking into account that the function $a(t)S^-(r^{-1}t)$ is uniformly bounded for $r\to 1-0$, we obtain

$$|I_2^{(1)}(r,t)|<\frac{C}{|\Pi(rt)|}\int_T\frac{|f(\tau)||\Pi(\tau)|(1-r^2)}{|\tau-rt|^2}|d\tau|,$$

$$|I_2^{(2)}(r,t)|<C\left|\frac{1}{\Pi(rt)}-\frac{1}{\Pi(r^{-1}t)}\right|\int_T\frac{f(\tau)\Pi(\tau)}{|\tau-rt|}|d\tau|.$$

Assuming that $\alpha_k>-1$, we can write

$$\|I_1(r,t)\|_{L^1(\rho)}\le C\int_T\frac{(1-r)^\delta|\rho(t)|}{|\Pi(rt)|}\int_T\frac{|f(\tau)||\Pi(\tau)|}{|\tau-rt|}|d\tau||dt|\le$$

$$\le C\int_T|f(\tau)|\,\rho(\tau)\rho^*(\tau)\int_T\frac{(1-r)^\delta}{\rho^*(t)|\tau-rt|}|d\tau||dt|,$$

where $\rho^*(t)$ is as in (1.1.2).

By assertion a) of Lemma 1.2.1 for $\delta\in(0,1)$ and $\tau\in T$, we have

$$\sup_{r\in(0,1)}\rho^*(\tau)\int_T\frac{(1-r)^\delta}{\rho^*(t)|\tau-rt|}|dt|<\infty.$$

Therefore

$$\|I_1(r,t)\|_{L^1(\rho)} \le C\|f\|_{L^1(\rho)}.$$

And we have

$$\left\|I_2^{(1)}(r,t)\right\|_{L^1(\rho)} \le C\int_T |f(\tau)|\,\rho(\tau)\rho^*(\tau)\int_T \frac{(1-r)^\delta}{\rho^*(t)|\tau-rt|^2}|d\tau||dt|.$$

By assertion b) of Lemma 1.2.1

$$\sup_{r\in(0,1)} \rho^*(\tau)\int_T \frac{(1-r^2)}{\rho^*(t)|\tau-rt|^2}|dt| < \infty.$$

Therefore

$$\left\|I_2^{(1)}(r,t)\right\|_{L^1(\rho)} \le C\|f\|_{L^1(\rho)}.$$

For evaluating $I_2^{(2)}(r,t)$ integral function at the interval T_k we firstly prove the following inequality:

$$\left|\frac{1}{\Pi(rt)} - \frac{1}{\Pi(r^{-1}t)}\right| \le C\left(\frac{1-r}{|t_k-rt|^{n_k}} + \frac{1-r}{|t_k-rt|^{n_k+1}}\right). \tag{1.2.6}$$

Let make the following notation:

$$\Pi^{(k)}(t) = \frac{\Pi(t)}{(t-t_k)^{n_k}}.$$

We obtain

$$\left|\frac{1}{\Pi(rt)} - \frac{1}{\Pi(r^{-1}t)}\right| =$$

$$= \left|\frac{1}{(t_k-rt)^{n_k}}\cdot\frac{1}{\Pi_k(rt)} - \frac{1}{(t_k-r^{-1}t)^{n_k}}\cdot\frac{1}{\Pi_k(r^{-1}t)}\right| =$$

$$= \left|\frac{1}{(t_k-rt)^{n_k}}\left(\frac{1}{\Pi_k(rt)} - \frac{1}{\Pi_k(r^{-1}t)}\right) + \frac{1}{\Pi_k(r^{-1}t)}\left(\frac{1}{(t_k-rt)^{n_k}} - \frac{1}{(t_k-r^{-1}t)^{n_k}}\right)\right| \le$$

$$\le \frac{1}{|t_k-rt|^{n_k}}\left|\frac{1}{\Pi_k(rt)} - \frac{1}{\Pi_k(r^{-1}t)}\right| + \frac{1}{|\Pi_k(r^{-1}t)|}\left|\frac{1}{(t_k-rt)^{n_k}} - \frac{1}{(t_k-r^{-1}t)^{n_k}}\right| \le$$

$$\le C\left(\frac{1-r}{|t_k-rt|^{n_k}} + \frac{1-r}{|t_k-rt|^{n_k+1}}\right).$$

So, we have

$$\left\|I_2^{(2)}(r,t)\right\|_{L^1(\rho)} \le C\int_T |f(\tau)|\,\rho(\tau)\rho^*(\tau)\int_T \frac{|\rho(t)|\left|\left(\Pi(rt)\right)^{-1}-\left(\Pi(r^{-1}t)\right)^{-1}\right|}{|\tau-rt|}|d\tau||dt|.$$

Taking into account that for any $\tau \in T_k$

$$\sup_{r\in(0,1)} \rho^*(\tau)\int_T \frac{\rho(t)\left|\left(\Pi(rt)\right)^{-1}-\left(\Pi(r^{-1}t)\right)^{-1}\right|}{|\tau-rt|}|dt| \le$$

$$\le \sum_{k=1}^{m}\int_{T_k} \frac{(1-r)\rho^*(\tau)}{|\tau-rt||t_k-rt|^{n_k-\alpha_k}}|dt| + \sum_{k=1}^{m}\int_{T_k} \frac{(1-r)\rho^*(\tau)}{|\tau-rt||t_k-rt|^{n_k-\alpha_k+1}}|dt|$$

we can apply assertion c) of Lemma 1.2.1 to obtain

$$\sup_{r\in(0,1)} \rho^*(\tau)\int_T \frac{\rho(t)\left|\left(\Pi(rt)\right)^{-1}-\left(\Pi(r^{-1}t)\right)^{-1}\right|}{|\tau-rt|}|dt| < \infty.$$

Hence

$$\left\|I_2^{(1)}(r,t)\right\|_{L^1(\rho)} \le C\|f\|_{L^1(\rho)}.$$

Now we assume $\alpha_k,\ k=1,\dots,m$ are arbitrary real numbers.

So we have

$$\|I_1(r,t)\|_{L^1(\rho_r)} \le C\int_T \frac{(1-r)^\delta|\rho_r(t)|}{|\Pi(rt)|}\int_T \frac{|f(\tau)||\Pi(\tau)||d\tau|}{|\tau-rt|}|dt|$$

$$\le C\int_T |f(\tau)|\,\rho(\tau)\rho^*(\tau)\int_T \frac{(1-r)^\delta|\rho_r(t)||dt|}{|\Pi(rt)||\tau-rt|}|d\tau|.$$

Finally, taking into account that

$$\rho^*(\tau)\int_T \frac{(1-r)^\delta|\rho_r(t)||dt|}{|\Pi(rt)||\tau-rt|} \le \rho^*(\tau)\int_T \frac{(1-r)^\delta|dt|}{\rho^*(t)|\tau-rt|}$$

we can apply Lemma 1.2.1 to complete the proof of Lemma 1.2.3.□

1.2.2 R^α class of functions

Definition

Given $\alpha=\{\alpha_1,\alpha_2,\dots,\alpha_m\}$, we say that a function $a(t)$ belongs to the class R^α if

$$\lim_{r\to 1-0} \|S^+(rt) - a(t)S^-(r^{-1}t)\|_{L^1(\rho_r)} = 0. \tag{1.2.7}$$

Let construct example of a function which belongs to the class R^α.

Example: The function

$$a(t) = cos^\gamma\left(\frac{\pi}{2}\left(\prod_{k=1}^{m}\left(1-\frac{t_k}{t}\right)^{\lambda_k}\right)\right), \tag{1.2.8}$$

where $t \in T$, γ is an integer and λ_k are nonnegative integers, belongs to R^α, provided that $\lambda_k > -n_k - 1$.

It is obvious that $\gamma = \operatorname{ind} a(t)$, and if $\gamma \geq 0$, then $S^+(z) \equiv 1$, and

$$S^-(z) = cos^{-\gamma}\left(\frac{\pi}{2}\left(\prod_{k=1}^{m}\left(1-\frac{t_k}{z}\right)^{\lambda_k}\right)\right).$$

We have

$$|S^+(rt) - a(t)S^-(r^{-1}t)| \leq C\left|\prod_{k=1}^{m}(t_k - r^{-1}t)^{\lambda_k} - \prod_{k=1}^{m}(t_k - t)^{\lambda_k}\right|.$$

Therefore for $t \in T_k$ we can write

$$|S^+(rt) - a(t)S^-(r^{-1}t)|\rho_r(t) \leq C(1-r)|t_k - rt|^{n_k+\lambda_k-1}|t_k - t|^{\alpha_k - n_k}.$$

Hence, taking into account that $\lambda_k + n_k > -1$, we obtain $a(t) \in R^\alpha$. Similarly we can show that $a(t) \in R^\alpha$ in $\gamma < 0$ case.

Lemma 1.2.4

Let $a(t) \in R^\alpha$ and $\alpha_j \leq -2$ for some $j \in 1, \dots, m$. If

$$P(z) = A_1(t_j - z) + A_2(t_j - z)^2 + \cdots + A_{-n_j-1}(t_j - z)^{-n_j-1}$$

satisfies the condition

$$\lim_{r\to 1-0}\int_{|t_j - t|<\delta} |S^+(rt)P(rt) - a(t)S^-(r^{-1}t)P(r^{-1}t)|\rho_r(t)\,|dt| = 0$$

for some $0 < \delta < min|t_j - t_k|,\ j \neq k,$ *then* $P(z) \equiv 0$.

Proof. Since

$$S^+(rt)P(rt) - a(t)S^-(r^{-1}t)P(r^{-1}t) = I_1(r,t) + I_2(r,t),$$

where

$$I_1(r,t) = \left(S^+(rt)P(rt) - a(t)S^-(r^{-1}t)\right)P(rt),$$

$$I_2(r,t) = a(t)S^-(r^{-1}t)\left(P(rt) - P(r^{-1}t)\right),$$

The condition $a(t) \in R^\alpha$ implies that $\|I_1(r,t)\|_{L^1(\rho_r)} \to 0$.

Let $A_1 \neq 0$, then we have

$$|P(rt) - P(r^{-1}t)| > (1-r),$$

where $|t_j - t| < C(1-r), C > 0$.

Hence

$$\int_T |I_2(r,t)|\rho_r(t)|dt| \geq \int_{|t_j - rt| < C(1-r)} \frac{(1-r)|dt|}{|t_j - rt|^{-n_j}|t_j - t|^{n_j - \alpha_j}}.$$

Since $|t_j - rt| = O(1-r)$ for $|t_j - t| < C(1-r)$, then we have

$$\int_{|t_j - rt| < C(1-r)} |I_2(r,t)|\rho_r(t)|dt| \geq (1-r)^{n_j+1} \int_0^{C(1-r)} \frac{d\theta}{\theta^{n_j - \alpha_j}} \geq C(1-r)^{2+\alpha_j},$$

implying that $A_1 = 0$.

Now assume that $A_k \neq 0$. Let k be an odd number and $A_1 = A_2 = \cdots = A_{k-1} = 0$. Then for $|t_j - rt| < C(1-r)$ we have

$$|P(rt) - P(r^{-1}t)| > (1-r)|t_j - t|^{k-1}.$$

Therefore,

$$\int_{|t_j - rt| < \delta} |I_2(r,t)|\rho_r(t)|dt| \geq \int_{|t_j - rt| < C(1-r)} \frac{(1-r)|t_j - t|^{k+\alpha_j - n_j - 1}|dt|}{|t_j - rt|^{-n_j}} >$$

$$> (1-r)^{n_k+1} \int_0^{C(1-r)} \theta^{k+\alpha_j - n_j - 1} d\theta = C(1-r)^{k+\alpha_j+1}.$$

Taking into account that $k + \alpha_j + 1 \leq 0$, we obtain

$$\lim_{r \to 1-0} \int_{|t_j - t| < \delta} |I_2(r,t)|\rho_r(t)\,|dt| > 0.$$

If k is an even number, then we have

$$\int_{|t_j - t|<\delta} |I_2(r,t)|\rho_r(t)|dt| > (1-r)^{n_j+1}\int_{c(1-r)^2}^{C(1-r)} \theta^{k+\alpha_j-n_j-1}d\theta =$$

$$= (1-r)^{k+\alpha_j+1}\left(A - B(1-r)^{k+\alpha_j-n_j}\right).$$

Taking into account that $k+\alpha_j-n_j>0$, we complete the proof of Lemma 1.2.4.□

In what follows in this chapter we assume that

$$\begin{cases} \alpha_1 \le \alpha_2 \le \cdots \le \alpha_m \\ \alpha_{m_0} \le -1 \\ \alpha_{m_0+1} > -1 \end{cases} \tag{1.2.9}$$

Lemma 1.2.5

Let $k_j \ge -n_j, j = 1,2,\dots,m_0$, *then*

$$\lim_{r\to 1-0}\left\| S^+(rt)\prod_{j=1}^{m_0}(t_j - rt)^{k_j} - a(t)S^-(r^{-1}t)\prod_{j=1}^{m_0}(t_j - r^{-1}t)^{k_j}\right\|_{L^1(\rho_r)} = 0 \tag{1.2.10}$$

Proof. Since

$$S^+(rt)\prod_{j=1}^{m_0}(t_j - rt)^{k_j} - a(t)S^-(r^{-1}t)\prod_{j=1}^{m_0}(t_j - r^{-1}t)^{k_j} = I_1(r,t) + I_2(r,t)$$

where

$$I_1(r,t) = \left(S^+(rt) - a(t)S^-(r^{-1}t)\right)\prod_{j=1}^{m_0}(t_j - rt)^{k_j},$$

$$I_2(r,t) = a(t)S^-(r^{-1}t)\left(\prod_{j=1}^{m_0}(t_j - rt)^{k_j} - \prod_{j=1}^{m_0}(t_j - r^{-1}t)^{k_j}\right).$$

In view of $|S^+(rt) - a(t)S^-(r^{-1}t)| < C(1-r)^{\delta}$ and $\delta > 0$, we obtain

$$\int_T |I_1(r,t)|\rho_r(t)|dt| \le C(1-r)^{\delta}\sum_{j=1}^{m}\int_{T_j}\frac{|t_j - rt|^{k_j}|dt|}{|t_j - rt|^{-n_j}|t_j - t|^{n_j-\alpha_j}}$$

So, we will get

$$\lim_{r\to 1-0}\int_T |I_1(r,t)||dt| = 0$$

Since for $t \in T_j$

$$\left|\prod_{j=1}^{m_0}(t_j - rt)^{k_j} - \prod_{j=1}^{m_0}(t_j - r^{-1}t)^{k_j}\right| \le C\left|(t_j - rt)^k - (t_j - r^{-1}t)^k\right| <$$

$$< C(1-r)|t_j - rt|^{k-1},$$

then we get

$$\int_T |I_2(r,t)|\rho_r(t)|dt| \le C(1-r)\sum_{j=1}^{m}\int_{T_j}\frac{|dt|}{|t_j - rt||t_j - t|^{n_j-\alpha_j}}.$$

Taking into consideration that $n_j - \alpha_j < 1$, we obtain

$$\lim_{r\to 1-0}\int_T |I_2(r,t)||dt| = 0.$$

So, lemma is proved.□

Lemma 1.2.6

Let $\alpha_k \le -2$ *for some* $k,\ 1 \le k \le m,\ a(t) \in R^{\alpha}$ *and*

$$\Phi_k(z) = (t_k - z)^{-n_k}\prod_{j=1}^{m}(t_j - z)^{n_j}. \qquad (1.2.11)$$

If the function

$$\varphi_k(z) = \left(\frac{A_1}{(t_k - z)} + \frac{A_2}{(t_k - z)^2} + \cdots + \frac{A_{-n_k}}{(t_k - z)^{-n_k}}\right)\Phi_k(z) \qquad (1.2.12)$$

satisfies the condition

$$\int_T |S^+(rt)\varphi_k^+(rt) - a(t)S^-(r^{-1}t)\varphi_k^-(rt)|\rho_r(t)|dt| \to 0,$$

then $\varphi_k(z)$ can be represented in the form

$$\varphi_k(z) = A_0 + (t_k - z)^{-n_k}\Phi_0(z), \qquad (1.2.13)$$

where $\Phi_0(z)$ is analytic at point t_k function, and A_0 is some complex number.

Proof. By Lemma 1.2.5,

$$\int_{|t_k-t|<\epsilon} |(t_k-rt)^{-n_k}S^+(rt)\varphi_k^+(rt) - a(t)(t_k-r^{-1}t)^{-n_k}S^-(r^{-1}t)\varphi_k^-(rt)|\rho_r(t)|dt| \to 0,$$

as $r \to 1-0$, we can conclude that the function

$$\phi_k(z) = \varphi_k'(t_k)(t_k-z) + \cdots + \frac{\varphi_k^{(-n_k-1)}(t_k)(t_k-z)^{-n_k-1}}{(-n_k)!}$$

also satisfies the condition

$$\int_{|t_k-t|<\epsilon} |S^+(rt)\phi_k^+(rt) - a(t)S^-(r^{-1}t)\phi_k^-(rt)|\rho_r(t)|dt| \to 0.$$

By lemma 1.2.4, the numbers $A_1, A_2, \dots, A_{-n_k-1}$ can uniquely be determined from the conditions:

$$\begin{cases} A_{-n_k}F'(t_k) + A_{-n_k-1}F(t_k) = 0 \\ A_{-n_k}F''(t_k) + A_{-n_k-1}F'(t_k) + A_{-n_k-2}F(t_k) = 0 \\ \dots \\ A_{-n_k}F^{-n_k}(t_k) + A_{-n_k-1}F^{-n_k-1}(t_k) + \cdots + A_1F(t_k) = 0 \end{cases} \tag{1.2.13}$$

Choosing A_{-n_k} arbitrarily, the numbers $A_1, A_2, \dots, A_{-n_k-1}$ can uniquely determined by the above system. Taking into account Lemmas 1.2.4 and 1.2.5 we complete the proof of Lemma 1.2.6.□

1.3 The main result

Theorem 1.3.1

Let $f \in L^1(\rho)$. Then

$$\lim_{r\to 1-0} \|K^+(f,rt) - a(t)K^-(f,r^{-1}t) - f(t)\|_{L^1(\rho_r)} = 0, \tag{1.3.1}$$

*where $K(f,z)$ is as in (1.2.5), that is, $K(f,z)$ is a solution of the Problem **R**, when $N+\kappa \geq 0$. If $N+\kappa < 0$, then $K(f,z)$ is a solution of the Problem **R** if and only if f satisfies the conditions (1.2.4).*

Proof. Let $f \equiv 0$ in some non-overlapping neighboarhoods $T_1(k) \in T$ of points t_k, and let T_1 denote the union of these intervals. Then the function

$$\Phi_1(z) = \int_T \frac{f(\tau)\Pi(\tau)d\tau}{S^+(\tau)(\tau - z)}$$

is analytic in T_1, and its Taylor series at points $t_i, i = 1,2,\dots,m$ has the form

$$\Phi_1(z) = A_0^{(i)} + A_1^{(i)}(z - t_i) + \dots + A_k^{(i)}(z - t_i)^k + \cdots,$$

where

$$A_k^{(i)} = \frac{k!}{2\pi i}\int_T \frac{f(\tau)\Pi(\tau)d\tau}{S^+(\tau)(\tau - t_i)^{k+1}}, \qquad i = 1,2,\dots,m.$$

We have

$$\mathrm{K}^+(f, rt) - a(t)\mathrm{K}^-(f, r^{-1}t) = I_1(r,t) + I_2(r,t),$$

where

$$I_1(r,t) = \frac{S^+(rt)\Phi_1(rt) - a(t)S^-(r^{-1}t)\Phi_1(r^{-1}t)}{\Pi(rt)},$$

$$I_2(r,t) = a(t)S^-(r^{-1}t)\Phi_1(r^{-1}t)\left(\frac{1}{\Pi(rt)} - \frac{1}{\Pi(r^{-1}t)}\right).$$

Since $|\Phi_1(rt) - \Phi_1(r^{-1}t)| < C(1 - r), t \in T_1$, then in view of

$$|S^+(rt)\Phi_1(rt) - a(t)S^-(r^{-1}t)\Phi_1(r^{-1}t)| < C(1 - r)^\delta$$

for $\delta > 0, r \to 1 - 0$ and every $t \in T_1$ we obtain

$$\left|\int_{T_1} I_1(rt)\rho_r(t)dt\right| \le C\int_{T_1} \frac{(1 - r)^\delta \rho_r(t)|dt|}{\Pi(rt)} \le C(1 - r)^\delta \int_{T_1} |\rho^*(t)dt|.$$

Also, for $t \in T_1$ we have

$$\left|\int_{T_1} I_2(rt)\rho_r(t)dt\right| \le C\sum_{k=1}^{m}\int_{T_1^k}(1 - r)\left(\frac{1}{|t_k - rt|^{n_k}} + \frac{1}{|t_k - rt|^{n_k+1}}\right)|\rho_r(t)||dt| \le$$

$$\le C(1 - r)\sum_{k=1}^{m}\left(\int_{T_1^k}|t - t_k|^{\alpha_k - n_k}|dt| + \int_{T_1^k}|t - t_k|^{\alpha_k - n_k - 1}|dt|\right).$$

Next, let $t \in T\backslash T_1$. Since $f(t)\Pi(t) \in L^1(T)$, the function

$$\Phi_2(z) = \frac{S(z)}{2\pi i}\int_T \frac{f(\tau)\Pi(\tau)d\tau}{S^+(\tau)(\tau - z)}$$

satisfies the condition

$$\|\Phi_2^+(rt) - a(t)\Phi_2^-(r^{-1}t) - f(t)\Pi(t)\|_{L^1} \to 0, \qquad r \to 1-0.$$

Thus, the theorem is proved for a function vanishing in neighboarhoods of points t_k, $k = 1, \dots, m$.

Now let f be an arbitrary function from $L^1(\rho)$. Then for any $\varepsilon > 0$ a function $f_\varepsilon \in L^1(\rho)$ can be found to satisfy $f_\varepsilon \equiv 0$ in T_1 and $\|f - f_\varepsilon\|_{L^1(\rho)} < \varepsilon$. In view of Lemma 1.2.3 we can write

$$\|\mathrm{K}^+(f, rt) - a(t)\mathrm{K}^-(f, r^{-1}t) - f(t)\|_{L^1(\rho_r)} \leq$$
$$\leq \|\mathrm{K}^+(f - f_\varepsilon, rt) - a(t)\mathrm{K}^-(f - f_\varepsilon, r^{-1}t)\|_{L^1(\rho_r)} +$$
$$+\|\mathrm{K}^+(f_\varepsilon, rt) - a(t)\mathrm{K}^-(f_\varepsilon, r^{-1}t) - f_\varepsilon\|_{L^1(\rho_r)} + \|f - f_\varepsilon\|_{L^1(\rho)} \leq$$
$$\leq C\|f - f_\varepsilon\|_{L^1(\rho)} + \|\mathrm{K}^+(f_\varepsilon, rt) - a(t)\mathrm{K}^-(f_\varepsilon, r^{-1}t) - f_\varepsilon\|_{L^1(\rho_r)}.$$

Taking into account that $\|\mathrm{K}^+(f_\varepsilon, rt) - a(t)\mathrm{K}^-(f_\varepsilon, r^{-1}t) - f_\varepsilon\|_{L^1(\rho_r)} \to 0, r \to 1-0,$ we complete the proof of Theorem 1.3.1.□

1.4 Investigation of the Problem R in the case $\alpha_k > -1, k = 1, 2, \dots, m$

1.4.1 General solution of the homogeneous Problem R

In the theorem 1.4.1 we state the general solution of the homogeneous Problem **R** in the case $\alpha_k > -1,\ k = 1,2,\dots,m$.

Theorem 1.4.1

The following assertions hold.

*a) If $\kappa \geq 0$, then the general solution of the homogeneous Problem **R** can be represented in the form:*

$$\Phi_0(z) = S(z)\left(\sum_{k=1}^{m}\sum_{j=1}^{n_k}\frac{A_j^{(k)}}{(t_k - z)^j} + P(z)\right), \tag{1.4.1}$$

where $P(z)$ is a polynomial of degree $\kappa - 1$ for $\kappa > 0$ and $P(z) \equiv 0$ for $\kappa \equiv 0$.

*b) If $\kappa < 0$ and $N + \kappa > 0$, then the general solution of the homogeneous Problem **R** can be represented in the form:*

$$\Phi_0(z) = \frac{S(z)P(z)}{\Pi(z)}, \tag{1.4.2}$$

where $P(z)$ is a polynomial of degree $N + \kappa - 1$.

c) If $N + \kappa \leq 0$, then the homogeneous problem has only trivial solution.

Proof. Let

$$P(z) = \sum_{k=0}^{\kappa-1} A_k z^k.$$

Since $|S^+(rt) - a(t)S^-(r^{-1}t)| < C(1-r)^\delta$, we have

$$|S^+(rt)P(rt) - a(t)S^-(r^{-1}t)P(r^{-1}t)| < C(1-r)^\delta.$$

Therefore

$$\|S^+(rt)P(rt) - a(t)S^-(r^{-1}t)P(r^{-1}t)\|_{L^1(\rho)} \to 0, r \to 1-0.$$

Setting $\Phi_{jk}^+(z) = S(z)(t_k - z)^{-j},\ j < n_k,\ k = 1, \ldots, m,$ we can write

$$\left|\Phi_{jk}^+(rt) - a(t)\Phi_{jk}^-(r^{-1}t)\right| \leq \left|S^+(rt)(t_k - rt)^{-j} - a(t)S^-(r^{-1}t)(t_k - r^{-1}t)^{-j}\right| \leq$$

$$\leq \frac{|S^+(rt) - a(t)S^-(r^{-1}t)|}{|t_k - rt|^j} + |a(t)S^-(r^{-1}t)| \left|\frac{1}{(t_k - rt)^j} - \frac{1}{(t_k - r^{-1}t)^j}\right| \leq$$

$$\leq C\left(\frac{(1-r)^\delta}{|t_k - rt|^j} + \left|\frac{1}{(t_k - rt)^j} + \frac{1}{(t_k - r^{-1}t)^j}\right|\right) \leq C\left(\frac{(1-r)^\delta}{|t_k - rt|^j} + \frac{(1-r)}{|t_k - rt|^{j+1}}\right).$$

Hence

$$\left\|\Phi_{jk}^+(rt) - a(t)\Phi_{jk}^-(r^{-1}t)\right\|_{L^1(\rho)} \leq C\left(\int_T \frac{(1-r)^\delta \rho(t)|dt|}{|t_k - rt|^j} + \int_T \frac{(1-r)\rho(t)|dt|}{|t_k - rt|^{j+1}}\right).$$

Since $j < n_k$ and $n_k - \alpha_k < 1$, the last integrals tend to zero as $r \to 1-0$. Thus, the assertion a) of the theorem is proved. Assertions b) and c) can be proved similarly. Theorem 1.4.1 is proved.□

1.4.2 General solution of the inhomogeneous Problem R

In the theorem 1.4.2 we state the general solution of the Problem **R** in the case $\alpha_k > -1,\ k = 1,2,\dots,m$.

Theorem 1.4.2

The following assertions hold.

*a) If $N + \kappa \geq 0$, then the general solution of the inhomogeneous Problem **R** can be represented in the form:*

$$\Phi(z) = K(f,z) + \Phi_0(z), \tag{1.4.3}$$

*where $K(f,z)$ is as in (1.2.5), and $\Phi_0(z)$ is the general solution of the homogeneous Problem **R**.*

*b) If $N + \kappa < 0$, then the Problem **R** is solvable if and only if f satisfies the conditions (1.2.4). And the solution can be represented in the form (1.2.5).*

1.5 Investigation of the Problem R in the case of arbitrary $\alpha_k, k = 1,2,\dots,m$

In the theorem 1.5.1 we state general solution of the Problem **R** in the case of arbitrary numbers $\alpha_k,\ k = 1,2,\dots,m$.

Theorem 1.5.1

Let $a \in R^{\alpha}$. The following assertions hold.

*a) If $N + \kappa \geq 0$, then the general solution of the inhomogeneous Problem **R** can be represented in the form:*

$$\Phi(z) = K(f,z) + S(z)\left(A_0 + \frac{P(z)}{\Pi(z)}\right), \tag{1.5.1}$$

where $K(f,z)$ is as in (1.2.5), A_0 is an arbitrary complex number for $\kappa > 0$ and $A_0 = 0$ for $\kappa \leq 0$, and $P(z)$ is a polynomial of degree $N + \kappa - 1$.

b) If $N+\kappa<0$ and $\kappa>0$, then $\Phi(z)$ can be represented in the form (1.4.3), where $\Phi_0(z)=0$ and f satisfies the following conditions:

$$\int_T \frac{f(t)}{S^+(t)} \Pi(t)\, t^k dt = 0, \quad k = 0,1,\dots,-(N+\kappa)-1. \tag{1.5.2}$$

*c) If $N<0$ and $\kappa\le 0$, then Problem **R** has a unique solution:*

$$\Phi(z) = K(f,z) + A_0\Phi_0(z), \tag{1.5.3}$$

where

$$A_0 = -\frac{1}{2\pi i}\int_T \frac{f(t)}{S^+(t)} \Pi(t)\, t^{-(N+1)} dt \tag{1.5.4}$$

and f satisfies conditions (1.5.2) if $\kappa \neq -N-1$.

*d) If $N+\kappa<0$ and $N\ge 0$, then Problem **R** has a single solution $\Phi(z)=K(f,z)$ and f satisfies conditions (1.5.2).*

Proof. Let Φ be a solution of the Problem **R**. Then we have

$$\lim_{r\to 1-0}\int_T \left|\frac{\Phi^+(rt)}{S^+(t)} - \frac{\Phi^-(r^{-1}t)}{S^-(t)} - \frac{f(t)}{S^+(t)}\right| |P_{jr}(t)||\tilde{P}_j(t)||dt| = 0,$$

where

$$P_{jr}(z) = \prod_{k=1}^{j}(t_k - rz)^{n_k}, \qquad \tilde{P}_j(z) = \prod_{k=j+1}^{m}(t_k - z)^{n_k}.$$

Denoting

$$\begin{cases} \Psi_r^+(z) = \dfrac{\Phi^+(rz)}{S^+(z)} P_{jr}(z)\tilde{P}_j(z) \\ \Psi_r^-(z) = \dfrac{\Phi^-(r^{-1}z)}{S^-(z)} P_{jr}(z)\tilde{P}_j(z) \end{cases}$$

we obtain the Hilbert problem for function $\Psi_r(z)$:

$$\Psi_r^+(t) - \Psi_r^-(t) = f_r(t),$$

where

$$f_r(t) = \frac{f(t)}{S^+(t)} P_{jr}(t)\tilde{P}_j(t).$$

Taking into account that the function $\Psi_r^-(z)$ has a pole of order $-n_k$ at point $t = r^{-1}t_k,\ k = 1,2,\dots,j$ and at infinity $|\Psi_r^+(z)| < C|z|^{N+\kappa-1}$. Also, denoting by

$$Q_{kr}(z) = \frac{A_1^k(r)}{(t_k - rz)} + \frac{A_2^k(r)}{(t_k - rz)^2} + \cdots + \frac{A_{n_k}^k(r)}{(t_k - rz)^{-n_k}}$$

the principal part of Laurent expansion of function $\Psi_r^-(z)$ at point $r^{-1}t_k$, we obtain

$$\Psi_r^+(t) - \left(\Psi_r^-(t) - \sum_{k=1}^{m_0} Q_{kr}(t)\right) = f_r(t) + \sum_{k=1}^{m_0} Q_{kr}(t). \qquad (1.5.5)$$

Next, assuming that $N + \kappa \geq 0$, we obtain

$$\Psi_r(z) = \frac{1}{2\pi i}\int_T \frac{f_r(t)}{(t-z)}dt + \sum_{k=1}^{m_0} Q_{kr}(z) + P_r(z),$$

where $P_r(z)$ is some polynomial of degree $N + \kappa - 1$ for $N + \kappa > 0$ and $P_r(z) \equiv 0$ for $N + \kappa = 0$.

Since $f_r(t) \to f(t)\Pi(t)(S^+(t))^{-1}$ in L^1, we have

$$\Phi(z) = K(f,z) + \frac{S(z)}{\Pi(z)}\sum_{k=1}^{m_0}(Q_k(z) + P(z)), \qquad (1.5.6)$$

where

$$Q_k(z) = \frac{C_1^k}{(t_k - z)} + \frac{C_2^k}{(t_k - z)^2} + \cdots + \frac{C_{-n_k}^k}{(t_k - z)^{-n_k}}, \qquad k = 1,2,\dots,m_0.$$

It is clear that the function

$$\frac{S(z)P(z)}{\Pi(z)}$$

satisfies condition (1.1.3). Hence the function

$$\frac{1}{\Pi(z)}\sum_{k=1}^{m_0} Q_k(z)$$

also satisfies condition (1.1.3). Therefore, in view of Lemma 1.2.6 we obtain the following representation:

$$\Phi(z) = K(f,z) + S(z)\left(A_0 + \frac{P(z)}{\Pi(z)}\right). \tag{1.5.7}$$

Assume that $N + \kappa < 0$, $\kappa > 0$. Since $a \in R^{\alpha}$, then $A_0 S(z)$ is a solution of the homogeneous Problem **R**. Hence the general solution of the inhomogeneous Problem **R** can be represented in the form:

$$\Phi(z) = K(f,z) + A_0 S(z),$$

where A_0 is an arbitrary complex number.

If $N > 0$, $\kappa \leq 0$, then the function $S(z)$ has a pole of order $-\kappa$ at infinity. Hence $A_0 S(z)$ is not a solution of the homogeneous Problem **R**. In order to satisfy the condition $\Phi(\infty) = 0$, it is necessary to find A_0 from (1.5.4), and to require f to satisfy the conditions (1.5.2) for $\kappa \neq -N-1$.

Finally, for $N + \kappa < 0$, $N \leq 0$, clearly we have $A_0 = 0$ and f satisfies the condition (1.5.2). Theorem 1.5.1 is proved.□

Theorem 1.5.2

Let $a \in R^{\alpha}$. The following assertions hold.

*a) If $N + \kappa > 0$, then the general solution of the homogeneous Problem **R** can be represented in the form:*

$$\Phi(z) = S(z)\left(A_0 + \frac{P(z)}{\Pi(z)}\right), \tag{1.5.8}$$

where A_0 is an arbitrary complex number for $\kappa > 0$ and $A_0 = 0$ for $\kappa \leq 0$, and $P(z)$ is a polynomial of degree $N + \kappa - 1$.

b) If $N + \kappa \leq 0$ and $\kappa > 0$, then $\Phi(z) = A_0 S(z)$, where A_0 is an arbitrary complex number.

*c) If $N + \kappa \leq 0$ and $\kappa \leq 0$, then the homogeneous Problem **R** has only trivial solution: $\Phi(z) \equiv 0$.*

CHAPTER 2.RIEMANN-HILBERT and DISCONTINUOUS RIEMANN PROBLEMS IN THE WEIGHTED SPACES $L^1(\rho)$

2.1Riemann-Hilbert Problem

2.1.1 Introduction

If not stated opposite we will use here the same notations as in Chapter 1.

Let $\Phi(z)$ is a function defined in D^+. By $\Phi_*(z)$ we understand the following (see [59]):

$$\Phi_*(z) = -\overline{\Phi\left(\frac{1}{\bar{z}}\right)}, \tag{2.1.1}$$

where $\Phi_*(z)$ is defined in D^-. It is well known that this operation is symmetric, thus

$$[\Phi_*(z)]_* = \Phi(z).$$

Properties

1. If $\Phi(z)$ is holomorphic (meromorphic) function D^+, then $\Phi_*(z)$ is holomorphic (meromorphic) in D^-.

Suppose

$$\Phi(z) = \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n},$$

then

$$\Phi_*(z) = -\overline{\Phi\left(\frac{1}{\bar{z}}\right)} = -\frac{\overline{a_0} z^{-m} + \overline{a_1} z^{-m+1} + \cdots + \overline{a_m}}{\overline{b_0} z^{-n} + \overline{b_1} z^{-n+1} + \cdots + \overline{b_n}}.$$

2. If

$$\Phi(z) = \sum_{-\infty}^{\infty} a_k z^k, \qquad z \in D^+,$$

then

$$\Phi_*(z) = -\sum_{-\infty}^{\infty} \overline{a_k} z^{-k}, \qquad z \in D^-.$$

Thus, if $\Phi(z)$ has a pole (zero) of order k at the point $z = 0$ $(z = \infty)$, then $\Phi_*(z)$ also has a pole (zero) of order k at the point $z = \infty$ $(z = 0)$. So, if Φ is analytic in unit circle, then Φ_* is bounded at infinity.

3. Let $\Phi^+(t) = \lim_{z\to t}\Phi(z), z \in D^+$. Then, there exists

$$\Phi_*^-(t) = \lim_{z\to t}\Phi_*(z) = \lim_{z\to t} -\overline{\Phi\left(\frac{1}{\bar{z}}\right)} = -\overline{\Phi^+(t)}.$$

So, if we define function $\Phi(z)$ as follows:

$$\Phi(z) = \begin{cases}\Phi(z), & z \in D^+,\\ \Phi_*(z), & z \in D^-,\end{cases}$$

where $\Phi(z)$ is holomorphic in D^+ and $\Phi_*(z)$ is holomorphic in D^-, then

$$\Phi^-(t) = -\overline{\Phi^+(t)}, \Phi^+(t) = -\overline{\Phi^-(t)}. \qquad (2.1.2)$$

4. Let $\Psi(z)$ is the Cauchy type integral of function $\varphi(t)$. So,

$$\Psi(z) = \frac{1}{2\pi i}\int_T \frac{\varphi(t)}{t-z}dt.$$

Then,

$$\Psi\left(\frac{1}{\bar{z}}\right) = \frac{1}{2\pi i}\int_T \frac{\varphi(t)}{t-\frac{1}{\bar{z}}}dt$$

and

$$\Psi_*(z) = -\overline{\Psi\left(\frac{1}{\bar{z}}\right)} = -\overline{\frac{1}{2\pi\iota}\int_T \frac{\varphi(t)}{t-\frac{1}{\bar{z}}}dt} = \frac{1}{2\pi i}\int_T \frac{\overline{\varphi(t)}}{\bar{t}-\frac{1}{z}}\overline{dt}.$$

By substituting

$$t = e^{i\theta}, dt = ie^{i\theta}d\theta, \overline{dt} = -ie^{-i\theta}d\theta = -\frac{dt}{t^2},$$

we have

$$\Psi_*(z) = -\frac{1}{2\pi i}\int_T \frac{\overline{\varphi(t)}}{t^2\left(\bar{t}-\frac{1}{z}\right)}dt = \frac{1}{2\pi i}\int_T \frac{z\overline{\varphi(t)}}{t(t-z)}dt = \frac{1}{2\pi i}\int_T \frac{\overline{\varphi(t)}}{t-z}dt - \frac{1}{2\pi i}\int_T \frac{\overline{\varphi(t)}}{t}dt,$$

thus

$$\Psi_*(z) = \frac{1}{2\pi i}\int_T \frac{\overline{\varphi(t)}}{t-z}dt - \frac{1}{2\pi i}\int_T \frac{\overline{\varphi(t)}}{t}dt. \qquad (2.1.3)$$

2.1.2 Statement of the problem

We consider the Riemann-Hilbert problem in the following setting:

Problem H

Let f be a real-valued, measurable on T function from the space $L^1(\rho)$. Determine an analytic in D^+ function $\Phi(z)$ to satisfy the boundary condition:

$$\lim_{r\to 1-0}\|\mathrm{Re}\{a(t)\Phi(rt)\} - f(t)\|_{L^1(\rho_r)} = 0, \tag{2.1.4}$$

where $a(t), a(t) \neq 0$ is an arbitrary function from the class $C^\delta(T),\ \delta > 0$.

By taking into consideration

$$\mathrm{Re}\{a(t)\Phi(rt)\} = \frac{1}{2}\left(a(t)\Phi(rt) + \overline{a(t)\Phi(rt)}\right)$$

equation, (2.1.2) can be represented as follows:

$$\lim_{r\to 1-0}\left\|a(t)\Phi(rt) + \overline{a(t)\Phi(rt)} - 2f(t)\right\|_{L^1(\rho_r)} = 0. \tag{2.1.5}$$

Our goal is to transform condition (2.1.4) such that it has the same representation as convergence condition (1.1.3) in Riemann boundary value problem (Problem **R**) which is fully investigated in Chapter 1. Then we will reconsider Riemann-Hilbert Problem (Problem **H**) as a complicated application of the Problem **R**. In other words, by applying fundamental results attained in Problem **R** we will solve the Problem **H**.

Taking into account (2.1.1) and (2.1.2) we come to the following contractions of function Φ on D^+ and D^- respectively:

$$\begin{cases} \Phi^+(z) = \Phi(z), & z \in D^+, \\ \Phi^-(z) = -\overline{\Phi\left(\frac{1}{\bar{z}}\right)}, & z \in D^-. \end{cases} \tag{2.1.6}$$

By taking into account (2.1.6) we can rewrite (2.1.5) as follows:

$$\lim_{r\to 1-0}\left\|\Phi^+(rt) - \frac{\overline{a(t)}}{a(t)}\Phi^-(r^{-1}t) - \frac{2f(t)}{a(t)}\right\|_{L^1(\rho_r)} = 0. \tag{2.1.7}$$

It remains to make some minor changes in the view of (2.1.7) to get exactly the same convergence condition as in Problem **R**. Denote,

$$\frac{\overline{a(t)}}{a(t)} = D(t), \qquad \frac{2f(t)}{a(t)} = g(t).$$

As $a(t) \in C^{\delta}(T)$ with some $\delta > 0$, obviously $g \in L^1(\rho)$ in T. Also, if $\operatorname{ind} a(t) = \kappa_0$, then

$$\operatorname{ind} D(t) = -2\kappa_0. \tag{2.1.8}$$

From (2.1.8) it is obvious that the function $D(t)$ has index of even number. We will denote it by κ and take into account that $\operatorname{ind} D(t) = \kappa$, where κ is even number.

So, we have come to the following Riemann Problem.

Problem R

Let g is some measurable on T function from the class $L^1(\rho)$. Determine an analytic in $D^+ \cup D^-$ function $\Phi(z), \Phi(\infty) = C$ to satisfy the boundary condition:

$$\lim_{r\to 1-0} \|\Phi^+(rt) - D(t)\Phi^-(r^{-1}t) - g(t)\|_{L^1(\rho_r)} = 0, \tag{2.1.9}$$

where $D(t), D(t) \neq 0$ is a function from the class $C^{\delta}(T)$, $\delta > 0$, and $\Phi^{\pm}$ are the contractions of function Φ on $D^{\pm}$ respectively.

Suppose $\Phi(z)$ is a solution of the Problem **R.** Then, generally it may not be a solution of the Problem **H** as well. For $\Phi(z)$ to be a solution of the Problem **H** it is necessary and sufficient that $\Phi(z)$ satisfy to the following condition:

$$\Phi_*(z) = \Phi(z), \qquad |z| \neq 1. \tag{2.1.10}$$

Besides, $\Phi_*(z)$ is also a solution of the Problem **R.** So, we will give the general solution of the Problem **H** with the following formula

$$\Omega(f, z) = \frac{1}{2}\big(\Phi(z) + \Phi_*(z)\big), \tag{2.1.11}$$

where $\Phi(z)$ is the general solution of the Problem **R**.

2.1.3 Solution of the Problem H

Let

$$D = \frac{u - iv}{u + iv}, \tag{2.1.12}$$

where u and v are real-valued functions, such that $a = u + iv$.

It is well known that the function D admits the representation (see [15])

$$D(t) = \frac{S^+(t)}{S^-(t)},$$

where

$$S^+(z) = exp\left\{\frac{1}{2\pi i}\int_T \frac{ln\left(t^{-\kappa}D(t)\right)}{t - z}dt\right\}, \quad z \in D^+,$$
$$S^-(z) = z^{-\kappa}\, exp\left\{\frac{1}{2\pi i}\int_T \frac{ln\left(t^{-\kappa}D(t)\right)}{t - z}dt\right\}, \quad z \in D^-, \tag{2.1.13}$$

$S^{\pm} \in C^{\delta}(\overline{D^{\pm}})$ and $|S^-(z)| = O(|z|^{-\kappa})$ as $z \to \infty$.

Taking into consideration (2.1.3) (see [59])

$$S_*(z) = \overline{S\left(\frac{1}{\bar{z}}\right)} = z^{\kappa}S(z). \tag{2.1.14}$$

Also, from (2.1.2)

$$\overline{S^+(t)} = S_*^-(t) = t^{\kappa}S^-(t). \tag{2.1.15}$$

Suppose

$$\Pi_*(z) = \overline{\Pi\left(\frac{1}{\bar{z}}\right)}, \quad z \in D^-.$$

Now we evaluate function $\Pi_*(z)$.

$$\Pi_*(z) = \prod_{k=1}^{m}\left(\frac{1}{z} - \bar{t}_k\right)^{n_k} = \frac{(-1)^N}{z^N}\prod_{k=1}^{m}(z - t_k)^{n_k}\prod_{k=1}^{m} {t_k}^{-n_k} = \frac{(-1)^N}{z^N}\Pi(z)\prod_{k=1}^{m} {t_k}^{-n_k}.$$

Thus we have

$$\Pi_*(z) = \frac{(-1)^N}{z^N}\Pi(z)\prod_{k=1}^{m} {t_k}^{-n_k}. \tag{2.1.16}$$

With the same manner we conclude

$$\overline{\Pi(t)} = \frac{(-1)^N}{t^N}\Pi(t)\prod_{k=1}^{m} {t_k}^{-n_k}.$$

Let

$$K(f,z)=\frac{S(z)}{\pi i\Pi(z)}\int_T\frac{f(t)\Pi(t)}{S^+(t)(u+iv)(t-z)}dt\,,\quad z\in D^+\cup D^-, \tag{2.1.17}$$

As in (2.1.1) suppose

$$K_*(f,z)=-\overline{K\left(f,\frac{1}{\bar{z}}\right)}.$$

Taking into consideration (2.1.12), (2.1.14), (2.1.16) and (2.1.3) we have:

$$K_*(f,z)=\frac{z^N S(z)}{\Pi(z)}\left(\frac{1}{\pi i}\int_T\frac{f(t)\Pi(t)dt}{\overline{S^+(t)}(u-iv)t^N(t-z)}-\frac{1}{\pi i}\int_T\frac{f(t)\Pi(t)dt}{\overline{S^+(t)}(u-iv)t^{N+1}}\right).$$

Obviously,

$$(u-iv)S^-(t)=(u+iv)S^+(t).$$

So, from (2.1.15) we get

$$K_*(f,z)=\frac{z^{N+\kappa}S(z)}{\Pi(z)}\left(\frac{1}{\pi i}\int_T\frac{f(t)t^{-(N+\kappa)}\Pi(t)dt}{S^+(t)(u+iv)(t-z)}-\frac{1}{\pi i}\int_T\frac{f(t)t^{-(N+\kappa)}\Pi(t)dt}{S^+(t)(u+iv)t}\right).$$

Now we evaluate the following:

$$\Omega(f,z)=\frac{1}{2}\big(K(f,z)+K_*(f,z)\big).$$

Obviously,

$$\Omega(f,z)=\frac{S(z)}{2\pi i\Pi(z)}\int_T\frac{f(t)\Pi(t)dt}{S^+(t)(u+iv)(t-z)}+$$
$$+\frac{z^{N+\kappa}S(z)}{2\pi i\Pi(z)}\left(\int_T\frac{f(t)t^{-(N+\kappa)}\Pi(t)dt}{S^+(t)(u+iv)(t-z)}-\int_T\frac{f(t)t^{-(N+\kappa)}\Pi(t)dt}{S^+(t)(u+iv)t}\right).$$

Then,

$$\Omega(f,z)=\frac{S(z)}{2\pi i\Pi(z)}\left(z^{N+\kappa}\int_T\frac{f(t)t^{-(N+\kappa)}\Pi(t)dt}{S^+(t)(u+iv)(t-z)}+\int_T\frac{f(t)\Pi(t)dt}{S^+(t)(u+iv)(t-z)}\right)-$$
$$-\frac{z^{N+\kappa}S(z)}{2\pi i\Pi(z)}\int_T\frac{f(t)t^{-(N+\kappa)}\Pi(t)dt}{S^+(t)(u+iv)t}.$$

By some easy calculations we will get

$$\Omega(f,z) = \frac{S(z)}{2\pi i\Pi(z)}\left(\int_T \frac{(z^{N+\kappa}+t^{N+\kappa})f(t)\Pi(t)dt}{S^+(t)(u+iv)t^{N+\kappa}(t-z)} - z^{N+\kappa}\int_T \frac{f(t)\Pi(t)dt}{S^+(t)(u+iv)t^{N+\kappa+1}}\right). \quad (2.1.18)$$

Theorem 2.1.1

The following assertions hold.

a) If $N+\kappa \geq -1$, *then* $\Omega(f,z)$ *is a solution of the Problem **H**, where* $\Omega(z)$ *is defined as in (2.1.18).*

b) If $N+\kappa < -1$, *then* $\Omega(f,z)$ *is a solution if and only if* f *satisfies the following conditions:*

$$\int_T \frac{f(t)}{(u(t)+iv(t))S^+(t)}\Pi(t)\,t^k dt = 0, \quad k = 0,1,\dots,-(N+\kappa)-2. \qquad (2.1.19)$$

Proof. By taking into consideration $\Omega(f,\infty) = C$, the proof of this theorem directly follows from the theorem 1.3.1 in Chapter 1 and from the definition of function $\Omega(f,z)$. □

Suppose $N+\kappa = n > 0$. Then, let

$$\Psi(z) = \frac{S(z)P(z)}{\Pi(z)},$$

where $P(z)$ is a polynomial of degree n.

Let evaluate the following function

$$\Psi_*(z) = -\overline{\Psi\left(\frac{1}{\bar{z}}\right)}, \qquad z \in D^-.$$

By taking into account (2.1.14) and (2.1.16) we have:

$$\Psi_*(z) = \frac{(-1)^{N+1}z^{N+\kappa}S(z)}{\Pi(z)} \cdot \prod_{k=1}^{m} t_k{}^{n_k}\,\overline{P\left(\frac{1}{\bar{z}}\right)}.$$

Let

$$P(z) = c_0 z^n + c_1 z^{n-1} + \dots + c_n,$$

then

$$\overline{P\left(\frac{1}{\bar{z}}\right)} = \bar{c}_0 z^{-n} + \bar{c}_1 z^{-n+1} + \dots + \bar{c}_n.$$

Thus,

$$\Psi_*(z) = \frac{(-1)^{N+1}S(z)}{\Pi(z)} \cdot \prod_{k=1}^{m} {t_k}^{n_k} (\overline{c_0} + \overline{c_1}z + \cdots + \overline{c_n}z^{N+\kappa}).$$

Suppose $\Psi_*(z) = \Psi(z),\ |z| \neq 1$. Then, we have

$$\frac{(-1)^{N+1}S(z)}{\Pi(z)} \cdot \prod_{k=1}^{m} {t_k}^{n_k} (\overline{c_0} + \overline{c_1}z + \cdots + \overline{c_n}z^{n}) = \frac{S(z)}{\Pi(z)}(c_0z^n + c_1z^{n-1} + \cdots + c_n).$$

So we get

$$(-1)^{N+1} \prod_{k=1}^{m} {t_k}^{n_k} (\overline{c_0} + \overline{c_1}z + \cdots + \overline{c_n}z^{n}) = (c_0z^n + c_1z^{n-1} + \cdots + c_n)$$

and

$$(-1)^{N+1}\overline{c_l} \prod_{k=1}^{m} {t_k}^{n_k} = c_{N+\kappa-l}, \qquad l = 0,1,\dots,N+\kappa. \tag{2.1.20}$$

For the homogeneous problem we will discuss two cases separately as in Riemann boundary value problem in Chapter 1.

1. Let $\alpha_k > -1, k = 1,2,\dots,m$.

Theorem 2.1.2

The following assertions hold.

*a) If $N + \kappa > -1$, then the general solution of the homogeneous Problem **H** can be represented in the form:*

$$\Phi_0(z) = \frac{S(z)}{\Pi(z)}(c_0z^{N+\kappa} + c_1z^{N+\kappa-1} + \cdots + c_{N+\kappa}), \tag{2.1.21}$$

where numbers $\{c_l\}_{l=0}^{N+\kappa}$ satisfy condition (2.1.20).

b) If $N + \kappa \leq -1$, then the homogeneous problem has only trivial solution.

Proof. By taking into consideration $\Phi_0(\infty) = C$, the proof of this theorem directly follows from evaluations above and the theorem 1.4.1 in Chapter 1. □

Theorem 2.1.3

The following assertions hold.

*a) If $N+\kappa \geq -1$, then the general solution of the Problem **H** can be represented in the form:*

$$\Phi(z) = \Omega(f,z) + \Phi_0(z), \tag{2.1.22}$$

*where $\Omega(f,z)$ is as in (2.1.18), and $\Phi_0(z)$ is the general solution of the homogeneous Problem **H**.*

*b) If $N+\kappa < -1$, then the Problem **H** is solvable if and only if f satisfies the conditions (2.1.19). And the solution can be represented in the form (2.1.18).*

2. Let $\alpha_k, k = 1,2,\dots,m$ are arbitrary real numbers

Suppose function $a(t)$ is such that function

$$D(t) = \frac{\overline{a(t)}}{a(t)}$$

satisfies (1.2.7) convergence condition. Thus, $D(t) \in R^{\alpha}$.

Theorem 2.1.4

Let $D(t) \in R^{\alpha}$. Then the following assertions hold.

a) If $N+\kappa > -1$, then the general solution of the homogeneous Problem H can be represented in the form (2.1.21), where numbers $\{c_l\}_{l=0}^{N+\kappa}$ satisfy conditions (2.1.20).

b) If $N+\kappa \leq -1$, then the homogeneous problem has only trivial solution.

Proof. From theorem 1.5.1 in chapter 1 we can see that in case of $\kappa \geq 0$, function $A_0 S(z)$, is a solution of the homogeneous problem (2.1.9), where A_0 is any complex number. As we concluded above not every solution of the problem (2.1.9) is a solution of the Riemann-Hilbert problem. For being a solution it should necessarily satisfy (2.1.10). As $S^*(z) = z^{\kappa} S(z)$, so $A_0 S^*(z) \neq A_0 S(z)$, hence it is not a solution of the Problem **H**. □

Theorem 2.1.5

Let $D(t) \in R^{\alpha}$. Then the following assertions hold.

*a) If $N+\kappa \geq -1$, then the general solution of the Problem **H** can be represented in the form (2.1.22), where $\Omega(f,z)$ is as in (2.1.18), and $\Phi_0(z)$ is the general solution of the homogeneous Problem **H**.*

*b) If $N+\kappa < -1$, then the Problem **H** is solvable if and only if f satisfies the conditions (2.1.19). And the solution can be represented in the form (2.1.18).*

2.1.4 Dirichlet Problem

Now let consider particular case, when $a(t) \equiv 1$. Then we get Dirichlet problem in the following setting:

Problem D

Let f be a real-valued, measurable on T function from the class $L^1(\rho)$. Determine an analytic in D^+ function $\Phi(z)$ to satisfy the boundary condition:

$$\lim_{r\to 1-0} \|\operatorname{Re}\Phi(rt) - f(t)\|_{L^1(\rho_r)} = 0. \tag{2.1.23}$$

In the same way (2.1.23) can be reduced to

$$\lim_{r\to 1-0} \|\Phi^+(rt) - \Phi^-(r^{-1}t) - 2f(t)\|_{L^1(\rho_r)} = 0. \tag{2.1.24}$$

Let

$$K(f,z) = \frac{1}{\pi i \Pi(z)} \int_T \frac{f(t)\Pi(t)}{(t-z)} dt\,, \quad z \in D^+ \cup D^-,$$

then

$$K_*(f,z) = \frac{z^N}{\Pi(z)} \left(\frac{1}{\pi i} \int_T \frac{f(t)\Pi(t)dt}{t^N(t-z)} - \frac{1}{\pi i} \int_T \frac{f(t)\Pi(t)dt}{t^{N+1}} \right).$$

Hence,

$$\Omega(f,z) = \frac{1}{2}\left(K(f,z) + K_*(f,z)\right)$$

and

$$\Omega(f,z) = \frac{1}{\pi i \Pi(z)} \left(\int_T \frac{f(t)(t^N + z^N)\Pi(t)dt}{t^N(t-z)} - \int_T \frac{f(t)\Pi(t)dt}{t^{N+1}} \right). \tag{2.1.25}$$

Theorem 2.1.6

The following assertions hold.

*a) If $N > -1$, then the general solution of the homogeneous Problem **D** can be represented in the form:*

$$\Phi_0(z) = \frac{1}{\Pi(z)}(c_0 z^N + c_1 z^{N-1} + \cdots + c_N), \tag{2.1.26}$$

where numbers $\{c_l\}_{l=0}^N$ satisfy conditions:

$$(-1)^{N+1}\overline{c_l}\prod_{k=1}^{m} t_k{}^{n_k} = c_{N-l}, \qquad l = 0,1,\dots,N. \tag{2.1.27}$$

b) If $N \leq -1$, then the homogeneous problem has only trivial solution.

Proof. The proof of this theorem directly follows from theorems 2.1.2 and 2.1.4. □

Theorem 2.1.7

The following assertions hold.

*a) If $N \geq -1$, then the general solution of the Problem **D** can be represented in the form:*

$$\Phi(z) = \Omega(f, z) + \Phi_0(z),$$

*where $\Omega(f, z)$ is as in (2.1.25), and $\Phi_0(z)$ is the general solution of the homogeneous Problem **D**.*

*b) If $N < -1$, then the Problem **D** is solvable if and only if f satisfies the following conditions:*

$$\int_T f(t)\Pi(t)\, t^k dt = 0, \quad k = 0,1,\dots,-N-2.$$

And the solution can be represented in the form (2.1.25).

Proof. This theorem directly follows from theorems 2.1.3 and 2.1.5. □

2.2 Discontinuous Riemann Problem

2.2.1 Statement of the Problem

Let $T = \{t, |t| = 1\}, \rho(t) = |t - t_0|^{\alpha}$, where $t_0 \in T$ and $\alpha > -1$ is arbitrary real number. By $H(T)$ we denote Holder class functions in T.

Definition

We say that $a(t) \in H_0(T; t_1, t_2, \dots, t_m)$, *if* a *belongs to Holder class in any interval from* T *not including* $t_k, k = 1,2, \dots, m$ *points and has jump discontinuity at those points.*

We will discuss the problem in the case

$$t_0 \neq t_k, \qquad k = 1,2, \dots, m$$

Let $a(t) \in H_0(T; t_1, t_2, \dots, t_m)$ and $a(t) \neq 0, t \in T$. By introducing $\varphi(t) = \ln a(t)$ function it is easy to get the following (see [15]):

$$\alpha_k + i\beta_k = \frac{1}{2\pi i}\left(\varphi(t_k - 0) - \varphi(t_k + 0)\right), \qquad k = 1,2, \dots, m \qquad (2.2.1)$$

Obviously, the following function

$$S_1(z) = exp\left\{\frac{1}{2\pi i}\int_T \frac{\varphi(t)dt}{t - z}\right\}, z \in D$$

can be represented in some small interval of the point $t_k, k = 1,2, \dots, m$ as follows:

$S_1(z) = (z - t_k)^{\alpha_k + i\beta_k}\Delta_k(z),$

where $\Delta_k(z)$ is analytic function in D and $\lim_{z\to t_k}\Delta_k(z) = A \neq 0$. For function $S_1(z)$ we make the following notation:

$$S_1(z) = \begin{cases} S_1^+(z), & z \in D^+, \\ S_1^-(z), & z \in D^-. \end{cases}$$

Let consider the following boundary problem. Find analytic $\Phi(z)$ function in D such that

$$\Phi^+(t) - \Phi^-(t) = 0, \qquad (2.2.2)$$

where

$$\Phi(z) = \begin{cases} \Phi^+(z), & z \in D^+, \\ \Phi^-(z), & z \in D^-. \end{cases}$$

There exist $\mu^+(z)$ and $\mu^-(z)(\mu(\infty)=\infty)$ conformal mapping functions from D^+ and D^- into some Δ_+ and Δ_- domains respectively such that they satisfy Lipschitz condition in $D^+\cup T$ and $D^-\cup T$ (see [34]). Let consider

$$\Phi_1^+(z)=\prod_{k=1}^{n}\left(\mu^+(z)-\mu^+(t_k)\right)^{\lambda_k}, \qquad z\in D^+,$$

$$\Phi_1^-(z)=\prod_{k=1}^{n}\left(\mu^+(z)-\mu^-(t_k)\right)^{\lambda_k}, \qquad z\in D^-,$$

where λ_k are integers such that $-1<\lambda_k+\alpha_k\leq 0,\ k=1,2,\dots,m.$

Making following notation

$$\Phi_1(z)=\begin{cases}\Phi_1^+(z), & z\in D^+,\\ \Phi_1^-(z), & z\in D^-,\end{cases}$$

we can easily conclude that function $\Phi_1(z)$ satisfies (2.2.2). Even further, $\Phi_1(z)=(z-t_k)^{\lambda_k}\Omega_k(z)$, where $\Omega_k(z)$ is analytic function in D and $\lim_{z\to t_k}\Omega_k(z)=B\neq 0.$

Let $S(z)=S_1(z)\Phi_1(z),\ z\in D^+\cup D^-$. It admits the following representation:

$$S(z)=\exp\left[\frac{1}{2\pi i}\int_T\frac{\ln a(t)}{t-z}dt\right]\prod_{k=1}^{m}(z-t_k)^{\lambda_k}. \qquad (2.2.3)$$

Lemma 2.2.1

For function $S(z)$ we have (see [34]):

$$S^+(t)-a(t)S^-(t)=0;$$

$$S^+(t),S^-(t)\in L^1(T)\text{ and }\left(S^+(t)\right)^{-1},\left(S^-(t)\right)^{-1}\in L^\infty(T);$$

in some small interval T_k of the point t_k the following is true:

$$S(z)=\frac{\lambda_k(z)}{(z-t_k)^{\delta_k-i\beta_k}}, \qquad z\in D\cap T_k,$$

where $\delta_k=-(\alpha_k+\lambda_k)$, thus $0\leq\delta_k<1$ and function $\lambda_k(z)=\Delta_k(z)\Omega_k(z)$. Obviously, $\lambda_k(z)\to AB\neq 0$ when $z\to t_k$.

By n we denote the following:

$$n=\begin{cases}[\alpha]+1, & if\ \alpha\ \text{is not integer},\\ \alpha, & if\ \alpha\ \text{is integer}.\end{cases} \tag{2.2.4}$$

We consider Riemann boundary value problem with the following setting:

Problem R

Let $a(t)\in H_0(T;t_1,t_2,\dots,t_m)$ *and* $a(t)\neq 0, t\in T, \rho(t)=|t-t_0|^\alpha$ *is the weight function, where* $\alpha>-1$ *is arbitrary real number and* $t_0\in T$. *Determine in* D *analytic function* $\Phi(z), \Phi(\infty)=0$, *such that the following condition holds:*

$$\lim_{r\to 1-0}\|\Phi^+(rt)-a(t)\Phi^-(r^{-1}t)-f(t)\|_{L^1(\rho)}=0. \tag{2.2.5}$$

Parallel, we will consider Problem **R** by extracting condition $\Phi(\infty)=0$ from function Φ and letting him to have some finite degree at infinity.

2.2.2 Solution of the Problem R

Theorem 2.2.1

Let $f\in L^1(\rho)$. *If* Φ *is a solution of the Problem* ***R*** *and has at infinity some finite degree, then the following representation is true:*

$$\Phi^+(z)=\frac{S^+(z)}{2\pi i(z-t_0)^n}\int_T\frac{g(t)dt}{t-z}, \qquad z\in D^+,$$

$$\Phi^-(z)=\frac{S^-(z)}{2\pi i(z-t_0)^n}\int_T\frac{g(t)dt}{t-z}+S^-(z)P(z), \qquad z\in D^-,$$

where P *is some polynomial and*

$$g(t)=\left(P(t)+\frac{f(t)}{S^+(t)}\right)(t-t_0)^n$$

Proof. Let Φ be a solution of the Problem **R** and has at infinity some finite degree. Also, $f_r(t)\in H(T)$ is a sequence such that $\lim_{r\to 1-0}\|f_r(t)-f(t)\|_{L^1(\rho)}=0$. From Lemma 2.2.1 we can easily obtain the following result:

$$\lim_{r\to 1-0}\left\|\left(\frac{\Phi^+(rt)}{S^+(t)}-\frac{\Phi^-(r^{-1}t)}{S^-(t)}-\frac{f_r(t)}{S^+(t)}\right)(t-t_0)^n\right\|_{L^1}=0.$$

Let

$$\Psi_r(t) = \left(\frac{\Phi^+(rt)}{S^+(t)} - \frac{\Phi^-(r^{-1}t)}{S^-(t)} - \frac{f_r(t)}{S^+(t)}\right)(t-t_0)^n,$$

$$F_r^{\ +}(z) = \frac{\Phi^+(rz)(z-t_0)^n}{S^+(z)}, \qquad z \in D^+, \tag{2.2.6}$$

$$F_r^{\ -}(z) = \frac{\Phi^-(r^{-1}z)(z-t_0)^n}{S^-(z)}, \qquad z \in D^-.$$

Thus, we get the following boundary problem

$$F_r^{\ +}(t) - F_r^{\ -}(t) = \Psi_r(t) + \frac{f_r(t)(t-t_0)^n}{S^+(t)},$$

where $t \in T,\ 0 < r < 1,\ \Psi_r(t) \in H_0(T)$ and $\lim_{r\to 1-0}\|\Psi_r(t)\|_{L^1} = 0$. Taking into consideration that functions $F_r^{\ +}(z)$ and $F_r^{\ -}(z)$ are bounded respectively in D^+ and D^- we conclude:

$$F_r^{\ +}(z) = \frac{1}{2\pi i}\int_T \frac{g_r(t)\,dt}{t-z}, \qquad z \in D^+,$$

$$F_r^{\ -}(z) = \frac{1}{2\pi i}\int_T \frac{g_r(t)\,dt}{t-z} + P_r(z), \qquad z \in D^-, \tag{2.2.7}$$

where $P_r(z)$ is a general part of Laurent expansion of function $F_r^{\ -}$ at infinity and

$$g_r(t) = \Psi_r(t) + \left(P_r(t) + \frac{f_r(t)}{S^+(t)}\right)(t-t_0)^n.$$

We have

$$\lim_{r\to 1-0} F_r^{\ +}(z) = \frac{\Phi^+(z)(z-t_0)^n}{S^+(z)},$$

$$\lim_{r\to 1-0} F_r^{\ -}(z) = \frac{\Phi^-(z)(z-t_0)^n}{S^-(z)}.$$

Besides, P_r uniformly converges to polynomial P, if $r \to 1-0$. So, we get

$$\lim_{r\to 1-0}\left\|\left(\Psi_r(t) + P_r(t) + \frac{f_r(t)}{S^+(t)} - P(t) - \frac{f(t)}{S^+(t)}\right)(t-t_0)^n\right\|_{L^1} = 0.$$

Hence,

$$\lim_{r\to 1-0}\|g_r(t) - g(t)\|_{L^1} = 0.$$

Taking into account (2.2.6), theorem 2.2.1 is proved.□

Theorem 2.2.2

*Let $f \in L^1(\rho)$. Then, the general solution of the Problem **R**, which has some finite degree at infinity, is given by the following formula:*

$$\begin{aligned} \Phi^+(z) &= \frac{S^+(z)}{2\pi i (z-t_0)^n} \int_T \frac{g(t)dt}{t-z}, \qquad z \in D^+, \\ \Phi^-(z) &= \frac{S^-(z)}{2\pi i (z-t_0)^n} \int_T \frac{g(t)dt}{t-z} + S^-(z)P(z), \qquad z \in D^-, \end{aligned} \tag{2.2.8}$$

where P is some polynomial and

$$g(t) = \left(P(t) + \frac{f(t)}{S^+(t)}\right)(t-t_0)^n. \tag{2.2.9}$$

Remark

In theorem 2.2.1 we showed that any solution of the Problem **R** which has some finite degree at infinity has the representation (2.2.8). Now we will prove that any function, which has the representation (2.2.8) is a solution of the Problem **R**. By doing so, we will eventually prove theorem 2.2.2 which states the general solution of the Problem **R** with the assumption of it having finite degree at infinity.

Proof of Theorem 2.2.2

Let $f_n(t) \in H(T)$ and $\lim_{n\to\infty} \|f_n(t) - f(t)\|_{L^1(\rho)} = 0$. Denote by g_n the following:

$$g_n(t) = \left(P(t) + \frac{f_n(t)}{S^+(t)}\right)(t-t_0)^n,$$

where P is some polynomial.

Also, suppose

$$\begin{aligned} \Phi_n^+(z) &= \frac{S^+(z)}{2\pi i (z-t_0)^n} \int_T \frac{g_n(t)dt}{t-z}, \qquad z \in D^+, \\ \Phi_n^-(z) &= \frac{S^-(z)}{2\pi i (z-t_0)^n} \int_T \frac{g_n(t)dt}{t-z} + S^-(z)P(z), \qquad z \in D^-. \end{aligned} \tag{2.2.10}$$

It is clear that the function $\Phi_n(z)$ is the solution of the following Riemann boundary value problem

$$\Phi_n^+(t) - a(t)\Phi_n^-(t) = f_n(t). \tag{2.2.11}$$

Even further, they have degree of $\delta,\ \delta \in (0,1)$ in some small interval of points t_k, so there exists some real number $p, (\ p > 1)$ such that for every $r, r \in (0,1)$ the following is true:

$$\int_T |\Phi_n^+(rt)|^p |t - t_0|^n |dt| < C,$$

$$\int_T |\Phi_n^-(r^{-1}t)|^p |t - t_0|^n |dt| < C.$$

Taking into consideration last result, for every $n, (n \geq 1)$ we have

$$\lim_{r\to 1-0} \|\Phi_n^+(rt) - a(t)\Phi_n^-(r^{-1}t) - f_n(t)\|_{L^1(\rho)} = \lim_{r\to 1-0} \|I_n^1(r)\|_{L^1(\rho)} = 0.$$

By denoting $\varepsilon_n(t) = g_n(t) - g(t)$, we will have

$$\int_T |\Phi_n^+(rt) - a(t)\Phi_n^-(r^{-1}t) - f(t)||t - t_0|^n |dt| \leq$$

$$\leq C\left(\int_T \frac{(1-r)|S^+(rt)|}{|t_0 - rt|^n} \int_T \frac{|\varepsilon_n(\tau)|}{|\tau - rt|^2} |d\tau|\, |\rho(t)||dt| + \right.$$

$$+ C\left(\int_T \frac{(1-r)|S^+(rt)|}{|t_0 - rt|^{n+1}} \int_T \frac{|\varepsilon_n(\tau)|}{|\tau - r^{-1}t|} |d\tau|\, |\rho(t)||dt|\right) +$$

$$+C\left(\int_T \frac{|S^+(rt) - a(t)S^-(r^{-1}t)|}{|t_0 - r^{-1}t|^n} \int_T \frac{|\varepsilon_n(\tau)|}{|\tau - r^{-1}t|} |d\tau|\, |\rho(t)||dt|\right) +$$

$$+\|I_n^1(r)\|_{L^1(\rho)} + \|f_n - f\|_{L^1(\rho)}.$$

All summands at the right side of the inequality tend to zero as $r \to 1-0$, so we get the proof of the theorem 2.2.2. □

Let introduce the following functions:

$$\Phi_k^+(z) = \frac{1}{2\pi i} \int_T \frac{t^k (t - t_0)^n}{t - z} dt, \qquad z \in D^+,$$

$$\Phi_k^-(z) = \frac{1}{2\pi i} \int_T \frac{t^k (t - t_0)^n}{t - z} dt + z^{n+k}, \qquad z \in D^-.$$

Then,

$$\tilde{g}(t) = \frac{f(t)(t-t_0)^n}{S^+(t)}.$$

Hence, for any polynomial $P(z) = c_0 + c_1 z + \cdots + c_m z^m$ (2.2.8) can be represented as follows:

$$\Phi^+(z) = \frac{S^+(z)}{2\pi i (z-t_0)^n} \int_T \frac{\tilde{g}(t)dt}{t-z} + \frac{S^+(z)}{(z-t_0)^n} \sum_{k=0}^{m} c_k \Phi_k^+(z), \qquad z \in D^+,$$

$$\Phi^-(z) = \frac{S^-(z)}{2\pi i (z-t_0)^n} \int_T \frac{\tilde{g}(t)dt}{t-z} + \frac{S^-(z)}{(z-t_0)^n} \sum_{k=0}^{m} c_k \Phi_k^-(z), \qquad z \in D^-. \tag{2.2.12}$$

We have $\kappa = -\sum_{k=1}^{m} \lambda_k$ (see [34]). Obviously, function $S(z)$ has at infinity $-\kappa$ degree.

Theorem 2.2.3

Let $f \in L^1(\rho)$. Then, the general solution of the Problem ***R*** *has the following representation:*

a) if $n + \kappa \geq 0$, then

$$\Phi^+(z) = \frac{S^+(z)}{2\pi i (z-t_0)^n} \int_T \frac{\tilde{g}(t)dt}{t-z} + \frac{S^+(z)}{(z-t_0)^n} \sum_{k=0}^{\kappa-1} c_k \Phi_k^+(z), \qquad z \in D^+,$$

$$\Phi^-(z) = \frac{S^-(z)}{2\pi i (z-t_0)^n} \int_T \frac{\tilde{g}(t)dt}{t-z} + \frac{S^-(z)}{(z-t_0)^n} \sum_{k=0}^{\kappa-1} c_k \Phi_k^-(z), \qquad z \in D^-, \tag{2.2.13}$$

where $c_0, c_1, \dots, c_{\kappa-1}$ are arbitrary complex numbers when $\kappa \geq 1$ and $c_0 = c_1 = \cdots = c_{\kappa-1} = 0$ when $\kappa = 0$.

b) if $n + \kappa < 0$, then the problem has a solution if and only if:

$$\int_T \frac{\tilde{g}(t)}{t-z} t^k dt = 0, \quad k = 0,1,\dots,-(n+\kappa)-1 \tag{2.2.14}$$

Besides, the solution has the representation (2.2.13), where $c_0 = c_1 = \cdots = c_{\kappa-1} = 0$.

CHAPTER 3. DIRICHLET PROBLEM FOR BIHARMONIC FUNCTIONS IN THE WEIGHTED SPACES

Let $G \in \mathbb{C}$ is any simply connected domain bounded by the curve Γ. Let consider the following equation in this domain:

$$\Delta^n u = 0, \tag{3.1}$$

where

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Solutions of the equation (3.1) which are continuous with their $2n-1$ power derivatives are called n –harmonic or polyharmonic functions. For this equation Dirichlet boundary value problem has been studied with different boundary conditions by Tovmasyan N.E. [77], [78], Bitsadze A.V. [12], [13], Soldatov A.P. [73], Hayrapetyan H.M. [29], [31], [32], [33], Babayan A.H. [79]. If boundary functions belong to the classes $C^{(n,\alpha)}(\Gamma)$, then Dirichlet problem in the classical statement has the following setting. Determine those solutions of the equation (3.1), which satisfy boundary conditions:

$$\begin{cases} u(t) = f_0(t), \\ \dfrac{\partial u}{\partial n} = f_1(t), \\ \quad \dots \\ \dfrac{\partial^{n-1} u}{\partial n^{n-1}} = f_{n-1}(t), \end{cases} \tag{3.2}$$

where $t \in \Gamma$, $f_0(t), f_1(t), \dots, f_{n-1}(t)$ are real-valued functions belonging to $C^{(n,\alpha)}(\Gamma)$ class of functions. Also, by $\frac{\partial}{\partial n}$ we mean normal derivative. Particularly, if G is a unit circle, then $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$. If we denote unit circle by D^+, then this problem in spaces L^p, $1 \le p < \infty$, can be formulated with the boundary conditions:

$$\begin{cases} \lim_{r\to 1-0} \|u(rt) - f_0(t)\|_{L^p} = 0, \\ \lim_{r\to 1-0} \left\| \frac{\partial u(rt)}{\partial r} - f_1(t) \right\|_{L^p} = 0, \\ \dots \\ \lim_{r\to 1-0} \left\| \frac{\partial^{n-1} u(rt)}{\partial r^{n-1}} - f_{n-1}(t) \right\|_{L^p} = 0, \end{cases} \tag{3.3}$$

It is shown that in case of $1 < p < \infty$ this statement is equivalent to the classical statement with boundary conditions (3.2), which is investigated by Khvedelidze B.V. [50], Soldatov A.P. [68] and others [48], [49]. Boundary value problem (3.1), (3.2) also is studied in the C −space of continuous functions [36], [37], in the L^∞ and in the W −space of measures [39]. In the space C boundary conditions are understood in the sense of uniformly convergence:

$$\begin{cases} \lim_{r\to 1-0} \|u(rt) - f_0(t)\|_{C} = 0, \\ \lim_{r\to 1-0} \left\| \frac{\partial u(rt)}{\partial r} - f_1(t) \right\|_{C} = 0, \\ \dots \\ \lim_{r\to 1-0} \left\| \frac{\partial^{n-1} u(rt)}{\partial r^{n-1}} - f_{n-1}(t) \right\|_{C} = 0, \end{cases} \tag{3.4}$$

where $\|\cdot\|_C$ is the norm of space C, and $f_0(t), f_1(t), \dots, f_{n-1}(t)$ are continuous functions. In the space L^∞ the same statement does not guarantee normal solvability of the Dirichlet problem, so in the works [63], [64] it is suggested new boundary conditions:

$$\frac{\partial^k u(rt)}{\partial r^k} \to f_k(t), \qquad k = 0,1,\dots,n-1,$$

where $f_k \in L^\infty, k = 0,1,\dots,n-1$ and convergence is understood in the sense of weak topology of L^∞. Thus,

$$\lim_{r\to 1-0} \int_T \frac{\partial^k u(rt)}{\partial r^k} g(t)dt = \int_T f_k(t) g(t) dt, \qquad k = 0,1,\dots,n-1,$$

where $g \in L^1(T)$ is any function. In the same way (3.1), (3.2) problem is studied in the space of measures W with boundary conditions:

$$\lim_{r\to 1-0} \int_T \frac{\partial^k u(rt)}{\partial r^k} g(t)dt = \int_T g(t) d\mu(t), \qquad k = 0,1,\dots,n-1$$

for any function $g \in C(T)$ [21].

Dirichlet problem in the weighted spaces $L^p(\rho), 1 \le p < \infty$ firstly was studied by Kazarian K. [45], [46], [47]. In his works he attained direct connections between Dirichlet problem and Fourier series. By Hayrapetyan H.M. Dirchlet problem is investigated for RO −varying weight functions [38].

3.1. Statement of the Problem

Let $\rho_0(t)$ is a weight function, $\rho_0(t) = |1-t|^\alpha; \alpha$ is arbitrary real number, $\rho_1(t) = \rho_0(t)|1-t|, \|\cdot\|_{L^1(\rho_k)}$ is the norm of space $L^1(\rho_k), k = 0,1;$ $f_k(t)$ are real valued functions in T such that $f_k(t) \in L^1(\rho_k),\ \ k = 0, 1.$

We investigate Dirichlet problem in the following setting:

Problem D

Determine function $u(z), z \in D^+$ *to satisfy the equation*

$$\Delta^2 u = 0 \tag{3.1.1}$$

and boundary condition

$$\lim_{r\to 1-0}\left\|\frac{\partial^k u(rt)}{\partial r^k} - f_k(t)\right\|_{L^1(\rho_k)} = 0, \qquad k = 0,1. \tag{3.1.2}$$

With this statement we prove that Problem (3.1.1), (3.1.2) is normally solvable, besides if $\alpha \le -1$, the problem is investigated with the following boundary condition:

$$\lim_{r\to 1-0}\left\|\frac{\partial^k u(rt)}{\partial r^k} - f_k(t)\right\|_{L^1(\rho_{kr})} = 0, \qquad k = 0, 1, \tag{3.1.3}$$

where $\rho_{kr}(t) = |1-rt|^{n+k}|1-t|^{\alpha-n}$.

It is known that (3.1.1) can be written in explicit form:

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2\,\partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0.$$

3.2. Solution of the homogeneous Problem D

Theorem 3.1

Let $\alpha \le -1$. *Then the homogeneous problem (3.1.1), (3.1.3) has only trivial solution*

$$u(z) \equiv 0.$$

Proof. It is well known that the general solution of the equation (3.1.1) can be represented as follows (see [30]):

$$u(z) = \mathrm{Re}\big(\Phi_0(z) + (1 - |z|^2)\Phi_1(z)\big), \qquad z \in D^+, \tag{3.1.4}$$

where $\Phi_k(z),\ k = 0,1$ are some analytic functions satisfying in D^+condition: $\mathrm{Im}\Phi_k(0) = 0,\ k = 0,1$. Besides, functions $\Phi_k(z)$ are uniquely determined with the function $u(z)$. Substituting $u(z)$ with its representation (3.1.4) in the first condition of (3.1.3), for $f_0 \equiv 0$ we get

$$\lim_{r\to 1-0} \|\mathrm{Re}\big(\Phi_0(rt) + (1 - r^2)\Phi_1(rt)\big)\|_{L^1(\rho_{0r})} = 0.$$

By denoting

$$\mathrm{Re}\big(\Phi_0(rt) + (1 - r^2)\Phi_1(rt)\big) = f_{0r}(t),$$

we get $\|f_{0r}(t)\|_{L^1(\rho_{0r})} \to 0$ and

$$\Phi_0(rz) + (1 - r^2)\Phi_1(rz) = \frac{1}{2\pi i}\int_T f_{0r}(t)\frac{t+z}{t-z}\frac{dt}{t} + iC_{0r},$$

where C_{0r} is some real number. Passing to the limit as $r \to 1 - 0$, for every fixed $z; |z| < 1$ we get $\Phi_0(z) = iC_0$, where C_0 is a real number. From the second condition (3.1.3) we obtain

$$\lim_{r\to 1-0}\left\|\mathrm{Re}\left(-2r\Phi_1(rt) + (1 - r^2)t\frac{\partial \Phi_1(rt)}{\partial r}\right)\right\|_{L^1(\rho_{1r})} = 0,$$

where $\rho_1(t) = \rho_0(t)|1 - t|$. Taking into consideration the results attained in [22], we can easily state

$$-2r\Phi_1(rz) + (1 - r^2)z\,\frac{\partial \Phi_1(rz)}{\partial r} = \frac{1}{2\pi i(1 - z)}\int_T f_{1r}(t)(1 - t)\frac{t+z}{t-z}\frac{dt}{t} + iC_{1r},$$

where

$$f_{1r}(t) = \mathrm{Re}\left(-2r\Phi_1(rt) + (1 - r^2)t\,\frac{\partial \Phi_1(rt)}{\partial r}\right),$$

$\|f_{1r}\|_{L^1(\rho_1)} \to 0$, C_{1r} is some real number. Passing to the limit as $r \to 1 - 0$, we get $\Phi_1(z) = iC_1$, where C_1 is some real number. Thus from (3.1.4) $u(z) \equiv 0$. □

Definition: *We say that,* $\{C_k\}_0^n$ *numbers belong to class* $S_0(n)$*,if*

$$C_k = (-1)^k \overline{C_{n-k}}, \qquad k = 0,1,\dots,n. \tag{3.1.5}$$

Theorem 3.2

Let $\alpha > -1$*. Then the general solution of the homogeneous problem (3.1.1), (3.1.2) can be represented as follows:*

$$u(z) = \mathrm{Re}\big(\Phi_0(z) + (1-|z|^2)\Phi_1(z)\big), \qquad z \in D^+,$$

where

$$\begin{cases} \Phi_0(z) = \sum_{k=0}^{n} \frac{A_k}{(1-z)^k}, \\ \Phi_1(z) = \frac{z}{2}\frac{\partial \Phi_0(z)}{\partial z} - \frac{1}{2}\sum_{k=0}^{n+1} \frac{B_k}{(1-z)^k}. \end{cases} \tag{3.1.6}$$

Besides, $\{A_k\}_0^n$ *numbers belong to the class* $S_0(n)$*, and* $\{B_k\}_0^{n+1}$ *numbers to the class* $S_0(n+1)$*.*

Proof. Taking into account (3.1.4), from the first condition of (3.1.2) we obtain

$$\Phi_0(rz) + (1-r^2)\Phi_1(rz) = \frac{1}{2\pi i(1-z)^n}\int_T f_{0r}(t)(1-t)^n \frac{t+z}{t-z}\frac{dt}{t} + \sum_{k=0}^{n} \frac{A_k}{(1-z)^k}.$$

Passing to the limit as $r \to 1-0$ and taking into account that $f_{0r} \to 0$ in $L^1(\rho_0)$, as $r \to 1-0$ we conclude

$$\Phi_0(rz) = \sum_{k=0}^{n} \frac{A_k}{(1-z)^k}.$$

From the second condition of (3.1.2) we have:

$$z\frac{\partial \Phi_0(rz)}{\partial z} - 2r\Phi_1(rz) + (1-r^2)\frac{\partial \Phi_1(rz)}{\partial z} =$$

$$= \frac{1}{2\pi i(1-z)^{n+1}}\int_T f_{1r}(t)\frac{t+z}{t-z}\frac{dt}{t} + \sum_{k=0}^{n} \frac{B_k}{(1-z)^k}.$$

Passing to the limit as $r \to 1-0$ we get

$$z\frac{\partial \Phi_0(z)}{\partial z} - 2\Phi_1(z) = \sum_{k=0}^{n+1} \frac{B_k}{(1-z)^k}.$$

Thus

$$\Phi_1(z) = z\sum_{k=0}^{n}\frac{A_k}{(1-z)^{k+1}} - \sum_{k=0}^{n+1}\frac{B_k}{(1-z)^k}.$$

Theorem 3.2 is proved. □

3.3. Solution of the inhomogeneous Problem D

Theorem 3.3

Let $\alpha \le -1$*. Then the general solution of the problem (3.1.1), (3.1.3) can be represented as follows:*

$$u(z) = \mathrm{Re}\big(\Phi_0(z) + (1-|z|^2)\Phi_1(z)\big),$$

where Φ_0, Φ_1 *are analytic functions in* D^+*, determined with the following formulas:*

$$\begin{cases} \Phi_0(z) = \dfrac{1}{2\pi i(1-z)^n}\displaystyle\int_T \frac{f_0(t)(1-t)^n}{t-z}dt, \\ \Phi_1(z) = \dfrac{z}{2}\dfrac{\partial\Phi_0(z)}{\partial z} + \dfrac{1}{2\pi i(1-z)^{n+1}}\displaystyle\int_T \frac{f_1(t)(1-t)^{n+1}}{t-z}dt. \end{cases} \qquad (3.1.7)$$

Besides, there are necessary and sufficient conditions for solvability of the problem, which have the following form:

$$\begin{cases} \displaystyle\int_T t^k f_0(t)\,dt = 0, \\ \displaystyle\int_T t^k f_1(t)\,dt = 0, \end{cases} \quad k = 0,1,\dots,n \qquad (3.1.8)$$

Proof. As the general solution of the problem (3.1.1) can be represented as follows:

$$u(z) = \mathrm{Re}\big(\Phi_0(z) + (1-|z|^2)\Phi_1(z)\big),$$

where Φ_0, Φ_1 are analytic functions in D^+, from the first condition (3.1.3) we can easily obtain

$$\lim_{r\to 1-0}\big\|\mathrm{Re}\big(\Phi_0(rt) + (1-r^2)\Phi_1(rt)\big) - f_0(t)\big\|_{L^1(\rho_{0r})} = 0. \qquad (3.1.9)$$

Hence (see [43])

$$\Phi_0(z) = \frac{1}{2\pi i(1-z)^n}\int_T \frac{f_0(t)(1-t)^n}{t-z}dt. \tag{3.1.10}$$

As

$$\lim_{r\to 1-0}\left\|\mathrm{Re}\left(t\frac{\partial\Phi_0(rt)}{\partial r} - 2r\Phi_1(rt) + (1-r^2)t\frac{\partial\Phi_1(rt)}{\partial r}\right) - f_1(t)\right\|_{L^1(\rho_1)} = 0,$$

then

$$z\frac{\partial\Phi_0(z)}{\partial z} - 2r\Phi_1(z) = \frac{1}{2\pi i(1-z)^{n+1}}\int_T \frac{f_1(t)(1-t)^{n+1}}{t-z}dt$$

and

$$\Phi_1(z) = \frac{z}{2}\frac{\partial\Phi_0(z)}{\partial z} - \frac{1}{2\pi i(1-z)^{n+1}}\int_T \frac{f_1(t)(1-t)^{n+1}}{t-z}dt. \tag{3.1.11}$$

Therefore, $u(z)$ has the form

$$u(z) = \mathrm{Re}\big(\Phi_0(z) + (1-|z|^2)\Phi_1(z)\big),$$

where $\Phi_0,\ \Phi_1$ are determined as in (3.1.10), (3.1.11).

Now we prove that the function $u(z)$ determined by (3.1.10) and (3.1.11) satisfies (3.1.1), (3.1.2) conditions for every $f_0 \in L^1(\rho_0),\ f_1 \in L^1(\rho_1)$ functions. Indeed, as

$$u(rt) = \mathrm{Re}\big(\Phi_0(rt) + (1-r^2)\Phi_1(rt)\big)$$

and (see [22])

$$\lim_{r\to 1-0}\left\|\mathrm{Re}\big(\Phi_0(rt)\big) - f_0(t)\right\|_{L^1(\rho_0)} = 0,$$

then the proof of the theorem follows from the following assertions (see [22]):

$$\lim_{r\to 1-0}(1-r^2)\left\|\mathrm{Re}\big(\Phi_1(rt)\big)\right\|_{L^1(\rho_1)} = 0,$$

$$\lim_{r\to 1-0}(1-r^2)\left\|\mathrm{Re}\left(\frac{rt}{2}\frac{\partial\Phi_0(rt)}{\partial t}\right)\right\|_{L^1(\rho_0)} = 0,$$

$$\lim_{r\to 1-0}(1-r^2)\left\|\mathrm{Re}\left(\frac{1}{2\pi i(1-rt)^{n+1}}\int_T \frac{f_1(\tau)(1-\tau)^{n+1}}{\tau-rt}d\tau\right)\right\|_{L^1(\rho_1)} = 0.$$

With the same manner next theorem can be proved. □

Theorem 3.4

Let $\alpha > -1$. Then the general solution of the problem (3.1.1), (3.1.2) can be represented in the form:

$$u(z) = \mathrm{Re}\left(\Phi_0(z) + (1 - |z|^2)\Phi_1(z)\right),$$

where Φ_0, Φ_1 are analytic functions in D^+, and are given by the following formulas:

$$\begin{cases} \Phi_0(z) = \dfrac{1}{2\pi i(1-z)^n} \displaystyle\int_T \dfrac{f_0(t)(1-t)^n}{t-z} dt + \sum_{k=0}^{n} \dfrac{A_k}{(1-z)^k}, \\ \Phi_1(z) = \dfrac{1}{2\pi i(1-z)^{n+1}} \displaystyle\int_T \dfrac{f_1(t)(1-t)^{n+1}}{t-z} dt + z\sum_{k=0}^{n} \dfrac{A_k}{(1-z)^{k+1}} - \sum_{k=0}^{n+1} \dfrac{B_k}{(1-z)^k}, \end{cases} \quad (3.1.12)$$

where $\{A_k\}_0^n$ numbers belong to the class $S_0(n)$, and $\{B_k\}_0^{n+1}$ numbers to the class $S_0(n+1)$.

CONCLUSION

PETROSYAN VAHE

BOUNDARY VALUE PROBLEMS IN HARDY WEIGHTED SPACES

In dissertation there are represented the following main results:

1. Investigated Riemann boundary value problem in the weighted spaces $L^1(\rho)$ in unit circle with a coefficient from Holder class $C^\delta, \delta \in (0;1]$, where weight is concentrated on finite number of singular points. In the case, when order of singularity of weight function at all singular points is greater than -1, it is shown normal solvability of the problem and the general solution of the problem is determined in explicit form. Besides, if the sum of index of coefficient and order of singularities at singular points is negative, then it is given necessary and sufficient conditions for solvability of the problem [87], [89], [90].

   When order of singularity at least at one singular point is less or equal -1, then it is proved that the number of linear independent solutions of the homogeneous problem does not only depend on the index of coefficient but also on the behavior of the coefficient at the corresponding singular points. Thus, it is introduced R^α class of coefficients and is shown that for the coefficients belonging to this class, the problem is normally solvable. As in the first case, here also there are given the same kind of necessary and sufficient conditions for solvability of the problem and the solutions are written in explicit form [85], [91], [92].
2. Investigated Riemann-Hilbert boundary value problem in the weighted spaces $L^1(\rho)$ in unit circle with a coefficient from Holder class $C^\delta, \delta \in (0;1]$, where weight is concentrated on finite number of singular points. With the help of the famous transformation this problem is reduced to Riemann boundary value problem which is investigated in chapter 1. It is known that if weight function is concentrated on one singular point then coefficients of the polynomial in the general solution of the homogeneous problem must satisfy some conditions. With this setting it is found new conditions for solvability of the problem which are direct generalizations of the famous

results. Thus, it is shown normal solvability of the problem and the general solution is given in explicit form.

3. Investigated Dirichlet boundary value problem in the weighted spaces $L^1(\rho)$ in unit circle as a particular case of Riemann-Hilbert boundary value problem. It is shown normal solvability of the problem, there are given necessary and sufficient conditions for solvability of the problem and the general solution is written in explicit form.
4. Investigated Riemann boundary value problem in the weighted spaces $L^1(\rho)$ in unit circle with a piecewise continuous in the sense of Holder coefficient, where weight function is concentrated on one singular point, besides the order of singularity at this point is greater than -1. It is shown normal solvability of the problem, there are given necessary and sufficient conditions for solvability of the problem which depend on the value of jump at the points of discontinuity and on the order of singularity of weight function at the singular point. Besides, the general solution is given in explicit form [86].
5. Investigated Dirichlet problem for biharmonic functions in unit circle in the weighted spaces, where weight function is concentrated on one singular point. This problem is investigated in the case when boundary conditions on the function and on the normal derivative of this function are considered in different weighted spaces. Convergence of biharmonic function to boundary function is understood with $L^1(\rho_0)$ and convergence for normal derivative in the norm $L^1(\rho_1)$, where $\rho_0(t) = |1-t|^\alpha$ and $\rho_1(t) = \rho_0(t)|1-t|$. When $\alpha > -1$ it is shown normal solvability of the problem and the general solution is given in explicit form. If $\alpha \leq -1$, then the homogeneous problem has only trivial solution, besides there are given necessary and sufficient conditions for solvability of the inhomogeneous problem [88].

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