ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Պետրոսյան Արմենակ Լավրենտիի

Ֆունկցիաների վերականգնումը տեղաշարժի նկատմամբ ինվարիանտ տարածություններում

ՍԵՂՄԱԳԻՐ

Ա.01.01-"Մաթեմատիկական անալիզ"մասնագիտությամբ ֆիզիկամաթեմատիկական գիտությունների թեկնածուի գիտական աստիճանի հայցման ատենախոսության

ԵՐԵՎԱՆ 2015

YEREVAN STATE UNIVERSITY

Armenak Petrosyan

Recovery of functions in shift invariant spaces

ABSTRACT

of dissertation submitted for a degree of candidate in phys-math sciences
Specialty: A.01.01-’"Mathematical Analysis"

YEREVAN 2015

Ատենախոսության թեման հաստատվել է ՀՀ ԳԱԱ Մաթեմատիկայի Ինստիտուտում

Գիտական ղեկավար՝	ֆիզ-մաթ գիտությունների դոկտոր ՀՀ ԳԱԱ թղթակից անդամ Ա.Ա. Սահակյան
	Ամերիկական մաթ. ընկերության գործող անդամ PhD (ԱՄՆ) Աքրամ Ալդրուբի
Պաշտոնական ընդիմախոսներ՝	ֆիզ-մաթ գիտությունների դոկտոր Գրիգորի Կարագուլյան
	ֆիզ-մաթ գիտությունների թեկնածու Իլյա Կրիշտալ (ԱՄՆ)
Առաջատար կազմակերպություն՝	Հայաստանի ազգային պոլիտեխնիկական համալսարան

Պաշտպանությունը կկայանա 2015թ. հունիսի 15-ին ժ. 15:00-ին Երևանի պետական համալսարանում գործող ԲՈՀ-ի 050 մասնագիտական խորհրդի նիստում (0025, Երևան, Ալեք Մանուկյան 1):

Ատենախոսությանը կարելի է ծանոթանալ Երևանի պետական համալսարանի գրադարանում:
Ատենախոսությունը առաքված է 2015թ. մայիսի 14-ին:

Մասնագիտական խորհրդի գիտական քարտուղար Տ. Հարությունյան

The topic of dissertation was approved by the Institute of Mathematics of NAS of RA

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Dissertation defense will take place on 15th of June, 2015 at 15:00, during the specialized meeting of the Higher Attestation Commission council 050 at YSU (1 Alex Manoogian, Yeravan 0025, Armenia).

The dissertation is available in the library of Yerevan State University.
The abstract of dissertation was distributed on 14th of May.

Scientific secretary of specialized council T. Harutyunyan

General description

Topicality: The signals we use in real world are called analog signals. They are usually modeled as functions depending on continuous variables of time, space, frequency etc. To be able to process those signals digitally they need to be discretized (analogue to digital conversion) and then converted back to analogue signals after being processed (digital to analogue conversion). Sampling theory is the mathematical model behind this conversion.

The classical Sampling theory takes its origins in late 1940's, in the works of Claude E. Shannon, where he introduced the famous Sampling theorem to communication engineering community [19, 14, 24]. In mathematical and engineering literature the theorem carries several names (Nyquist - Shannon theorem, Shannon - Kotelnikov theorem, Whittaker - Shannon theorem etc) due to being discovered by several people in different countries, decades and even centuries. Since then the Sampling theory has seen a lot of development and become more mathematical, having connections to Functional analysis, Frame theory, Wavelets, Shift-invariant spaces, Time-frequency analysis etc [4, 6, 21, 22, 15, 25].

Dynamical sampling is a new development in Sampling theory, where not only the function (signal) of interest is being sampled but also its evolved states over time. The problem is to recover the initial function by combining these spatio-temporal samples. The dynamical sampling was introduced in the works of A. Aldroubi, J. Davis and I. Krishtal where they consider the problem in spaces $l^2(\mathbb{Z}_d)$ and $l^2(\mathbb{Z})$ on uniform grid [2, 3]. They find a reconstruction procedure and treat the issues of additional sampling locations under certain symmetry assumptions on the evolution operator. In the dissertation we consider the problem in shift-invariant spaces, as well as in two dimensional settings $l^2(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2})$ and $l^2(\mathbb{Z}^2)$. In two dimensions there are more options with the symmetries on the evolution kernel. We consider the case with quadrantal symmetry and construct additional sampling sets of minimal size. Also the case when the positions of sampling points is allowed to change at different time levels is considered. If the number of sampling points at any time is constant, a necessary and sufficient condition is found for the existence of positions that allow full recovery of any function by samples taken at those positions. Also, for single measurement per time level, a lower bound on the number of such sampling configurations is computed.

Objective:

1) Stating the dynamical sampling problem for different settings and providing reconstruction procedures

2) Finding additional sampling locations when the samples taken on the uniform grid won't allow unique and stable reconstruction

3) Estimating the number of configurations that allow recovery for general operators acting on functions defined on a finite domain

Research methods: Methods from Functional analysis, Harmonic analysis and Linear algebra are used.

Scientific novelty: All results presented in the dissertation are new.

Practical and theoretical significance: The main results of the work are of theoretical nature but can have possible practical applications as well. The problem originally stems from signal processing; geophysical and medical imaging are other possible fields of application.

Approbation: Main results of the dissertation have been presented during one of the seminars of the chair of Function theory in Department of Mathematics and Mechanics at YSU, also during the following conferences

- February Fourier Talks (FFT) 2015, Norbert Wiener Center, University of Maryland, USA, February 15-20, 2015

- 11th International Conference Sampling Theory and Applications (SampTA) 2015, American University, Washington DC, USA, May 25-29, 2015 (special session presentation)

Main results of the dissertation are published in three papers; references can be found in the end of this booklet

Structure and volume of dissertation: the dissertation is written on 74 pages; consists of an introduction, four chapters, conclusion and the list of 35 cited references.

Overview and main results

The classical sampling problem is to, under certain assumptions, recover a function f defined on some set X from its values (often called samples) on a smaller discrete subset $\Omega \subset X$.

In many applications, taking samples on an appropriate sampling set Ω is not practical or even possible (it can be that measuring devices are expensive or only a few are available). However, if f is an initial state of a physical process evolving in time under the action of an operator A, then we may be able to recover f from its samples on Ω and the subsequent samples of its evolutions on, possibly different from Ω, sampling sets (if we allow the measuring devices to move). This new problem is related to sensing networks [18, 17] and the work in [13, 8]. In [13] Lu and Vetterli study the problem of dynamical sampling for the specific case of bandlimited functions and when the evolution operator is the heat kernel.

The mathematical formulation of the problem we are considering is as follows: let V and V', $V \subseteq V'$, be spaces of functions defined on a set X. We assume the initial state of a (linear time-invariant dynamical) system $f_n = A^n f_{n-1}$, is given by an unknown function $f \in V$, i.e. $f_0 = f$, and $A : V' \to V'$ is a known linear operator. At each time instance n $(0 \leq n < L)$ the values (samples) of the evolved function $A^n f$ are measured on some subset $\Omega_n \subseteq X$, i.e. $(A^n f)|_{\Omega_n}$ are given. The main problem in dynamical sampling is to uniquely reconstruct the function $f \in V$ from these samples, preferably in a stable way. That is, if one of the following properties is satisfied:

Definition. *(1) The operator A, the sampling sets $\Omega_n \subseteq X$, $0 \leq n < L$ and the number of repeated sampling procedures L satisfy the invertibility sampling property (ISP) condition within a space of functions V, if any $f \in V$ is uniquely determined by*

its samples $(A^n f)|_{\Omega_n}$, $0 \le n < L$.
(2) The operator A*, the sampling sets* $\Omega_n \subseteq X$, $0 \le n < L$ *and the number of repeated sampling procedures* L *satisfy the stability sampling property condition (SSP) within a space of functions* V*, if for any two signals* $f, f_1 \in V$

$$\|f - f_1\|^2 \asymp \sum_{0 \le n < L} \sum_{x \in \Omega_n} |A^n(f - f_1)(x)|^2.$$

SSP clearly implies the ISP.

The dynamical sampling problem was first studied in [2]. There the authors assume that $X = \mathbb{Z}_d$ $(d = mJ)$, $\Omega_0 = \dots = \Omega_{L-1} = m\mathbb{Z}_d$, where $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$ is the cyclic group of d elements, $V = V' = l^2(\mathbb{Z}_d)$ and the operators A is given as convolution with some kernel $a \in l^2(\mathbb{Z}_d)$:

$$(Af)(k) = (a * f)(k) = \sum_{s=0}^{d-1} f(s)a(k - s), \ k \in \mathbb{Z}_d$$

($k - s$ is understood in sense of summation operation in $\mathbb{Z}_d$, and $m\mathbb{Z}_d$ is understood in sense of multiplication operation in $\mathbb{Z}_d$).

Let S_m be the subsampling operator with rate m on $\mathbb{Z}_d$: $g \in l^2(\mathbb{Z}_d)$ $S_m g(j) = g(j)$ for $j \in m\mathbb{Z}_d$ and 0 otherwise. The Discrete Fourier Transform (DFT) of the vector $g \in l^2(\mathbb{Z}_d)$ is defined as

$$\hat{g}(k) = \sum_{s=0}^{d-1} g(s) e^{\frac{-2\pi i k s}{d}}.$$

Proposition ([2]). *For* $L = m$*, any vector* $f \in l^2(\mathbb{Z}_d)$ *can be uniquely recovered from the samples*

$$y_0 = S_m f, \ y_1 = S_m(a * f), \ \dots \ y_m = S_m(a^{m-1} f)$$

if and only if $det(\mathcal{A}_m(k)) \neq 0$ *for every* $k = 0, \dots, J - 1$*, where*

$$\mathcal{A}_m(k) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \hat{a}\,(k) & \hat{a}\,(k+J) & \dots & \hat{a}\,(k+(m-1)J) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{a}^{L-1}\,(k) & \hat{a}^{L-1}\,(k+J) & \dots & \hat{a}^{L-1}\,(k+(m-1)J) \end{pmatrix}.$$

The case when $X = \mathbb{Z}$, $\Omega_0 = \dots = \Omega_{L-1} = m\mathbb{Z}$, $V = V' = l^2(\mathbb{Z})$ and the operators A is given as convolution with some kernel a, $Af = a*f$, has been studied in [3]. Let S_m denote the downsampling operator with rate m, such that $S_m(k)g = g(mk)$, $\forall k \in \mathbb{Z}$. The discrete Fourier transform(DFT) of $g \in l^2(\mathbb{Z})$ is defined as

$$\hat{g}(\xi) = \sum_{s \in \mathbb{Z}} g(s) e^{-2\pi i \xi s}, \ \ \xi \in [0, 1].$$

Put

$$\mathcal{A}_m(\xi) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \hat{a}(\frac{\xi}{m}) & \hat{a}(\frac{\xi+1}{m}) & \dots & \hat{a}(\frac{\xi+m-1}{m}) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{a}^{(L-1)}(\frac{\xi}{m}) & \hat{a}^{(L-1)}(\frac{\xi+1}{m}) & \dots & \hat{a}^{(L-1)}(\frac{\xi+m-1}{m}) \end{pmatrix}.$$

Proposition ([3]). *If $L = m$ then*
(a) the invertibility sampling property is equivalent to the condition $\det \mathcal{A}_m(\xi) \neq 0$ *for a.e* $\xi \in [0,1]$.
(b) the stability sampling property is equivalent to the condition $|\det \mathcal{A}_m(\xi)| > \alpha\}$ *a.e.* $\xi \in [0,1]$ *for some number* $\alpha > 0$.

Although there are convolution operators A that satisfy the assumptions of the above theorem, many natural operators in practice do not satisfy these conditions. For example, when a is real, symmetric and $\hat{a}$ is strictly monotonic on $[0, \frac{1}{2}]$, it can be shown that the matrices $\mathcal{A}_m(0)$ and $\mathcal{A}_m(\frac{1}{2})$ are singular. Note that, even if there is a single singularity the recovery will not be stable, since all the terms in $\mathcal{A}_m(\xi)$ are continuous. To remove the singularities and achieve stable recovery, some extra sampling locations need to be added. Let T_c be the operator that shifts a vector in $\ell^2(\mathbb{Z})$ to the right by c units so that, for $g \in l^2(\mathbb{Z})$, $T_c g(k) = z(k-c)$.

Theorem ([3]). *Suppose $\hat{a}$ is real, symmetric* $[0,1]$ *around* $\frac{1}{2}$, *continuous, and strictly decreasing on* $\left[0, \frac{1}{2}\right]$, *$m$ and J are odd, and* $\Omega = \{1, \dots, \frac{m-1}{2}\}$. *Then any* $f \in l^2(\mathbb{Z})$ *can be uniquely and stably recovered from the samples*

$$\{S_{mJ} T_c\}_{c \in \Omega} \cup \{S_m f, S_m(a * f), \ \dots \ S_m(a^{m-1} f)\}.$$

In [1] authors assume that $\Omega_0 = \dots = \Omega_{L-1}$ and at every $x \in \Omega$ the samples are taken up to some time level $l(x)$: $f(x), Af(x), \dots, A^{l(x)-1}$. For a special choice of $l(x)$ (the degree of minimal annihilator) and finite X they are able to fully describe when the ISP will be satisfied. For $X = \mathbb{N}$, finite Ω_0 and diagonalizable A among many other things they show that the system $\{(A^*)^n \delta_x\}_{x \in \Omega_0}, 0 \leq n < l(x)$ is never a basis or even minimal, and construct a case when it is a frame.

In [5] the operator A is allowed to be unknown as well. The generalization of Prony's method is used to reconstruct the spectrum of A and eventually the function itself.

In the dissertation we consider the dynamical sampling problem for four different settings:
In chapter 1 of the dissertation we let $V = V(\phi)$ be a well-defined shift-invariant space SIS via an appropriately chosen atom ϕ such that the pointwise evaluation is well defined in $V(\phi)$ (Wiener amalgam spaces are a standard choice for working with sampling in shift-invariant spaces [12, 4, 9]). Any element f in V is represented by a semi-discrete convolution, namely $f(t) = \sum_{k \in \mathbb{Z}} c_k \phi(t-k)$ for some $c = (c_n)_{n \in \mathbb{Z}} \in l^2(\mathbb{Z})$, and to recover the function f is equivalent to recovering the coefficients $c = (c_n)_{n \in \mathbb{Z}}$. We consider the case when the evolution operator A, acting on $V' = L^2(\mathbb{R})$, is a convolution with some kernel $a \in L^1(\mathbb{R})$, $Af(t) = a * f(t)$, and the measurements are given by samples by uniform rate m: $\Omega_0 = \dots = \Omega_{L-1} = m\mathbb{Z}$.

If the function $f \in V$ has the expansion $f(t) = \sum_{k \in \mathbb{Z}} c_k \phi(t-k)$ then

$$Af(t) = \sum_{k \in \mathbb{Z}} c(k) A\phi(t-k).$$

A maps $V(\phi)$ onto another SIS $V(a * \phi)$ and the function f and its evolved states $A^n f$ are elements of a respective SIS $A^j V = V(a^n * \phi)$, for $n \geq 0$.

After subsampling each state $A^n f$, we will have the sets of samples

$$y_n(k) = (A^n f(mk))_{k\in\mathbb{Z}}, \quad n = 0, 1, 2, ..., L-1.$$

Let Φ_n be the restriction of $A^n\phi$ to the set of integers: $\Phi_n = A^n\phi|_{\mathbb{Z}}$. Then under some conditions $\Phi_n \in l^1(\mathbb{Z})$ so we can consider the corresponding DFT $\hat{\Phi}_n(\xi)$, $\xi \in \mathbb{T}$. Let

$$\Phi_L(\xi) = \begin{pmatrix} \hat{\Phi}_0(\frac{\xi}{m}) & \hat{\Phi}_0(\frac{\xi+1}{m}) & \dots & \hat{\Phi}_0(\frac{\xi+m-1}{m}) \\ \hat{\Phi}_1(\frac{\xi}{m}) & \hat{\Phi}_1(\frac{\xi+1}{m}) & \dots & \hat{\Phi}_1(\frac{\xi+m-1}{m}) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{\Phi}_{L-1}(\frac{\xi}{m}) & \hat{\Phi}_{L-1}(\frac{\xi+1}{m}) & \dots & \hat{\Phi}_{L-1}(\frac{\xi+m-1}{m}) \end{pmatrix},$$

$$\bar{\mathbf{C}}(\xi) = \begin{pmatrix} \hat{c}\left(\frac{\xi}{m}\right) \\ \hat{c}\left(\frac{\xi+1}{m}\right) \\ \vdots \\ \hat{c}\left(\frac{\xi+m-1}{m}\right) \end{pmatrix} \text{ and } \bar{Y}(\xi) = \begin{pmatrix} \hat{y}_0(\xi) \\ \hat{y}_1(\xi) \\ \vdots \\ \hat{y}_{L-1}(\xi) \end{pmatrix}.$$

Proposition ([26]). *$\Phi_L(\xi)C(\xi) = Y(\xi)$ for $\xi \in [0,1]$.*

If $C(\xi)$ can be solved for every $\xi \in [0,1]$, then we can find the Fourier transform $\hat{c}(\xi)$ of the coefficients $c = (c_n)_{n\in\mathbb{Z}}$ and by taking the Fourier coefficients of $\hat{c}$, find the c itself. To be able to solve $C(\xi)$, we need $L \geq m$.

Corollary. *(a) Let $f \in V = V(\phi)$. We can recover the coefficients sequence $\{c_\lambda\}_{\lambda\in\mathbb{Z}^d}$ in the expansion of f from the collection of samples*

$$y_n = (A^n f(n))_{n\in m\mathbb{Z}} \text{ for } n = 0, 1, \dots, m-1,$$

a.e. invertibilty sampling property is satisfied, if and only $\det(\Phi_m(\xi)) \neq 0$ for a.e. $\xi \in [0,1]$.
(b) the stability sampling property is equivalent to the condition $|\det(\Phi_m(\xi))| > \alpha$ a.e. for some $\alpha > 0$.

Theorem ([26]). *If $\Phi_m(\xi)$ is singular only when $\xi \in \{\xi_i\}_{i\in I}$, where $|I| < \infty$, and J is a positive integer such that $|\xi_i - \xi_j| \neq \frac{k}{J}$ for any $i, j \in I$ and $k \in \{1, \dots, m-1\}$, then any function $f \in V(\phi)$ can be stably recovered from its samples*

$$\left\{(T_c f)|_{(mJ)\mathbb{Z}}\right\} \cup \left\{(a^n * f)|_{m\mathbb{Z}}\right\}_{n=0,1,2,\dots,m-1}.$$

In Chapter 2 of the dissertation we assume the domain, where the sampled functions are defined on, is the direct sum of two cyclic groups, $X = \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2}$, $d_1, d_2 \in \mathbb{N}^+$, and the evolution operator is given as a convolution with a kernel $a = (a_{k,l})_{(k,l)\in X}$:

$$Af(k_1,k_2) = a * f(k_1,k_2) = \sum_{(l_1,l_2)\in X} a_{l_1,l_2} f(k_1 - l_1, k_2 - l_2) \quad \text{for all} \quad (k_1,k_2) \in X.$$

Here $(k_1 - l_1, k_2 - l_2)$ is understood in terms of summation operations in cyclic groups $\mathbb{Z}_{d_1}$ and $\mathbb{Z}_{d_2}$.

We additionally assume that $d_1 = J_1 m_1$, $d_2 = J_2 m_2$, where d_1, d_2 are odd numbers and the initial state f and its temporally evolved states $Af,\ A^2 f, \ldots, A^{L-1} f$ are sampled on a uniform grid $\Omega = m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}$.

Let $S_{m_1,m_2} g = \mathbf{1}_{m_1\mathbb{Z}_{d_1}\times m_2\mathbb{Z}_{d_2}} g$ be the subsampling operator on $m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}$. Our objective is to reconstruct f from the samples set

$$\begin{cases} y_0 = S_{m_1,m_2} f \\ y_1 = S_{m_1,m_2} A f \\ \quad\vdots \\ y_{L-1} = S_{m_1,m_2} A^{L-1} f. \end{cases}$$

Denote by $\hat{g}$ the discrete Fourier transform (DFT) of two dimensional array $g \in l^2\left(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2}\right)$ defined by the following formula:

$$\hat{g}(\kappa_1,\kappa_2) = \sum_{k_1=0}^{d_1-1}\sum_{k_2=0}^{d_2-1} g(k_1,k_2) e^{\frac{-i2\pi\kappa_1 k_1}{d_1}} e^{\frac{-i2\pi\kappa_2 k_2}{d_2}} \quad \text{for all} \quad (\kappa_1,\kappa_2) \in \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2}.$$

We make use of the block-matrices

$$A_{s_2,m_1,m_2}(\kappa_1,\kappa_2) =$$

$$\begin{pmatrix} \hat{a}(\kappa_1,\kappa_2+s_2J_2) & \hat{a}(\kappa_1+1J_1,\kappa_2+s_2J_2) & \cdots & \hat{a}(\kappa_1+(m_1-1)J_1,\kappa_2+s_2J_2) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{a}^{L-1}(\kappa_1,\kappa_2+s_2J_2) & \hat{a}^{L-1}(\kappa_1+1J_1,\kappa_2+s_2J_2) & \ldots & \hat{a}^{L-1}(\kappa_1+(m_1-1)J_1,\kappa_2+s_2J_2) \end{pmatrix},$$

where $s_2 = 0, 1, ..., m_2-1$, to define for all $(\kappa_1,\kappa_2) \in I = \{0,\ldots,J_1-1\}\times\{0,\ldots,J_2-1\}$

$$A_{m_1,m_2}(\kappa_1,\kappa_2) = \left[A_{0,m_1m_2}(\kappa_1,\kappa_2)\ A_{1,m_1m_2}(\kappa_1,\kappa_2)...A_{m_2-1,m_1m_2}(\kappa_1,\kappa_2))\right].$$

For every $(\kappa_1,\kappa_2) \in I$ put $\bar{\mathbf{y}}(\kappa_1,\kappa_2) = [\hat{y}_0(\kappa_1,\kappa_2)\ \ \hat{y}_1(\kappa_1,\kappa_2)\ \ \ldots\ \ \hat{y}_{L-1}(\kappa_1,\kappa_2)]^T$, and let

$$\bar{\mathbf{f}}(\kappa_1,\kappa_2) = \begin{pmatrix} \hat{f}(\kappa_1,\kappa_2) \\ \hat{f}(\kappa_1+1J_1,\kappa_2) \\ \vdots \\ \hat{f}(\kappa_1+(m_1-1)J_1,\kappa_2) \\ \hat{f}(\kappa_1,\kappa_2+J_2) \\ \vdots \\ \hat{f}(\kappa_1+(m_1-1)J_1,\kappa_2+J_2) \\ \vdots \\ \vdots \\ \hat{f}(\kappa_1,\kappa_2+(m_2-1)J_2) \\ \vdots \\ \hat{f}(\kappa_1+(m_1-1)J_1,\kappa_2+(m_2-1)J_2) \end{pmatrix}.$$

Proposition ([27]). *For the $\mathcal{A}_{m_1,m_2}(\kappa_1,\kappa_2)$,$\bar{\mathbf{f}}(\kappa_1,\kappa_2)$ and $\bar{\mathbf{y}}(\kappa_1,\kappa_2)$ as defined above the following matrix equation holds true*

$$\bar{\mathbf{y}}(\kappa_1,\kappa_2) = \frac{1}{m_1m_2}\mathcal{A}_{m_1,m_2}(\kappa_1,\kappa_2)\,\bar{\mathbf{f}}(\kappa_1,\kappa_2).$$

Note that, to be able to recover the vector f, we need to take samples at least $m_1 m_2$ times and, when $L = m_1 m_2$, $\mathcal{A}_{m_1,m_2}(\kappa_1, \kappa_2)$ becomes a square matrix.

Corollary. *For $L = m_1 m_2$, any $f \in l^2(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2})$ can be uniquely recovered from its samples $\{S_{m_1,m_1} A^n f\}_{n=0,\dots,m-1}$ if and only if $\det(\mathcal{A}_{m_1,m_2}(\kappa_1, \kappa_2)) \neq 0$. for every $(\kappa_1, \kappa_2) \in I$.*

Because $\mathcal{A}_{m_1,m_2}(\kappa_1, \kappa_2)$ is a Vandermonde matrix, it is singular at an $(\kappa_1, \kappa_2) \in I$ if and only if

$$\hat{a}(\kappa_1 + s_1 J_1, \kappa_2 + s_2 J_2) = \hat{a}\left(\kappa_1 + s_1' J_1, \kappa_2 + s_2' J_2\right)$$

for some $(s_1, s_2), (s_1', s_2') \in \{0, \cdots, m_1 - 1\} \times \{0, \cdots, m_2 - 1\}$. Hence, taking samples after the first $m_1 m_2$ measurements is not going to add anything new in terms of recovery. In that case, we need to consider adding extra sampling points to overcome the singularities of $\mathcal{A}_{m_1,m_2}(\kappa_1, \kappa_2)$.

If the operator $\mathcal{A}_{m_1,m_2}(\kappa_1, \kappa_2)$ is singular at some $(\kappa_1, \kappa_2) \in I$, we want to be able to find a set

$$\Omega_{add} \subset X \setminus (m_1 \mathbb{Z}_{d_1} \times m_2 \mathbb{Z}_{d_2})$$

such that, for the related sampling operator $S_{\Omega_{add}}$, any function can be uniquely recovered from the samples $\{S_{\Omega_{add}} f, S_{m_1,m_2} f, \cdots, S_{m_1,m_2} A^{m_1 m_2 - 1} f\}$.

We consider one special class of filters for which we are able to explicitly construct an additional sampling set of possible minimal size.

Definition. *The kernel $\hat{a}$ is called (strictly) quadrantally symmetric, if*

$$\hat{a}(\kappa_1, \kappa_2) = \hat{a}(d_1 - \kappa_1, \kappa_2) = \hat{a}(\kappa_1, \kappa_2) = \hat{a}(d_1 - \kappa_1, d_2 - \kappa_2)$$

for all $(\kappa_1, \kappa_2) \in \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2}$, and $\hat{a}(\kappa_1, \kappa_2) \neq \hat{a}(\lambda_1, \lambda_2)$ for any other pairs (λ_1, λ_2).

If, for the kernel a, $\hat{a}$ is quadrantally symmetric then it can be easily verified that (in particular) $\mathcal{A}_{m_1,m_2}(0,0)$ is singular.

Theorem ([27]). *Let the DFT $\hat{a}$ of the kernel a be quadrantally symmetric and let*

$$\Omega_{add} = \left\{1, \cdots, \frac{m_1 - 1}{2}\right\} \times \mathbb{Z}_{d_2} \cup \mathbb{Z}_{d_1} \times \left\{1, \cdots, \frac{m_2 - 1}{2}\right\}.$$

Then, any $f \in l^2(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2})$ can be uniquely recovered from the expanded set of samples

$$\left\{S_{\Omega_{add}} f, S_{m_1,m_2} f, \cdots, S_{m_1,m_2} A^{m_1 m_2 - 1} f\right\}.$$

Moreover, Ω_{add} above has possible minimal size among the sets of additional sampling points which allow unique recovery of every f from the extended set of samples.

In chapter 3 we let the underlying domain be the set $X = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$. The signal of interest $f \in l^2(\mathbb{Z}^2)$ evolves over time under the repeated influence of an evolution operator. The assumption is that the evolution operator A is given as a discrete convolution operator, described by a two-dimensional vector $a \in l^1(\mathbb{Z}^2)$, namely

$$(a * f)(k_1, k_2) = \sum_{(l_1,l_2) \in \mathbb{Z} \times \mathbb{Z}} a_{l_1,l_2} f(l_1 - k_1, l_2 - k_2), \qquad (k_1, k_2) \in \mathbb{Z}^2.$$

The initial state f and its temporally evolved states Af, $A^2f, \ldots, A^{L-1}f$ are subsampled at fixed rates m_1, $m_2 \geq 1$ on the uniform grid $\Omega = m_1\mathbb{Z} \times m_2\mathbb{Z}$. In order to avoid unnecessary technical issuses, we assume that m_1, m_2 both are odd numbers. The samples of the function and its evolved states can be given by the downsampling operator $(S_{m_1,m_2}g)(k_1,k_2) = g(m_1k_1, m_2k_2)$ for $(k_1,k_2) \in \mathbb{Z}^2$. Our objective is to reconstruct f from

$$y_n = S_{m_1,m_2}A^n f, \; n = 0\ldots, L-1.$$

Denote by $\hat{g} \in L^2(\mathbb{T}\times\mathbb{T})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the discrete Fourier transform (DFT) of $g \in l^2\left(\mathbb{Z}^2\right)$, which is just the Fourier series with coefficients f:

$$\hat{g}(\xi_1,\xi_2) = \sum_{(k_1.k_2)\in\mathbb{Z}\times\mathbb{Z}} g(k_1.k_2)e^{-2\pi i k_1\xi_1}e^{-2\pi i k_2\xi_2}, \; (\xi_1,\xi_2) \in \mathbb{T}\times\mathbb{T}.$$

For $s_2 = 0, 1, \cdots, m_2-1$, we define $L\times m_1$ matrices

$$A_{s_2,m_1,m_2}(\xi_1,\xi_2) = \begin{pmatrix} \hat{a}^1\left(\frac{\xi_1}{m_1}, \frac{\xi_2+s_2}{m_2}\right) & \hat{a}^1\left(\frac{\xi_1+1}{m_1}, \frac{\xi_2+s_2}{m_2}\right) & \cdots & \hat{a}^1\left(\frac{\xi_1+m_1-1}{m_1}, \frac{\xi_2+s_2}{m_2}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{a}^{L-1}\left(\frac{\xi_1}{m_1}, \frac{\xi_2+s_2}{m_2}\right) & \hat{a}^{L-1}\left(\frac{\xi_1+1}{m_1}, \frac{\xi_2+s_2}{m_2}\right) & \cdots & \hat{a}^{L-1}\left(\frac{\xi_1+m_1-1}{m_1}, \frac{\xi_2+s_2}{m_2}\right) \end{pmatrix},$$

and denote by $\mathcal{A}_{m_1,m_2}(\xi_1,\xi_1)$ the block matrix

$$[A_{0,m_1,m_2}(\xi_1,\xi_2)\; A_{1,m_1,m_2}(\xi_1,\xi_2) \ldots A_{m_2-1,m_1,m_2}(\xi_1,\xi_2)].$$

Let $\bar{\mathbf{y}}(\xi_1,\xi_2) = (\hat{y}_0(\xi_1,\xi_2)\; \hat{y}_1(\xi_1,\xi_2)\; \ldots \hat{y}_{L-1}(\xi_1,\xi_2)\;)^T$ and $\bar{\mathbf{f}}(\xi_1,\xi_2)$ be the vector

$$\begin{pmatrix} \hat{f}(\frac{\xi_1}{m_1}, \frac{\xi_2}{m_2}) \\ \hat{f}(\frac{\xi_1+1}{m_1}, \frac{\xi_2}{m_2}) \\ \vdots \\ \hat{f}(\frac{\xi_1+m_1-1}{m_1}, \frac{\xi_2}{m_2}) \\ \hat{f}(\frac{\xi_1}{m_1}, \frac{\xi_2+1}{m_2}) \\ \vdots \\ \hat{f}(\frac{\xi_1+m_1-1}{m_1}, \frac{\xi_2+1}{m_2}) \\ \vdots \\ \vdots \\ \hat{f}(\frac{\xi_1}{m_1}, \frac{\xi_2+m_2-1}{m_2}) \\ \vdots \\ \hat{f}(\frac{\xi_1+m_1-1}{m_1}, \frac{\xi_2+m_2-1}{m_2}) \end{pmatrix}.$$

Proposition ([23]). *For the $\mathcal{A}_{m_1,m_2}, \bar{\mathbf{y}}$ and $\bar{\mathbf{f}}$ defined above the following holds true*

$$\bar{\mathbf{y}}(\xi_1,\xi_2) = \frac{1}{m_1m_2}\mathcal{A}_{m_1,m_2}(\xi_1,\xi_2)\bar{\mathbf{f}}(\xi_1,\xi_2).$$

Corollary. *(a) Property ISP is satisfied if and only if* $\mathcal{A}_{m_1,m_2}(\xi,\omega)$ *is full column rank at a.e.* $(\xi_1,\xi_2)\in\mathbb{T}\times\mathbb{T}$.
(b) Let $\sigma_s(\mathcal{A}_{m_1,m_2}(\xi_1,\xi_2))$ *be the smallest singular value of* $\mathcal{A}_{m_1,m_2}(\xi_1,\xi_2)$*. Property SSP is satisfied if, there exists some constant* $\alpha>0$ *such that* $|\sigma_s(\mathcal{A}_{m_1,m_2}(\xi_1,\xi_2))|>\alpha$ *a.e.* $(\xi_1,\xi_2)\in\mathbb{T}^2$.

We can immediately conclude that if the invertibility sampling property holds then $L\geq m_1m_2$. In particular, if $L=m_1m_2$, then $\mathcal{A}_{m_1,m_2}(\xi_1,\xi_2)$ is a square matrix, and we can examine its singularity.

Corollary. *When* $L=m_1m_2$*, the invertibility sampling property is equivalent to the condition:*

$$\det\mathcal{A}_{m_1,m_2}(\xi_1,\xi_2)\neq 0 \textit{ for a.e. } (\xi_1,\xi_2)\in\mathbb{T}\times\mathbb{T}.$$

Since $\mathcal{A}_{m_1,m_2}(\xi_1,\xi_2)$ *has continuous entries* ($a\in l^1(\mathbb{Z}^2)$)*, the stable sampling property is equivalent to*

$$\det\mathcal{A}_{m_1,m_2}(\xi_1,\xi_2)\neq 0 \textit{ for all } (\xi_1,\xi_2)\in\mathbb{T}\times\mathbb{T}.$$

We will assume $L=m_1m_2$. By its structure, $\mathcal{A}_{m_1,m_2}(\xi_1,\xi_2)$ is a Vandermonde matrix, thus it is singular at $(\xi,\omega)\in\mathbb{T}\times\mathbb{T}$ if and only if some of its columns coincide. In case $\mathcal{A}_{m_1,m_2}(\xi_1,\xi_2)$ is singular, no matter how many times we resample the evolved states A^nf, $n>L$, on the grid $\Omega=m_1\mathbb{Z}\times m_2\mathbb{Z}$, the additional data is not going to add anything new in terms of recovery and stability. If $\mathcal{A}_{m_1,m_2}(\xi_1,\xi_2)$ is singular at some (ξ_1,ξ_2) then, by the corollary above, the recovery of $f\in l^2(\mathbb{Z}^2)$ is not stable. To remove the singularities of the matrix $\mathcal{A}_{m_1,m_2}(\xi_1,\xi_2)$ and achieve stable recovery, some extra sampling locations need to be added. The additional sampling locations depend on the position of the singularities of $\mathcal{A}_{m_1,m_2}(\xi,\omega)$ to be removed.

Let the additional sampling set be given of the form

$$\Omega_W=\{m_1\mathbb{Z}\times m_2\mathbb{Z}+(c_1,c_2)\ \mid\ (c_1,c_2)\in W\subset\mathbb{Z}_{m_1}\times\mathbb{Z}_{m_2}\}.$$

Definition. $\hat{a}$ *is said to satisfy the qudrantal symmetry property if* $\hat{a}(\xi_1,\xi_2)=\hat{a}(\omega_1,\omega_2)$ *for* $(\xi_1,\xi_2),(\omega_1,\omega_2)\in\mathbb{T}\times\mathbb{T}=\mathbb{T}^2$ *if and only if one of the following conditions is satisfied:*

1. $\xi_1=\omega_1,\ \xi_2+\omega_2=1$
2. $\xi_1+\omega_1=1,\ \xi_2=\omega_2$
3. $\xi_1+\omega_1=1,\ \xi_2+\omega_2=1.$

Consider the set $W_{add}=W_1\cup W_2$, where

$$W_1=\{1,\cdots\frac{m_1-1}{2}\}\times\{0,\cdots,m_2-1\},$$

$$W_2=\{0,\cdots,m_1-1\}\times\{1,\cdots,\frac{m_2-1}{2}\}\}.$$

Theorem ([23]). *Let* $a\in l^1(\mathbb{Z}^2)$ *be the filter such that the evolution operator is given by* $Ax=a*x$*. Suppose* $\hat{a}$ *satisfies the strictly quadrantal symmetric property,* $\Omega_{W_{add}}$ *is the set of extra sampling locations corresponding to* W_{add} *above. Then any* $f\in l^2(\mathbb{Z}^2)$

can be recovered in a stable way i.e. property SSP is satisfied, from the expanded set of samples

$$\{S_{\Omega_W} f, S_{m_1,m_2} f, S_{m_1,m_2} Af, \cdots, S_{m_1,m_2} A^{m_1 m_2 - 1} f\}.$$

Moreover, $W_a dd$ has the minimal possible size among all $W \subset \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ sets that Ω_W removes the singularities of the matrix $\mathcal{A}_{m_1,m_2}(\xi_1, \xi_1)$.

Finally, in chapter 4 of the dissertation we consider the case when the set X is finite and $V = l^2(X)$. Without loss of generality it can assumed that $X = \{1, \ldots, d\}$, $V = \mathbb{C}^d$ and the operator A is given as a $d \times d$ matrix. Later we will also assume that there are fixed M number of measuring devices, i.e., $|\Omega_i| = M$ for every i, and we are allowed to move them anywhere in X after each measurement. For the classical sampling problem with moving devices in the class of bandilimited functions see [10].

Definition. *We say that a system of subsets of X $(\Omega_0, \ldots, \Omega_{L-1})$ allows recovery if any function f can be uniquely reconstructed from samples*

$$y_0 = f|_{\Omega_0},\ y_1 = (Af)|_{\Omega_1},\ \ldots\ y_{L-1} = (A^{L-1} f)|_{\Omega_{L-1}}.$$

When $|\Omega_n| = 1$ for every n, we call the system a path.

Theorem ([28]). *For given $M \in \mathbb{N}$ there is a system $(\Omega_0, \ldots, \Omega_{L-1})$ of subsets of $\{1, \ldots, d\}$, with $|\Omega_n| = M$ $n = 0, \ldots, L-1$, that allows recovery if and only if $LM \geq d$ and $dim(ker(A)) \leq M$.*

Theorem ([28]). *For a.e. matrix $A \in M_d(\mathbb{C})$ every path of length d allows recovery.*

Theorem ([28]). *For a given matrix A, there is a path that allows recovery if and only if in the Jordan form of A there is at most one Jordan block corresponding to 0-eigenvalue. If the size of that Jordan block is κ, then there are at least $(d-\kappa)!$ paths of length d that allow recovery.*

Theorem ([28]). *For any non-singular matrix A there are at least $d!$ paths that allow recovery. Moreover, there are exactly $d!$ such paths if and only if A is a diagonal matrix up to permutation of rows.*

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Ամփոփագիր

Ատենախոսությունում դիտարկվում է դինամիկ նմուշների միջոցով ֆունկցիաների վերականգնման խնդիրը որոշ տեղաշարժերի նկատմամբ ինվարիանտ տարածություններում: Ատենախոսությունում ստացվել են հետևյալ արդյունքները.

- Տեղաշարժերի նկատմամբ ինվարիանտ տարածություններում ձևակերպվել է դինամիկ նմուշառության խնդիրը և առաջարկվել է նախնական ֆունկցիայի վերականգնման մեթոդ: Ցույց է տրվել, որ այն դեպքում երբ առաջացած մատրիցը շրջելի չէ, ապա լրացուցիչ նմուշների ճիշտ տեղակայման դեպքում հնարավոր է ստանալ միակ և կայուն վերականգնում:

- Այն դեպքում, երբ տարածությունը $l^2\left(\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}\right)$-ն է, նմուշները վերցվում են $m_1\mathbb{Z}_{d_1}\times m_2\mathbb{Z}_{d_2}$ ցանցի վրա և օպերատորը տրվում է $a\in l^2\left(\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}\right)$ կորիզի հետ փաթույթի միջոցով, ստացվել է անհրաժեշտ և բավարար պայման ցանկացած $f\in l^2\left(\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}\right)$ ֆունկցիա իր

  $$f|_{m_1\mathbb{Z}_{d_1}\times m_2\mathbb{Z}_{d_2}},\ a*f|_{m_1\mathbb{Z}_{d_1}\times m_2\mathbb{Z}_{d_2}},\cdots,\ a^{m_1m_2-1}*f|_{m_1\mathbb{Z}_{d_1}\times m_2\mathbb{Z}_{d_2}}$$

  նմուշների միջոցով վերակազնելու համար: Խիստ քառորդային համաչափություն ունեցող կորիզների դեպքում, երբ նման վերակարգնումը հնարավոր չէ, կառուցվել է հնարավոր մինիմալ չափսի լրացուցիչ նմուշառման Ω_{add} բազմություն, որը ապահովում է ցանկացած f ֆունկցիայի միարժեք վերականգնումը

  $$f|_{\Omega_{add}},\ f|_{m_1\mathbb{Z}_{d_1}\times m_2\mathbb{Z}_{d_2}},\ a*f|_{m_1\mathbb{Z}_{d_1}\times m_2\mathbb{Z}_{d_2}},\cdots,\ a^{m_1m_2-1}*f|_{m_1\mathbb{Z}_{d_1}\times m_2\mathbb{Z}_{d_2}}$$

  ընդլայնված նմուշների միջոցով:

- $l^2\left(\mathbb{Z}\times\mathbb{Z}\right)$ տարածության դեպքում, երբ նմուշները վերցված են $m_1\mathbb{Z}\times m_2\mathbb{Z}$ ցանցի վրա և օպերատորը տրված է որպես $a\in l^1\left(\mathbb{Z}\times\mathbb{Z}\right)$ կորիզի հետ փաթույթ, ստացվել է անհրաժեշտ և բավարար պայման, որի դեպքում ցանկացած $f\in l^2\left(\mathbb{Z}\times\mathbb{Z}\right)$ ֆունկցիա հնարավոր է վերականգնել իր

  $$f|_{m_1\mathbb{Z}\times m_2\mathbb{Z}},\ a*f|_{m_1\mathbb{Z}\times m_2\mathbb{Z}},\cdots,\ a^{m_1m_2-1}*f|_{m_1\mathbb{Z}\times m_2\mathbb{Z}}$$

  նմուշների միջոցով: Խիստ քառորդային համաչափություն ունեցող կորիզների դեպքում, երբ կայուն վերականգնումը հնարավոր չէ, կառուցվել է հնարավոր մինիմալ չափսի $W\subset\mathbb{Z}_{m_1}\times\mathbb{Z}_{m_2}$ բազմություն, այպես որ լրացուցիչ նմուշառության

  $$\Omega_W=\{m_1\mathbb{Z}\times m_2\mathbb{Z}+(c_1,c_2):\ \ (c_1,c_2)\in W\subset\mathbb{Z}_{m_1}\times\mathbb{Z}_{m_2}\}$$

  բազմությունը ապահովում է կամայական f ֆունկցիայի կայուն վերակազնումն իր

  $$f|_{\Omega_W},\ f|_{m_1\mathbb{Z}\times m_2\mathbb{Z}},\ a*f|_{m_1\mathbb{Z}\times m_2\mathbb{Z}},\cdots,\ a^{m_1m_2-1}*f|_{m_1\mathbb{Z}\times m_2\mathbb{Z}}$$

  ընդլայյված նմուշների միջոցով:

- Այն դեպքում. երբ փիրույթը վերջավոր է և ժամանակի տարբեր պահերին փիրույթի տարբեր ենթաբազմությունների վրա են ընտրանքերը վերցվում, ստացվել է անհրաժեշտ և բավարար պայման այնպիսի բազմությունների համակարգի գոյության համար, որոնցում վերցված ընտրանքները թույլ են տալիս ցանկացած ֆունկցիայի միարժեք վերականգնումը: Նմուշառության մի կետանոց բազմությունների դեպքում գտնվել է բոլոր տեղաբաշխումների մինիմալ քանակը, որոնք ապահովում են կամայական ֆունկցիայի վերակագնումը:

Аннотация

В диссертации рассматривается проблема восстановления функции по ее динамическим выборкам в некоторых пространствах инвариантных относительно сдвигов. Доказываются следующие результаты:

- формулирована проблема динамических выборок в пространствах инвариантных относительно сдвигов, и получен метод для восстановления функции по ее временно пространственным выборкам. Когда полученная матрица являться вырожденной, показано что при правильном расположении дополнительных выборок можно достичь стабильного восстановления.

- в пространстве $l^2(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2})$, когда выборки берутся на $m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}$ и оператор ответственный за изменение функции есть свертка с ядром $a \in l^2(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2})$, получено необходимое и достаточное условие при котором каждая функция $f \in l^2(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2})$ можно восстановить по ее выборкам

$$f|_{m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}},\ a * f|_{m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}}, \cdots,\ a^{m_1 m_2 - 1} * f|_{m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}}.$$

  В случае ядра со строгой квадратичной симметрией, когда единственное восстановление не возможно, построено множество Ω_{add} дополнительных выборок с минимальным размером, которое позволяет восстановление любой функции по ее расширенным выборкам

$$f|_{\Omega_{add}},\ f|_{m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}},\ a * f|_{m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}}, \cdots,\ a^{m_1 m_2 - 1} * f|_{m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}}.$$

- в пространстве $l^2(\mathbb{Z} \times \mathbb{Z})$, когда выборки даны на $m_1\mathbb{Z} \times m_2\mathbb{Z}$ и оператор ответственный за изменения функции есть свертка с ядром $a \in l^1(\mathbb{Z} \times \mathbb{Z})$, получено необходимое и достаточное условие для восстановления любой функции $f \in l^2(\mathbb{Z} \times \mathbb{Z})$ по ее выборкам

$$f|_{m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}},\ a * f|_{m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}}, \cdots,\ a^{m_1 m_2 - 1} * f|_{m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}}.$$

  В случае ядра с квадратичной симметрий, когда каждая функция определяется единственным образом, но восстановление не являться стабильным, построено множество $W \subset \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ минимального размера, такое что множество дополнительных выборок

$$\Omega_W = \{m_1\mathbb{Z} \times m_2\mathbb{Z} + (c_1, c_2) : \ (c_1, c_2) \in W \subset \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}\},$$

позволяет стабильное восстановление любой функции по ее расширенным выборкам

$$f|_{\Omega_W},\ f|_{m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}},\ a * f|_{m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}}, \cdots,\ a^{m_1 m_2 - 1} * f|_{m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2}}.$$

- если область определения конечна и допускается что множества на которых даны отчеты могут меняться со временем, найдено необходимое и достаточное условие для существования системы подмножеств области на которых отчеты однозначно определяют любую функцию. В случае когда множество выборок состоит из одного элемента, которое может меняться со временем, найдено минимальное число конфигураций которые допускают восстановление любой функций.