ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Գասպարյան Սամվել Բագրատի

Պատահական պրոցեսների վիճակագրության որոշ հարցեր

ՍԵՂՄԱԳԻՐ

Ա.01.05-“Հավանականությունների տեսություն և մաթեմատիկական վիճակագրություն”մասնագիտությամբ ֆիզիկամաթեմատիկական գիտությունների թեկնածուի գիտական աստիճանի հայցման ատենախոսության

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Samvel B. Gasparyan

Some problems of statistical inference for stochastic processes

ABSTRACT

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Ատենախոսության թեման հաստատվել է Երևանի պետական համալսարանում:

Գիտական ղեկավարներ՝ ֆիզ-մաթ գիտությունների դոկտոր
Յու.Ա. Կուտոյանց
ֆիզ-մաթ գիտությունների դոկտոր
Վ.Կ. Օհանյան

Պաշտոնական ընդդիմախոսներ՝ ֆիզ-մաթ գիտությունների դոկտոր
Ս.Յու. Դաշյան
ֆիզ-մաթ գիտությունների թեկնածու
Կ.Վ. Գասպարյան

Առաջատար կազմակերպություն՝ Հայ-ռուսական (Սլավոնական) համալսարան

Պաշտպանությունը կկայանա 2015թ. հունիսի 15-ին ժ. 15:00-ին Երևանի պետական համալսարանում գործող ԲՈՀ-ի 050 մասնագիտական խորհրդի նիստում (0025, Երևան, Ալեք Մանուկյան 1):

Ատենախոսությանը կարելի է ծանոթանալ Երևանի պետական համալսարանի գրադարանում:
Ատենախոսությունը առաքված է 2015թ. մայիսի 14-ին:

Մասնագիտական խորհրդի գիտական քարտուղար Տ.Ն. Հարությունյան

The topic of dissertation was approved by the Yerevan State University

Scientific adviser: Doctor of Phys. math sciences
Yu.A. Kutoyants
Doctor of Phys. math sciences
V.K. Ohanyan

Official opponents: Doctor of Phys. math sciences
S.Yu. Dashian
Candidate of Phys. math sciences
K.V. Gasparyan

Leading institution: Russian-Armenian (Slavonic) University

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The dissertation is available in the library of Yerevan State University.
The abstract of dissertation was distributed on 14th of May.

Scientific secretary of specialized council T.N. Harutyunyan

General description

Topicality: Asymptotic efficiency in parametric models goes back to the beginning of the twentieth century and based on the works of Fisher. He proposed a program how to define asymptotic efficiency by comparing asymptotic error of an estimator in a point and proposed an estimator, which is called the maximum likelihood estimator, as an efficient estimator. Initial program of Fisher was not exactly true, since Hoges constructed an estimator which had a lower asymptotic error in a point than the asymptotic error of the maximum likelihood estimator. Such estimators were called super-efficient estimators. But, in realty these estimators were not better than the maximum likelihood estimator, since, for improving asymptotic behavior of an estimator in a point we damage the behavior of the estimator in neighbor points. To exclude such estimators Hajèk proved a lower bound which compares behavior of estimators locally uniformly. Theorem was proved in the models where there was a LAM (local asymptotic normality) condition, which was introduced by Le Cam. Hence, the lower bound was called Hajèk-Le Cam lower bound. For this definition the maximum likelihood estimator is asymptotically efficient for various models. Later Ibragimov and Khasminskii proved convergence of moments of the maximum likelihood estimator in the model of independent identically distributed random variables, hence they proved asymptotic efficiency of the maximum likelihood estimator for polynomial loss functions. Jeganathan generalized the notion of local asymptotic normality by defining local asymptotic mixed normality (LAMN) and for such models proved asymptotic lower bound for all possible estimators. Dohnal, for multidimensional case Gobet, proved that for a stochastic differential equation (SDE) with parameter in the diffusion coefficient, we have local asymptotic mixed normality property. Efficient estimators for this model were constructed by Genon-Catalot and Jacod.

Same ideas of asymptotic efficiency transferred to the non-parametric estimation problems. First a such result was proved by Pinsker in the model of signal estimation in the presence of Gaussian white noise. In such problems the role of the inverse of the Fisher information in parametric statistics plays a constant, which is called the Pinsker constant. Later, such results where proved for other models too, particularly, for the intensity function of an inhomogeneous Poisson process the result was proved by Kutoyants. To compare asymptotic efficient estimators Golubev and Levit introduced the concept of second order efficiency for the model of independent identically distributed random variables and constructed an estimator which is asymptotically the best one among asymptotically efficient estimators, hence it is called second order asymptotically efficient estimator.

Objective:

1) Construct an easy calculable, asymptotically efficient estimator for the parameter in the diffusion coefficient for some diffusion process.

2) Efficiently estimate the solution of a forward-backward stochastic differential equation (FBSDE).

3) Prove second order efficiency result for the mean function of an inhomogeneous Poisson process.

Research methods: Methods from Asymptotic Theory of Statistics, Probability Theory and Functional Analysis.

Scientific novelty: All result presented in the dissertation are new.

Practical and theoretical significance: The main results of the work are of theoretical nature but can have possible practical applications as well. In each model we are searching for asymptotically the best estimator, hence in practical applications it is the most preferable estimator.

Approbation: The results are presented in the scientific seminars in the Chair of Probability and Mathematical Statistics of Yerevan State University and in the Laboratory of Mathematics of University of Maine, Le Mans, France. Also, results are presented in the poster session during the international conference "Statistique Asymptotique des Processus Stochastiques X" in Le Mans, France, March 17-20, 2015.

Main results of the dissertation are published in three papers; references can be found in the end of this booklet.

Structure and volume of dissertation: the dissertation is written on 79 pages; consists of an introduction, two main chapters, conclusion and the list of 40 cited references.

Overview and main results

First chapter of this work is devoted to the approximation problem of the solution of a forward-backward stochastic differential equation (forward BSDE or FBSDE). We suppose that the diffusion coefficient of the forward equation depends on an unknown one-dimensional parameter, therefore the solution of the backward equation also depends on that parameter. It is well-known that if we observe the solution of a stochastic differential equation on the whole interval, even bounded and very small, then we can estimate the unknown parameter in the diffusion coefficient "without an error", hence it is not a statistical problem. Our considerations are based on the observations of the solution of the forward equation on a finite interval, at equidistant discrete points. As the number of observations tends to infinity, the distance between the observation times tends to zero. Such a statement of problem is called *high frequency asymptotics*. At a given point of time we have to construct an estimator for the unknown parameter based on the observation times before that time only, hence we have to construct an estimator process. We are seeking an estimator which is computationally simple, but of course, we do not want to lose in the performance of the estimator. For that reason we are looking for an estimator which is also asymptotically efficient.

Recall that the BSDE was first introduced in the linear case by Bismuth [3] and in general case this equation was studied by Pardoux and Peng [31]. Since that time the BSDE attracts attention of probabilists working in financial mathematics and obtained an intensive developement (see, e.g. El Karoui *et al.* [8], Ma and Yong [28] and the references therein). The detailed exposition of the current state of this theory can be found in Pardoux and Răscanu [33].

Formally, we consider the following problem. Suppose that we have a stochastic differential equation (called *forward*)

$$\mathrm{d}X_t = S\left(t, X_t\right)\mathrm{d}t + \sigma\left(\vartheta, t, X_t\right)\mathrm{d}W_t, \quad X_0, \quad 0 \leq t \leq T$$

and two functions $f\left(t, x, y, z\right)$ and $\Phi\left(x\right)$ are given. We have to find a couple of stochastic processes $\left(Y_t, Z_t\right)$ such that it satisfies the stochastic differential equation (called *backward*)

$$\mathrm{d}Y_t = -f\left(t, X_t, Y_t, Z_t\right)\mathrm{d}t + Z_t \mathrm{d}W_t, \quad 0 \leq t \leq T$$

with the final value $Y_T = \Phi\left(X_T\right)$.

The solution of this problem is well-known. We have to solve a special partial differential equation, to find its solution $u\left(t, x, \vartheta\right)$ and to put $Y_t = u\left(t, X_t, \vartheta\right)$ and $Z_t = \sigma\left(\vartheta, t, X_t\right) u'_x\left(t, X_t, \vartheta\right)$.

We are interested in the problem of approximation of the solution $\left(Y_t, Z_t\right)$ in the situation where the parameter ϑ is unknown. Therefore we first estimate this parameter with help of some good estimator $\vartheta^\star_{t,n}, 0 < t \leq T$ based on the discrete time observations (till time t) of the solution of the forward equation and then we propose the approximations $Y^\star_t = u\left(t, X_t, \vartheta^\star_{t,n}\right), \quad Z^\star_t = \sigma\left(\vartheta^\star_{t,n}, t, X_t\right) u'_x\left(t, X_t, \vartheta^\star_{t,n}\right)$. Moreover we show that the proposed approximations are in some sense asymptotically optimal.

The main difficulty in the construction of this approximation is to find an estimator-process $\vartheta^\star_{t,n}, 0 < t \leq T$ which can be easily calculated for all $t \in (0, T]$ and at the same time has asymptotically optimal properties. Unfortunately we cannot use the well-studied pseudo-MLE (maximum likelihood estimator) based on the pseudo-maximum likelihood function because its calculation is related to the solution of nonlinear equations and numerically is sufficiently difficult problem.

The MLE (maximum likelihood estimator) was proposed in the local asymptotically normal statistical models by Le Cam [24] as a method to improve an arbitrary estimator with the optimal rate up to the one which has the smallest variance. Volatility parameter estimation has another asymptotic property. Under regularity condition the volatility parameter estimation model is asymptotically mixed normal ([7], [9], [11]). We propose here a one-step MLE-process, which was recently introduced in the case of ergodic diffusion [21] and diffusion process with small noise [22], [23]. As in the construction of the MLE-estimator we take a preliminary estimator and improve its asymptotic performance by transforming that preliminary estimator to an optimal estimator, with the difference that the MLE-process allows us to improve even the rate of the preliminary estimator. Similar technique was introduced by Kamatani, Uchida [18]. The review of statistical problems for the BSDE model of observations can be found in [20].

Note that the problem of volatility parameter estimation by discrete time observations is actually a well developed branch of statistics (see, for example, [35] and references therein). The particularity of our approach is due to the need of updated *on-line* estimator $\vartheta^\star_{t,n}$ which depends on the first observations up till time t.

Let us fix some (small) $\tau > 0$, we call the interval $[0, \tau]$ *the learning interval.* We construct the estimator process for the values of $t \in [\tau, T]$. Based on this learning interval we construct a preliminary estimator, then we improve this estimator up to an optimal (asymptotically efficient) estimator which, on the other hand, is computationally easy calculable. As a preliminary estimator $\hat\vartheta_{\tau,n}$ we take a particular *minimum contrast estimator* (MCE) (for a general method of constructing MCE estimators for the diffusion parameter see [9]) which is called the pseudo-maximum likelihood estimator (PMLE), constructed by the observations $X^{\tau,n}=\left(X_0,X_{t_{1,n}},\ldots,X_{t_{N,n}}\right)$, where $t_{N,n}\leq\tau<t_{N+1,n}$. For defining the PMLE introduce the log pseudo-likelihood ratio

$$
\begin{aligned}
L_{t,k}\left(\vartheta,X^k\right)=&-\frac{1}{2}\sum_{j=0}^{k}\ln\left[2\pi\sigma^2\left(\vartheta,t_{j-1},X_{t_{j-1}}\right)\delta\right]\\
&-\sum_{j=1}^{k}\frac{\left[X_{t_j}-X_{t_{j-1}}-S\left(t_{j-1},X_{t_{j-1}}\right)\delta\right]^2}{2\sigma^2\left(\vartheta,t_{j-1},X_{t_{j-1}}\right)\delta}
\end{aligned}
$$

and define the PMLE $\hat\vartheta_{t,n}$ by the equation

$$
L_{t,k}(\hat\vartheta_{t,n},X^k)=\sup_{\theta\in\Theta}L_{t,k}\left(\theta,X^k\right),\qquad .
$$

This estimator is consistent and asymptotically conditionally normal ([9])

$$
\begin{aligned}
\sqrt{\frac{n}{T}}\left(\hat\vartheta_{\tau,n}-\vartheta_0\right)&=\mathbb{I}_{\tau,n}\left(\vartheta_0\right)^{-1}\sqrt{2}\sum_{j=1}^{N}\frac{\dot\sigma(\vartheta_0,t_{j-1},X_{t_{j-1}})}{\sigma(\vartheta_0,t_{j-1},X_{t_{j-1}})}\,w_j+o\left(1\right)\\
&\Longrightarrow\xi_\tau\left(\vartheta_0\right)=\mathbb{I}_\tau\left(\vartheta_0\right)^{-1}\sqrt{2}\int_0^\tau\frac{\dot\sigma\left(\vartheta_0,s,X_s\right)}{\sigma\left(\vartheta_0,s,X_s\right)}{\rm d}w\left(s\right).
\end{aligned}
$$

Here the random Fisher information matrix is

$$
\mathbb{I}_\tau\left(\vartheta_0\right)=2\int_0^\tau\frac{\dot\sigma\left(\vartheta_0,s,X_s\right)\dot\sigma\left(\vartheta_0,s,X_s\right)^{\mathbb{T}}}{\sigma^2\left(\vartheta_0,s,X_s\right)}\,{\rm d}s,
$$

where *dot* means the derivative with respect to the unknown parameter ϑ.

Introduce the pseudo score-function $\left(A_{j-1}\left(\vartheta\right)=\sigma^2\left(\vartheta,t_{j-1},X_{t_{j-1}}\right)\right)$

$$
\begin{aligned}
\Delta_{k,n}\left(\vartheta,X^k\right)&=\sum_{j=1}^{k}\dot\ell\left(\vartheta,X_{t_{j-1}},X_{t_j}\right)\\
&=\sum_{j=1}^{k}\frac{\left[\left(X_{t_j}-X_{t_{j-1}}-S_{j-1}\,\delta\right)^2-A_{j-1}\left(\vartheta\right)\delta\right]\dot A_{j-1}\left(\vartheta\right)}{2A_{j-1}^2\left(\vartheta\right)\sqrt{\delta}}.
\end{aligned}
$$

For any $t\in[\tau,T]$ define k by the condition $t_k\leq t<t_{k+1}$ and the one-step PMLE-process by the relation

$$
\vartheta^\star_{k,n}=\hat\vartheta_{\tau,n}+\sqrt{\delta}\,\mathbb{I}_{k,n}(\hat\vartheta_{\tau,n})^{-1}\Delta_{k,n}(\hat\vartheta_{\tau,n},X^k),\qquad k=N+1,\ldots,n.
$$

Our goal is to show that the corresponding approximation

$$
Y^\star_{t_k,n}=u\left(t_k,X_{t_k},\vartheta^\star_{k,n}\right),k=N+1,\ldots,n
$$

is asymptotically efficient. To do this we need to present the lower bound on the risks of all estimators and then to show that for the proposed approximation this lower bound is reached. Our first result is

Theorem 1. *The one-step MLE-process $\vartheta^\star_{k,n}, k = N+1,\ldots,n$ is consistent, asymptotically conditionally normal (stable convergence)*

$$\delta^{-1/2}\left(\vartheta^\star_{k,n}-\vartheta_0\right)\Longrightarrow \xi_t\left(\vartheta_0\right),\; \xi_t(\vartheta_0)=\frac{\Delta_t(\vartheta_0)}{\mathrm{I}_t(\vartheta_0)}$$

and is asymptotically efficient for $t\in[\tau_*,T]$ where $\tau<\tau_*<T$ and a bounded loss functions.

Then we prove a lower bound for the approximation of the solution of a FBSDE

Theorem 2. *Suppose that the coefficients of the diffusion process satisfies $\mathcal{R}$ conditions, then, for the loss function $\ell(u)=|u|^p$, $p>0$, the following lower bound is true*

$$\varliminf_{\varepsilon\to 0}\varliminf_{n\to+\infty}\sup_{|\vartheta-\vartheta_0|<\varepsilon}\mathbf{E}_\vartheta\ell\left(\delta^{-1/2}\left(\bar Y_{t_k,n}-Y_{t_k}\right)\right)\geq \mathbf{E}_{\vartheta_0}\ell(\dot u(\vartheta_0,t,X_t)\xi_t(\vartheta_0)).$$

Here $u\left(\vartheta,t,x\right)$ satisfies the equation

$$\frac{\partial u}{\partial t}+S\left(t,x\right)\frac{\partial u}{\partial x}+\frac{\sigma\left(\vartheta,t,x\right)^2}{2}\frac{\partial^2 u}{\partial x^2}=-f\left(t,x,u,\sigma\left(\vartheta,t,x\right)\frac{\partial u}{\partial x}\right).$$

Theorem 3. *Suppose that the conditions of regularity hold, then the estimators*

$$Y^\star_{t_k,n}=u(t_k,X_{t_k},\vartheta^\star_{k,n}),\qquad Z^\star_{t_k,n}=u'_x(t_k,X_{t_k},\vartheta^\star_{k,n})\sigma(t_k,X_{t_k},\vartheta^\star_{k,n}),\quad t_k\in\left[\tau,T\right],$$

are consistent

$$Y^\star_{t_k,n}\longrightarrow Y_t,\qquad Z^\star_{t_k,n}\longrightarrow Z_t,$$

and are asymptotically conditionally normal (stable convergence)

$$\begin{aligned}\delta^{-1/2}\left(Y^\star_{t_k,n}-Y_{t_k}\right)&\Longrightarrow\langle\dot u\left(t,X_t,\vartheta_0\right),\ \xi_t\left(\vartheta_0\right)\rangle,\\ \delta^{-1/2}\left(Z^\star_{t_k,n}-Z_{t_k}\right)&\Longrightarrow\sigma\left(t,X_t,\vartheta_0\right)\ \langle\dot u'_x\left(t,X_t,\vartheta_0\right),\ \xi_t\left(\vartheta_0\right)\rangle\\ &\qquad+u'_x\left(t,X_t,\vartheta_0\right)\langle\dot\sigma\left(t,X_t,\vartheta_0\right),\ \xi_t\left(\vartheta_0\right)\rangle.\end{aligned}$$

These results are presented in the work [39].

It have to be mentioned here that we could construct the approximation of Y_t and Z_t as follows

$$\hat Y_{t,n}=u(t,X_t,\hat\vartheta_{k,N})\text{ and }Z^\star_t=\sigma\left(\hat\vartheta_{t,N},t,X_t\right)u'_x\left(t,X_t,\hat\vartheta_{t,N}\right),$$

that is, using only the preliminary estimator. Note that this type of approximation is not asymptotically efficient, since we use only part of the observations (only the learning interval) and our estimator does not change depending on time. That is the reason why we are looking for another estimator of ϑ which can provide smaller error of estimation.

Then, we are considering a Pearson diffusion

$$\mathrm{d}X_t=-X_t\mathrm{d}t+\sqrt{\vartheta+X_t^2}\,\mathrm{d}W_t,\quad X_0,\quad 0\leq t\leq T.$$

For this model, using the preliminary estimator

$$\bar{\vartheta}_N = \frac{n}{TN}\left[X_{t_N}^2 - X_0^2 - 2\sum_{j=1}^{N} X_{t_{j-1}}\ \left[X_{t_j} - X_{t_{j-1}}\right] - \sum_{j=1}^{N} X_{t_{j-1}}^2\delta\right],$$

we propose the one-step MLE process

$$\vartheta_{t_k,n}^{\star} = \bar{\vartheta}_N + \sqrt{\delta}\sum_{j=1}^{k}\frac{\left[X_{t_j} - X_{t_{j-1}} + X_{t_{j-1}}\delta\right]^2 - \left(\bar{\vartheta}_N + X_{t_{j-1}}^2\right)\delta}{2\mathbb{I}_{t_k,n}\left(\bar{\vartheta}_N\right)\left(\bar{\vartheta}_N + X_{t_{j-1}}^2\right)^2\sqrt{\delta}},\ \tau \leq t_k \leq T.$$

and prove the theorem

Theorem 4. *The one-step MLE-process $\vartheta_{t_k,n}^{\star}$ is consistent: for any $\nu > 0$*

$$\mathbf{P}_{\vartheta_0}\left(\max_{N\leq k\leq n}\left|\vartheta_{t_k,n}^{\star} - \vartheta_0\right| > \nu\right) \to 0$$

and for all $t \in (\tau, T]$ the convergence

$$\delta^{-1/2}\left(\vartheta_{t_k,n}^{\star} - \vartheta_0\right) \Longrightarrow \zeta_t\left(\vartheta_0\right)$$

holds. Moreover, this estimator is asymptotically efficient.

The result is presented in [40].

The second chapter is devoted to non-parametric estimation. The difference from parametric estimation is that the unknown object to be estimated is infinite dimensional. As a model we have continuous time observations of an inhomogeneous Poisson process on the real line with a periodic intensity function. It is supposed that the period is known. To each Poisson process is associated a positive measure on the state space, called the mean measure of the Poisson process. If the state space is the real line and the mean function $\Lambda(\cdot)$ (which generates the mean measure) is absolutely continuous with respect to the Lebesgue measure

$$\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s$$

then the function $\lambda(\cdot)$ is called the intensity function of the Poisson process. We are considering the case when the intensity function is periodic with the known period.

The mean function estimation problem is very close to the distribution function estimation problem from independent, identically distributed random observations. More precisely we can construct consistent estimators without regularity conditions on the unknown object. It is well known that, for example, in the density or intensity estimation problems, even for constructing a consistent estimator we have to impose regularity conditions (existence and Hölder continuity of some derivative, see, for example, [36]) on the unknown object. In the works of Kutoyants [19] (for integral-type quadratic loss functions see the forthcoming book *Introduction to Statistics of Poisson Processes* by Kutoyants) it was shown that the empirical mean function

$$\hat{\Lambda}(t) = \frac{1}{n}\sum_{j=1}^{n} X_j(t)$$

is consistent, asymptotically normal estimator with the optimal rate and even is asymptotically efficient for a large number of loss functions (including polynomials). But for example, there are many estimators which are asymptotically efficient with respect to the integral-type quadratic loss function. Asymptotic efficiency we understand in a way that, as for all estimators the following lower bound is true

$$\underline{\lim}_{\delta\to 0}\ \underline{\lim}_{n\to+\infty}\ \sup_{\Lambda\in V_\delta} n\int_0^\tau \mathbf{E}_\Lambda(\bar\Lambda_n(s)-\Lambda(s))^2\mathrm{d}s \geq \int_0^\tau \Lambda^*(s)\mathrm{d}s,$$

the estimator $\Lambda^*(t)$ for which we have equality

$$\lim_{\delta\to 0}\ \lim_{n\to+\infty}\ \sup_{\Lambda\in V_\delta} n\int_0^\tau \mathbf{E}_\Lambda(\Lambda_n^*(s)-\Lambda(s))^2\mathrm{d}s = \int_0^\tau \Lambda^*(s)\mathrm{d}s.$$

To compare these asymptotically efficient estimators (which we call first order efficient) we prove an inequality for all possible estimators.

Theorem 5.

$$\underline{\lim}_{n\to+\infty}\ \sup_{\Lambda\in\mathcal{F}_m(R,S)} n^{\frac{2m}{2m-1}}\left(\int_0^\tau \mathbf{E}_\Lambda(\bar\Lambda_n(t)-\Lambda(t))^2\mathrm{d}t-\frac{1}{n}\int_0^\tau \Lambda(t)\mathrm{d}t\right)\geq -\Pi,$$

which compares second order asymptotic term of maximal loss over some non-parametric class of functions (under additional regularity conditions on the unknown mean function). Hence the estimators reaching that lower bound will be called second order efficient. In our work we explicitly calculate asymptotic minimal error for the second order estimation. This constant

$$\Pi=\Pi_m(R,S)=(2m-1)R\left(\frac{2S}{R}\frac{\tau}{2\pi}\frac{m}{(2m-1)(m-1)}\right)^{\frac{2m}{2m-1}}$$

plays the same role in second order estimation as the Pinsker constant in density estimation problem or the inverse of the Fisher information in the regular parametric estimation problems. But unlike mentioned problems here the constant is negative. This is due to the fact that for the empirical mean function

$$\mathbf{E}_\Lambda\int_0^\tau(\hat\Lambda(t)-\Lambda(t))^2\mathrm{d}t=\frac{1}{n}\int_0^\tau\Lambda(t)\mathrm{d}t,$$

the second term is equal to zero. We propose also an estimator

$$\Lambda_n^*(t)=\hat\Lambda_{1,n}\phi_1(t)+\sum_{l=1}^{N_n}K_{2l,n}\hat\Lambda_{2l,n}\phi_{2l}(t)+$$
$$+\sum_{l=1}^{+\infty}\left[K_{2l,n}(\hat\Lambda_{2l+1,n}+a_{2l+1})-a_{2l+1}\right]\phi_{2l+1}(t),$$

where $\{\phi_l\}_{l=1}^{+\infty}$ is the trigonometric basis on $\mathrm{L}_2[0,\tau]$, $\hat{\Lambda}_{l,n}$ are the Fourier coefficients of the empirical mean function with respect to this basis and

$$K_{2l,n} = \left(1 - \left|\frac{2\pi l}{\tau}\right|^m \alpha_n\right)_+,$$

$$\alpha_n = \left[\frac{1}{n}\frac{\tau}{2\pi}\frac{2S}{R}\frac{m}{(2m-1)(m-1)}\right]^{\frac{m}{2m-1}},$$

$$a_{2l+1} = \sqrt{\frac{\tau}{2}}\frac{\tau}{2\pi l}S,$$

$$N_n = \left[\frac{\tau}{2\pi}\frac{1}{\alpha_n^{\frac{1}{m}}}\right] \approx \mathrm{C}n^{\frac{1}{2m-1}}.$$

Here $(x_+ = \max(x,0))$. Our next result is the following theorem.

Theorem 6. *The estimator $\Lambda_n^*(t)$ attains the lower bound described above, that is,*

$$\lim_{n\to+\infty}\sup_{\Lambda\in\mathcal{F}_m(R,S)} n^{\frac{2m}{2m-1}}\left(\int_0^\tau \mathbf{E}_\Lambda(\Lambda_n^*(t)-\Lambda(t))^2\mathrm{d}t - \frac{1}{n}\int_0^\tau \Lambda(t)\mathrm{d}t\right) = -\Pi,$$

that is, our proposed estimator is asymptotically second order efficient. The estimator is linear, that is, its Fourier coefficients are linear combinations of the coefficients of the empirical mean function. We consider Fourier expansions with respect to the trigonometric basis. The theorem was presented in [38].

The non-parametric estimation problems where we can explicitly calculate the asymptotic error was first done by Pinsker [34] in the model of observation of a signal in the white Gaussian noise. The idea was to consider the minimax risk of integral-type quadratic loss functions on a Sobolev ellipsoid. The concept of second order efficiency was introduced by Golubev and Levit [13] in the problem of distribution function estimation for the model of independent, identically distributed random variables. In the paper [13] authors proved a lower bound which allows to compare second term of the expansion of the maximal loss over some set of functions and minimize that term. They proposed also an estimator which attains that lower bound, hence that lower bound is sharp.

Later, second order efficiency was considered for some other models. For example, Dalalyan and Kutoyants [5] proved second order asymptotic efficiency in the estimation problem of the invariant density of an ergodic diffusion process. Golubev and Härdle [12] proved second order asymptotic efficiency in partial linear models.

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Ամփոփագիր

Ատենախոսությունում դիտարկվում են հակադարձ ստոխաստիկ դիֆերենցիալ հավասարման լուծման մոտարկման խնդիրը, ինչպես նաև ոչ համասեռ պուասոնյան պատահական պրոցեսի միջին ֆունկցիայի համար երկրորդ կարգի էֆեկտիվության ապացուցման խնդիրը: Ատենախոսությունում ստացվել են հետևյալ արդյունքները:

- Դիտարկվում է նմուշ

$$\mathrm{d}X_t = S(t, X_t)\mathrm{d}t + \sigma(\vartheta, t, X_t)\mathrm{d}W(t),$$

ստոխաստիկ դիֆերենցիալ հավասարման լուծումից՝ ժամանակի դիսկրետ պահերին $X^n = (X_{t_1}, X_{t_2}, \cdots, X_{t_n})$, $t_i - t_{i-1} = \frac{T}{n}$: Սահմանում ենք պսևդո-ճշմարտանմանության ֆունկցիա (այստեղ $A_{j-1}(\vartheta) = \sigma^2\left(\vartheta, t_{j-1}, X_{t_{j-1}}\right)$)

$$\Delta_{k,n}\left(\vartheta, X^k\right) = \sum_{j=1}^{k} \frac{\left[\left(X_{t_j} - X_{t_{j-1}} - S_{j-1}\,\delta\right)^2 - A_{j-1}(\theta)\,\delta\right]\dot{A}_{j-1}(\vartheta)}{2A_{j-1}^2(\vartheta)\sqrt{\delta}}$$

և ստոխաստիկ Ֆիշերի ինֆորմացիա

$$\mathbb{I}_\tau(\vartheta_0) = 2\int_0^\tau \frac{\dot{\sigma}(\vartheta_0, s, X_s)\,\dot{\sigma}(\vartheta_0, s, X_s)^{\mathbb{T}}}{\sigma^2(\vartheta_0, s, X_s)}\,\mathrm{d}s.$$

Ժամանակի ցանկացած $t \in [\tau, T]$ պահի համար սահմանենք k այնպես, որ $t_k \leq t < t_{k+1}$ և ներմուծենք մեկ քայլանոց պսևդո մաքսիմում ճշմարտանմանության գնահատական-պրոցեսը հետևյալ առնչությունից ելնելով

$$\vartheta^\star_{k,n} = \hat{\vartheta}_{\tau,n} + \sqrt{\delta}\,\mathbb{I}_{k,n}(\hat{\vartheta}_{\tau,n})^{-1}\Delta_{k,n}(\hat{\vartheta}_{\tau,n}, X^k), \qquad k = N+1, \ldots, n.$$

Հետևաբար, մեկ քայլանոց պսևդո մաքսիմում ճշմարտանմանության գնահատական-պրոցեսը՝ $\vartheta^\star_{k,n}, k = N+1, \ldots, n$, ունակային է և ասիմպտոտիկ պայմանական նորմալ (կայուն զուգամիտություն)

$$\delta^{-1/2}\left(\vartheta^\star_{k,n} - \vartheta_0\right) \Longrightarrow \xi_t(\vartheta_0),\ \xi_t(\vartheta_0) = \frac{\Delta_t(\vartheta_0)}{\mathrm{I}_t(\vartheta_0)},$$

ինչպես նաև ասիմպտոտիկ էֆեկտիվ է բոլոր սահմանափակ կորստի ֆունկցիաների համար: Այստեղ $t \in [\tau_*, T]$ և $\tau < \tau_* < T$.

- Ենթադրենք ստոխաստիկ դիֆերենցիալ հավասարման գործակիցները բավարարում են $\mathcal{R}$ պայմաններին, հետևաբար, $\ell(u) = |u|^p$, $p > 0$ կորստի ֆունկցիաների համար տեղի ունի հետևյալ ստորին սահմանը՝

$$\underline{\lim}_{\varepsilon\to 0}\ \underline{\lim}_{n\to+\infty}\ \sup_{|\vartheta-\vartheta_0|<\varepsilon} \mathbf{E}_\vartheta \ell\left(\delta^{-1/2}\left(\bar{Y}_{t_k,n} - Y_{t_k}\right)\right) \geq \mathbf{E}_{\vartheta_0}\ell(\dot{u}(\vartheta_0, t, X_t)\xi_t(\vartheta_0)).$$

Այստեղ $u(\vartheta, t, x)$ բավարարում է

$$\frac{\partial u}{\partial t} + S(t,x)\frac{\partial u}{\partial x} + \frac{\sigma(\vartheta,t,x)^2}{2}\frac{\partial^2 u}{\partial x^2} = -f\left(t, x, u, \sigma(\vartheta,t,x)\frac{\partial u}{\partial x}\right),$$

հավասարմանը, իսկ $\dot{u}(\vartheta, t, X_t)$ հանդիսանում է $u(\vartheta, t, X_t)$ ֆունկցիայի ածանցյալը ըստ ϑ.

- Ենթադրենք ռեգուլյարության պայմանները բավարարված են: Այդ դեպքում $Y^{\star}_{t_k,n} = u(t_k, X_{t_k}, \vartheta^{\star}_{k,n})$, $Z^{\star}_{t_k,n} = u'_x(t_k, X_{t_k}, \vartheta^{\star}_{k,n})\sigma(t_k, X_{t_k}, \vartheta^{\star}_{k,n})$, $t_k \in [\tau, T]$ մոտարկումները ունակային են $Y^{\star}_{t_k,n} \longrightarrow Y_t$, $Z^{\star}_{t_k,n} \longrightarrow Z_t$ և ասիմպտոտիկ պայմանական նորմալ (կայուն զուգամիտություն)

$$\delta^{-1/2}\left(Y^{\star}_{t_k,n} - Y_{t_k}\right) \Longrightarrow \langle \dot{u}\left(t, X_t, \vartheta_0\right), \ \xi_t\left(\vartheta_0\right)\rangle,$$
$$\delta^{-1/2}\left(Z^{\star}_{t_k,n} - Z_{t_k}\right) \Longrightarrow \sigma\left(t, X_t, \vartheta_0\right) \ \langle \dot{u}'_x\left(t, X_t, \vartheta_0\right), \ \xi_t\left(\vartheta_0\right)\rangle + u'_x\left(t, X_t, \vartheta_0\right)\langle \dot{\sigma}\left(t, X_t, \vartheta_0\right), \ \xi_t\left(\vartheta_0\right)\rangle.$$

Արդյունքները ներկայացված են [39], [40].

- Եթե պուասոնյան պրոցեսի միջին ֆունկցիան պատկանում է հետևյալ բազմությանը

$$\mathcal{F}_m(R,S) = \left\{\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s : \ \lambda \in \tilde{\mathcal{C}}_{m-1}(\mathbf{R}_+), \right.$$
$$\left. \int_0^{\tau} [\Lambda^{(m)}(t)]^2 \mathrm{d}t \leq R, \ \frac{2}{\tau}\Lambda(\tau) = S\right\},$$

որտեղ $R > 0$, $S > 0$, $m > 1$, $m \in \mathcal{N}$ հաստատունները տրված են, ապա

$$\varliminf_{n\to+\infty} \sup_{\Lambda\in\mathcal{F}_m(R,S)} n^{\frac{2m}{2m-1}} \left(\int_0^{\tau} \mathbf{E}_{\Lambda}(\bar{\Lambda}_n(t) - \Lambda(t))^2 \mathrm{d}t - \frac{1}{n}\int_0^{\tau} \Lambda(t)\mathrm{d}t\right) \geq -\Pi,$$

այստեղ Π հաստատունն ունի հետևյալ տեսքը

$$\Pi = \Pi_m(R,S) = (2m-1)R\left(\frac{2S}{R}\frac{\tau}{2\pi}\frac{m}{(2m-1)(m-1)}\right)^{\frac{2m}{2m-1}}.$$

- Ներմուծում ենք հետևյալ գնահատականը

$$\Lambda^*_n(t) = \hat{\Lambda}_{1,n}\phi_1(t) + \sum_{l=1}^{N_n} K_{2l,n}\hat{\Lambda}_{2l,n}\phi_{2l}(t) +$$
$$+ \sum_{l=1}^{+\infty}\left[K_{2l,n}(\hat{\Lambda}_{2l+1,n} + a_{2l+1}) - a_{2l+1}\right]\phi_{2l+1}(t),$$

որտեղ $\{\phi_l\}_{l=1}^{+\infty}$ եռանկյունաչափական բազիսն է $\mathrm{L}_2[0,\tau]$ տարածությունում, իսկ $\hat{\Lambda}_{l,n}$ էմպիրիկ միջին ֆունկցիայի Ֆուրիեի գործակիցներն են այդ բազիսի նկատմամբ, ինչպես նաև (այստեղ $x_+ = \max(x,0)$)

$$K_{2l,n} = \left(1 - \left|\frac{2\pi l}{\tau}\right|^m \alpha_n\right)_+, \ N_n = \left[\frac{\tau}{2\pi}\frac{1}{\alpha_n^{\frac{1}{m}}}\right] \approx \mathrm{C}n^{\frac{1}{2m-1}},$$
$$\alpha_n = \left[\frac{1}{n}\frac{\tau}{2\pi}\frac{2S}{R}\frac{m}{(2m-1)(m-1)}\right]^{\frac{m}{2m-1}}, \ a_{2l+1} = \sqrt{\frac{\tau}{2}}\frac{\tau}{2\pi l}S.$$

Ներմուծված $\Lambda^*_n(t)$ գնահատականի համար տեղի ունի

$$\lim_{n\to+\infty} \sup_{\Lambda\in\mathcal{F}_m(R,S)} n^{\frac{2m}{2m-1}} \left(\int_0^{\tau} \mathbf{E}_{\Lambda}(\Lambda^*_n(t) - \Lambda(t))^2 \mathrm{d}t - \frac{1}{n}\int_0^{\tau} \Lambda(t)\mathrm{d}t\right) = -\Pi.$$

Արդյունքները ներկայացված են [38].

Аннотация

В диссертации рассматриваются проблема аппроксимации решения обратного стохастического дифференциального уравнения и проблема оценивания второго порядка средней функции неоднородного пуассоновского процесса. Получены следующие результаты.

- Рассматривается выборка из решения стохастического дифференциального уравнения
$$\mathrm{d}X_t = S(t, X_t)\mathrm{d}t + \sigma(\vartheta, t, X_t)\mathrm{d}W(t),$$
в дискретные моменты времени $X^n = (X_{t_1}, X_{t_2}, \cdots, X_{t_n}), \quad t_i - t_{i-1} = \frac{T}{n}$. Вводится производная функции логарифмического отношения псевдоправдоподобия (здесь $A_{j-1}(\vartheta) = \sigma^2\left(\vartheta, t_{j-1}, X_{t_{j-1}}\right)$)
$$\Delta_{k,n}\left(\vartheta, X^k\right) = \sum_{j=1}^{k} \frac{\left[\left(X_{t_j} - X_{t_{j-1}} - S_{j-1}\,\delta\right)^2 - A_{j-1}(\theta)\,\delta\right]\dot{A}_{j-1}(\vartheta)}{2A_{j-1}^2(\vartheta)\sqrt{\delta}}$$
и стохастическая информация Фишера
$$\mathbb{I}_\tau(\vartheta_0) = 2\int_0^\tau \frac{\dot{\sigma}(\vartheta_0, s, X_s)\,\dot{\sigma}(\vartheta_0, s, X_s)^{\mathbb{T}}}{\sigma^2(\vartheta_0, s, X_s)}\,\mathrm{d}s.$$
Для всех моментов времени $t \in [\tau, T]$ определяется k, так что $t_k \leq t < t_{k+1}$ и вводится одношаговая оценка псевдомаксимума правдоподобия (ОПМП) равенством
$$\vartheta^\star_{k,n} = \hat{\vartheta}_{\tau,n} + \sqrt{\delta}\,\mathbb{I}_{k,n}(\hat{\vartheta}_{\tau,n})^{-1}\Delta_{k,n}(\hat{\vartheta}_{\tau,n}, X^k), \qquad k = N+1, \ldots, n.$$
В этом случае, одношаговая оценка псевдомаксимума правдоподобия (ОПМП) $\vartheta^\star_{k,n}, k = N+1, \ldots, n$, является состоятельной и асимптотически условно нормальной (устойчивая сходимость) оценкой, т.е.
$$\delta^{-1/2}\left(\vartheta^\star_{k,n} - \vartheta_0\right) \Longrightarrow \xi_t(\vartheta_0),\ \xi_t(\vartheta_0) = \frac{\Delta_t(\vartheta_0)}{\mathrm{I}_t(\vartheta_0)}.$$
Более того, эта оценка является асимптотически эффективной для всех ограниченных функций потерь. Здесь $t \in [\tau_*, T]$ и $\tau < \tau_* < T$.

- Доказывается следующее неравенство для всех аппроксимаций решения обратного стохастического дифференциального уравнения:
$$\varliminf_{\varepsilon\to 0}\ \varliminf_{n\to+\infty}\ \sup_{|\vartheta-\vartheta_0|<\varepsilon} \mathbf{E}_\vartheta \ell\left(\delta^{-1/2}\left(\bar{Y}_{t_k,n} - Y_{t_k}\right)\right) \geq \mathbf{E}_{\vartheta_0}\ell(\dot{u}(\vartheta_0, t, X_t)\xi_t(\vartheta_0)),$$
для всех функций потерь вида $\ell(u) = |u|^p$, $p > 0$. Здесь $u(\vartheta, t, x)$ удовлетворяет уравнению
$$\frac{\partial u}{\partial t} + S(t,x)\frac{\partial u}{\partial x} + \frac{\sigma(\vartheta,t,x)^2}{2}\frac{\partial^2 u}{\partial x^2} = -f\left(t, x, u, \sigma(\vartheta,t,x)\frac{\partial u}{\partial x}\right),$$
а $\dot{u}(\vartheta, t, X_t)$ является производной $u(\vartheta, t, X_t)$ по ϑ. Предполагается что коэффициенты стохастического дифференциального уравнения удовлетворяют условиям $\mathcal{R}$.

- Если условия регулярности выполнены, тогда следующие аппроксимации
$$Y^\star_{t_k,n} = u(t_k, X_{t_k}, \vartheta^\star_{k,n}), \qquad Z^\star_{t_k,n} = u'_x(t_k, X_{t_k}, \vartheta^\star_{k,n})\sigma(t_k, X_{t_k}, \vartheta^\star_{k,n}), \quad t_k \in [\tau, T],$$
состоятельны, т.е. $Y^\star_{t_k,n} \longrightarrow Y_t$, $Z^\star_{t_k,n} \longrightarrow Z_t$, асимптотически условно нормальны (устойчивая сходимость)
$$\delta^{-1/2}\left(Y^\star_{t_k,n} - Y_{t_k}\right) \Longrightarrow \langle \dot u\left(t, X_t, \vartheta_0\right), \ \xi_t\left(\vartheta_0\right)\rangle,$$
$$\delta^{-1/2}\left(Z^\star_{t_k,n} - Z_{t_k}\right) \Longrightarrow \sigma\left(t, X_t, \vartheta_0\right) \ \langle \dot u'_x\left(t, X_t, \vartheta_0\right), \ \xi_t\left(\vartheta_0\right)\rangle + u'_x\left(t, X_t, \vartheta_0\right)\langle \dot\sigma\left(t, X_t, \vartheta_0\right), \ \xi_t\left(\vartheta_0\right)\rangle.$$

Результаты представлены в работах [39], [40].

- Если неизвестная средняя функция пуассоновского процесса принадлежит множеству
$$\mathcal{F}_m(R,S) = \Big\{\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s : \ \lambda \in \tilde{\mathcal{C}}_{m-1}(\mathbf{R}_+), \int_0^\tau [\Lambda^{(m)}(t)]^2\mathrm{d}t \le R, \ \frac{2}{\tau}\Lambda(\tau) = S\Big\},$$
где постоянные $R > 0$, $S > 0$, $m > 1$, $m \in \mathcal{N}$ известны, тогда
$$\varliminf_{n\to+\infty} \sup_{\Lambda\in\mathcal{F}_m(R,S)} n^{\frac{2m}{2m-1}} \left(\int_0^\tau \mathbf{E}_\Lambda(\bar\Lambda_n(t) - \Lambda(t))^2\mathrm{d}t - \frac{1}{n}\int_0^\tau \Lambda(t)\mathrm{d}t\right) \ge -\Pi,$$
здесь константа Π имеет вид
$$\Pi = \Pi_m(R,S) = (2m-1)R\left(\frac{2S}{R}\frac{\tau}{2\pi}\frac{m}{(2m-1)(m-1)}\right)^{\frac{2m}{2m-1}}.$$

- Для оценки $\Lambda^*_n(t)$ вышеприведенная нижняя граница достигается, то есть
$$\lim_{n\to+\infty} \sup_{\Lambda\in\mathcal{F}_m(R,S)} n^{\frac{2m}{2m-1}} \left(\int_0^\tau \mathbf{E}_\Lambda(\Lambda^*_n(t) - \Lambda(t))^2\mathrm{d}t - \frac{1}{n}\int_0^\tau \Lambda(t)\mathrm{d}t\right) = -\Pi.$$
где мы используем следующие обозначения
$$\Lambda^*_n(t) = \hat\Lambda_{1,n}\phi_1(t) + \sum_{l=1}^{N_n} K_{2l,n}\hat\Lambda_{2l,n}\phi_{2l}(t) + \sum_{l=1}^{+\infty}\left[K_{2l,n}(\hat\Lambda_{2l+1,n} + a_{2l+1}) - a_{2l+1}\right]\phi_{2l+1}(t),$$
$\{\phi_l\}_{l=1}^{+\infty}$ является тригонометрическим базисом в пространстве $\mathrm{L}_2[0,\tau]$, а $\hat\Lambda_{l,n}$ являются коэффициентами Фурье эмирической средней по отношению к этому базису, а (здесь $x_+ = \max(x,0)$)
$$K_{2l,n} = \left(1 - \left|\frac{2\pi l}{\tau}\right|^m \alpha_n\right)_+, \ N_n = \left[\frac{\tau}{2\pi}\frac{1}{\alpha_n^{\frac{1}{m}}}\right] \approx \mathrm{C}n^{\frac{1}{2m-1}},$$
$$\alpha_n = \left[\frac{1}{n}\frac{\tau}{2\pi}\frac{2S}{R}\frac{m}{(2m-1)(m-1)}\right]^{\frac{m}{2m-1}}, \ a_{2l+1} = \sqrt{\frac{\tau}{2}}\frac{\tau}{2\pi l}S.$$
Результаты представлены в работе [38].