ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Վահագն Կարենի Վարդանյան

Ուղիղների օգտագործումը GC բազմություններում

Ա 01.01 «Մաթեմատիկական անալիզ» մասնագիտությամբ
ֆիզիկամաթեմատիկական գիտությունների թեկնածուի
գիտական աստիճանի հայցման ատենախոսության

ՍԵՂՄԱԳԻՐ

ԵՐԵՎԱՆ – 2019

YEREVAN STATE UNIVERSITY

Vahagn Vardanyan

Usage of lines in GC sets

SYNOPSIS

Of dissertation for requesting the degree of candidate of
physical and mathematical sciences specializing in
A.01.01 - “Mathematical Analysis”

YEREVAN - 2019

Ատենախոսության թեման հաստատվել է ՀՀ ԳԱԱ Մաթեմատիկայի ինստիտուտում

Գիտական ղեկավար՝ ֆիզ. մաթ. գիտ. դոկտոր Հ. Ա. Հակոբյան

Պաշտոնական ընդդիմախոսներ՝ ֆիզ. մաթ. գիտ. դոկտոր Ղ.Ս. Ղազարյան
ֆիզ. մաթ. գիտ. դոկտոր Ա.Ա. Սահակյան

Առաջատար կազմակերպություն՝ Հայաստանի ազգային պոլիտեխնիկական համալսարան

Պաշտպանությունը կայանալու է 2019թ. հուլիսի 2-ին, ժամը 15:00-ին, ԵՊՀ-ում գործող ՀՀ ԲՈԿ-ի 050 մասնագիտական խորհրդի նիստում, հետևյալ հասցեով՝ 0025, Երևան, Ա. Մանուկյան 1:

Ատենախոսությանը կարելի է ծանոթանալ ԵՊՀ գրադարանում:
Սեղմագիրը առաքված է 2019թ. մայիսի 20-ին:

Մասնագիտական խորհրդի գիտական քարտուղար
ֆիզ. մաթ. գիտ. դոկտոր Տ.Ն. Հարությունյան

The topic of dissertation is approved in the Institute of Mathematics NAS RA

Scientific adviser: doctor of phys. math. sciences H. A. Hakopian

Official opponents: doctor of phys. math. sciences K.S. Kazarian
doctor of phys. math. sciences A.A. Sahakian

Leading institution: National Polytechnic University of Armenia

The dissertation defense will take place on July 2nd, 2019, at 15:00, during the specialized meeting of the Higher Attestation Commission Council 050 at YSU (1 A. Manukyan, Yerevan 0025, Armenia).

The dissertation is available in the library of Yerevan State University.
The Synopsis was sent on May 20th, 2019.

Scientific secretary of specialized council
doctor of phys. math. sciences T.N. Harutyunyan

Actuality of the subject. It is a classic problem in mathematics to estimate the value of a function at certain points from known values at other points. The process of reconstructing a function, curve, surface from certain known data is called interpolation. In some sense, polynomials are the simplest type of interpolants to work with, as their definition only involves a finite number of additions, subtractions, and multiplications. Also they can be easily differentiated or integrated.

Univariate polynomial interpolation is a classical subject, with a long history and a well settled theory. It dates back to the Newton and the Lagrange fundamental solutions of the interpolation problem.

Compared to that, the multivariate counterpart is much more complicated. It has only been systematically considered in the second half of the 20th century due to the development of computers. Another reason for the interest of multivariate problems was the emergence of new mathematical methods, as cubature formulae, finite element methods. Another connection is Algebraic Geometry, since the solvability of a multivariate interpolation problem relies on the fact that the interpolation points do not lie on an algebraic surface of a certain degree.

These days, applications of multivariate polynomial interpolation range over many different fields of pure and applied mathematics. Interpolation theory finds applications in many problems, as numerical differentiation and integration, numerical solution of differential equations, evaluation of transcendental functions, typography, and the computer-aided geometric design of cars, ships, and airplanes.

Purpose and goals of the thesis. The thesis consists of two parts. The first part is dedicated to the factorization of the fundamental polynomials of two and three variables.

In the second part we provide an adjustment to the formulation and then prove a conjecture proposed by V. Bayramyan and H. Hakopian in a recent paper. The conjecture characterizes the number of usages of a line in a GC set, provided that the Gasca-Maeztu conjecture is true.

The object of research. Multivariate polynomial spaces, n-independent and n-poised sets, GC_n sets, maximal lines, Gasca-Maeztu conjecture, general principal lattices, Chung-Yao lattices, Carnicer-Gasca lattices.

The methods of research. The methods of univariate and multivariate polynomial interpolations are used. Also some methods of linear algebra and algebraic geometry are used.

Scientific novelty. Necessary and sufficient conditions are provided for the factorization of fundamental polynomials of node sets in $\mathbb{R}^2$ of cardinality $\leq 2n + [n/2] + 1$ and of node sets in $\mathbb{R}^3$ of cardinality $\leq 3n + 1$. A new simple proof of the Gasca-Maeztu conjecture for the case of $n = 4$ is provided. Then a correct formulation of a property established by V. Bayramyan and H. Hakopian on the usage of n-node lines in GC_n sets is provided and then proved. Finally, an adjustment to the formulation of a conjecture proposed by V. Bayramyan and H. Hakopian on the usage of k-node lines in GC_n sets, $2 \leq k \leq n + 2$, is provided and then proved. Also counterexamples for the cases when the lines are not used at all are presented.

Practical significance. Main results in the thesis are of theoretical nature but at the same time some of them can have practical application. For instance finding fundamental polynomials in the simplest possible form is significant from the point of view of applications.

The characterization of independent and poised sets can be used in the mathematical problems where multivariate polynomial interpolation is considered.

The following provisions are presented for the defence.

- Necessary and sufficient conditions are provided for the factorization of fundamental polynomials of node sets in $\mathbb{R}^2$ of cardinality not exceeding $2n + [n/2] + 1$ as a product of factors of at most second degree.

- Independence of node sets with $3n+1$ nodes in $\mathbb{R}^3$. Necessary and sufficient conditions are provided for the factorization of fundamental polynomials of such node sets as a product of linear factors.

- A correction of a property on the usage of n-node lines in GC_n sets established by V. Bayramyan and H. Hakopian.

- An adjustment of the formulation of a conjecture proposed by V. Bayramyan and H. Hakopian on the usage of k-node lines in GC_n sets, $2 \leq k \leq n+2$, is provided and then proved. Namely, by assuming that the Gasca-Maeztu conjecture is true, we prove that any k-node line ℓ is not used at all, or it is used by exactly $\binom{s}{2}$ nodes, where s satisfies the condition $2k - n - 1 \leq s \leq k$.

The approbation of obtained results. The results of the thesis were reported in

- the scientific seminars held in the department of Numerical Analysis of the Faculty of Informatics and Applied Mathematics of Yerevan State University,

- the International Conference Dedicated to 90th Anniversary of Sergey Mergelyan, 20-25 May, 2018, Yerevan, Armenia, Abstracts, pp. 37-38 ([5*]).

- the Emil Artin International Conference, Dedicated to 120th Anniversary Emil Artin, 29 May - 02 June, 2018, Yerevan, Armenia, Abstracts, p. 67 ([6*]).

- the International Conference Harmonic Analysis and Approximations VII, Dedicated to 90th Anniversary of Alexandr Talalyan, 16-22 September, 2018, Tsaghkadzor, Armenia, Abstracts, pp. 45-47 ([7*]).

Publications. The results of the thesis were published in 4 scientific articles and reported in 3 international conferences which we bring at the end of the Synopsis.

The structure and the content of thesis. The thesis consists of introduction, two parts each of which contains three chapters, summary and bibliography. The puplications of the author are [1*] - [7*]. The paper [8*] is accepted for publication. The number of references is 32. The content of the thesis is 101 pages.

THE CONTENT OF THE THESIS

In Chapter 1 we present univariate and multivariate interpolation and some basic known facts.

In Section 1.2 we present multivariate interpolation problem. Denote

$$\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d,\ \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d,\ \mathbf{x}^\alpha = {x_1}^{\alpha_1} \cdot \cdots \cdot {x_d}^{\alpha_d},\ |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

Let Π_n^d be the space of polynomials of d variables of total degree at most n :

$$\Pi_n^d = \left\{ \sum_{|\alpha| \leq n} a_\alpha \mathbf{x}^\alpha : a_\alpha \in \mathbb{R}^d \right\}.$$

The dimension of this space is given by $N(n, d) := \dim \Pi_n^d = \binom{n+d}{d}$.

Below we state the Lagrange multivariate interpolation problem. Suppose we have a set of s distinct nodes $\mathcal{X}_s = \{\mathbf{x}_1, \ldots, \mathbf{x}_s\} \subset \mathbb{R}^d$ and s arbitrary values $c_1, \ldots c_s$. The problem of finding a (unique) polynomial satisfying the following conditions:

$$p(\mathbf{x}_i) = c_i, \quad i = 1, \ldots s, \tag{1.2.1}$$

is called interpolation problem. The conditions (1.2.1) are called interpolation conditions.

First, we define fundamental polynomials and independent sets. A polynomial $p \in \Pi_n^d$ is called an n-fundamental polynomial for a node $A = \mathbf{x}_k \in \mathcal{X}_s$ if

$$p(\mathbf{x}_i) = \delta_{ik}, \quad i = 1, \ldots, s,$$

where δ is the Kronecker symbol.

Definition 1.2.1. A set of nodes $\mathcal{X}$ is called *n-independent* if all its nodes have fundamental polynomials. Otherwise, $\mathcal{X}$ is called *n-dependent.*

Fundamental polynomials are linearly independent. Therefore, a necessary condition of n-independence is $|\mathcal{X}| \leq N(n, d)$.

Definition 1.2.2. The set of nodes $\mathcal{X}_s$ is called *n-poised* if for any data $\{c_1, \ldots, c_s\}$ there exists a unique polynomial $p \in \Pi_n^d$, satisfying the conditions (1.2.1).

A necessary condition for n-poisedness of the set $\mathcal{X}_s$ is $s = |\mathcal{X}_s| = N(n, d)$.

We have that a set $\mathcal{X}_s$, with $s = N(n, d)$ is n-poised if and only if it is n-independent.

In Section 1.2.2 we present basic known results on bivariate interpolation.

For the brevity we denote the space of bivariate polynomials of total degree at most n by $\Pi_n := \Pi_n^2$. Similarly we set $N := N(n, 2) = \binom{n+2}{2}$.

Proposition 1.2.3. The set of nodes $\mathcal{X}_N$ is n-poised if and only if the following condition holds:

$$p \in \Pi_n,\ p(x_i, y_i) = 0,\ i = 1, \ldots, N \implies p = 0. \tag{1.2.4}$$

A plane algebraic curve is the zero set of some bivariate polynomial. To simplify notation, we shall use the same letter, say p, to denote the polynomial p of degree $n \geq 1$ and the curve (of the same degree) given by the equation $p(x, y) = 0$. More precisely, suppose p is a polynomial without multiple factors. Then the algebraic curve defined by the equation $p(x, y) = 0$ shall also be denoted by p. So lines, conics, and cubics are equivalent to polynomials of degree 1, 2, and 3, respectively; a reducible conic is a pair of lines, and a reducible cubic is a triple of lines or consists of a line and an irreducible conic. We denote lines, conics and cubics by α, β and γ, respectively.

We get from Proposition 1.2.3 the following geometric interpretation of n-poisedness.

Proposition 1.2.4. The set of nodes $\mathcal{X}_N$ is not n-poised if and only if there is an algebraic plane curve of degree n passing through all the nodes of $\mathcal{X}_N$.

As it follows from Proposition 1.2.4 the construction of a set $\mathcal{X}_N$ which is not n-poised is very easy. One just needs to choose an algebraic curve p of degree n and locate all the nodes of $\mathcal{X}_N$ in that curve. While the construction of an n-poised set $\mathcal{X}_N$ is more difficult, since the n-poisedness means that there is no curve p of degree n passing through all the nodes of $\mathcal{X}_N$.

A general construction of n-poised sets was introduced by Berzolari [3] and Radon [23]. This construction is described in Section 1.2.2.

In Chapter 2 we start the presentation of the results of this thesis.

First the factorization of fundamental polynomials is studied. In the case of univariate interpolation fundamental polynomials can be presented as products of linear factors. But in the case of bivariate interpolation this is not always possible. Here we characterize n-independent node sets for which all fundamental polynomials are products of lines or conics.

Theorem 2.1.1 ([2*])**.** Let $\mathcal{X}$ be an n-independent set of nodes with $|\mathcal{X}| \leq 2n + 1$. Then for each node of $\mathcal{X}$ there is an n-fundamental polynomial, which is a product of lines. Moreover, this statement is not true in general for n-independent node sets $\mathcal{X}$ with $|\mathcal{X}| \geq 2n + 2$.

The first part follows from the following proposition, which covers a wider setting.

Proposition 2.1.2 ([2*])**.** Let $\mathcal{X}$ be a set of nodes with $|\mathcal{X}| \leq 2n + 1$ and $A \in \mathcal{X}$. Then the following three statements are equivalent:

i) The node A has an n-fundamental polynomial;

ii) The node A has an n-fundamental polynomial, which is a product of linear factors;

iii) No $n + 1$ nodes of $\mathcal{X} \setminus \{A\}$ are collinear together with the node A.

In Section 2.3 we present the main result of Chapter 2:

Theorem 2.3.1 ([2*])**.** Let $\mathcal{X}$ be an n-independent set of nodes with $|\mathcal{X}| \leq 2n + [n/2] + 1$. Then for each node of $\mathcal{X}$ there is an n-fundamental polynomial, which is a product of lines and conics. Moreover, this statement is not true in general for n-independent node sets $\mathcal{X}$ with $|\mathcal{X}| \geq 2n + [n/2] + 2$ and $n \geq 3$.

We provide a counterexample to prove the "Moreover" part of this Theorem. The following proposition covers a wider setting.

Proposition 2.3.2 ([2*]). Let $\mathcal{X}$ be a set of nodes with $|\mathcal{X}| \le 2n + [n/2] + 1$ and $A \in \mathcal{X}$. Then the following three statements are equivalent:

i) The node A has an n-fundamental polynomial;

ii) The node A has an n-fundamental polynomial, which is a product of lines and conics;

iii) a) No $n+1$ nodes of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

b) If $n+1$ nodes of $\mathcal{X} \setminus \{A\}$ are collinear and are lying in a line α then no n nodes of $\mathcal{X} \setminus (A \cup \alpha)$ are collinear together with A;

c) No $2n+1$ nodes of $\mathcal{X} \setminus \{A\}$ are lying on an irreducible conic together with A.

In Chapter 3 we present our results concerning factorization of trivariate fundamental polynomials. We provide the following

Proposition 3.1.1 ([3*]). Let $\mathcal{X}$ be a set of knots in $\mathbb{R}^2$ with $|\mathcal{X}| = 3n - k$, where $k, n \ge 1$ and $A \in \mathcal{X}$. Suppose that the following three conditions hold:

i) No $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

ii) If $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear and are lying in a line α, $A \notin \alpha$, then no n knots of $\mathcal{X} \setminus \alpha$ are collinear together with A;

iii) No $2n+1$ knots of $\mathcal{X} \setminus \{A\}$ belong to an irreducible conic together with A.

Then there exists a fundamental polynomial of A of form $p^{\star}_{A,\mathcal{X}} = \alpha_1\alpha_2...\alpha_k q$, where $\alpha_1, ..., \alpha_k$ are lines and $q \in \Pi_{n-k}$.

We denote by $\Pi_n(L)$ the set of restrictions of polynomials of total degree at most n on a plane L. The generalization is the following.

Bellow we present the main results of Chapter 3.

Proposition 3.2.2 ([3*]). Let $\mathcal{X}$ be a set of knots in $\mathbb{R}^3$ with $|\mathcal{X}| \le 3n + 1$ and $A \in \mathcal{X}$. Then the knot A has an n-fundamental polynomial, which is a product of linear factors, if and only if the following two assertions hold:

i) No $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

ii) If at least $2n+1$ of $\mathcal{X} \setminus \{A\}$ knots are lying in a plane L_A passing through A then all the knots of $\mathcal{X} \cap L_A$ different from A lie in n lines not passing through A.

In Section 3.3 we prove the following

Theorem 3.3.1 ([3*]). Let $\mathcal{X}$ be a set of non-coplanar knots in $\mathbb{R}^3$ with $|\mathcal{X}| \le 3n + 1$. Then $\mathcal{X}$ is n-independent if and only if the following three statements hold:

i) No $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

ii) There are no $2n+2$ coplanar knots of $\mathcal{X}$, which belong to a conic(reducible or irreducible);

iii) If $3n$ knots belong to a plane L, then there are no curves $\sigma_n \in \Pi_n(L)$ and $\gamma \in \Pi_3(L)$ such that $\mathcal{X} \cap L = \sigma_n \cap \gamma$.

The statement of Theorem follows from the following result which covers a wider setting.

Proposition 3.3.2 ([3*])**.** Let $\mathcal{X}$ be a set of non-coplanar knots in $\mathbb{R}^3$ with $|\mathcal{X}| \leq 3n+1$. Then a knot $A \in \mathcal{X}$ has an n-fundamental polynomial if and only if the following four statements hold:

i) No $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

ii) If $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear and are lying in a line α, $A \notin \alpha$, then no n knots of $\mathcal{X} \setminus \alpha$ are collinear together with A;

iii) If $2n+1$ knots of $\mathcal{X} \setminus \{A\}$ are coplanar together with A and belong to a plane L, then there is no conic in the plane passing through A and $2n+1$ other knots of $\mathcal{X} \cap L$;

iv) If A and $3n-1$ other knots belong to a plane L, then there are no curves $\sigma_n \in \Pi_n(L)$ and $\gamma \in \Pi_3(L)$ such that $\mathcal{X} \cap L = \sigma_n \cap \gamma$.

Next we present the following

Corollary 3.3.4 ([3*])**.** Let $\mathcal{X}$ be a set of knots in $\mathbb{R}^3$ with $|\mathcal{X}| \leq 3n+1$. Then $\mathcal{X}$ is n-independent if and only if for any plane L the set $\mathcal{X} \cap L$ is n-independent.

Part II of the thesis is devoted to GC_n sets, i.e, n-poised sets where each node possesses a fundamental polynomial which is a product of n lines.

In Chapter 4 we define GC_n sets, present their classification and bring some known results.

Definition 4.1.1. Given an n-poised set $\mathcal{X}$. We say that a node $A \in \mathcal{X}$ *uses a line* $\ell \in \Pi_1$, if $p_A^\star = \ell q,$ where $q \in \Pi_{n-1}$.

Next we bring the following proposition concerning the factorization of polynomials vanishing at some points of lines

Proposition 4.1.2. Suppose that a polynomial $p \in \Pi_n$ vanishes at $n+1$ points of a line ℓ. Then we have that $p = \ell r,$ where $r \in \Pi_{n-1}$.

As it follows from Proposition 4.1.2, at most $n+1$ nodes of an n-poised set $\mathcal{X}$ can be collinear. Thus we arrive to the following

Definition 4.1.3 ([4])**.** A line passing through $n+1$ nodes is called a *maximal line.*

Clearly, in view of Proposition 4.1.2 a maximal line λ is used by all the nodes in $\mathcal{X} \setminus \lambda$.

In Section 4.1, we define GC sets introduced by K.C. Chung and T.H. Yao.

Definition 4.1.6 ([14])**.** An n-poised set $\mathcal{X}$ is called GC_n *set* (or GC set) if the n-fundamental polynomial of each node $A \in \mathcal{X}$ is a product of n linear factors.

So, GC_n sets are n-poised sets such that each of its nodes uses exactly n lines.

Next, the Gasca-Maeztu conjecture, briefly called GM conjecture is presented:

Conjecture 4.1.7 ([16], Sect. 5)**.** Any GC_n set possesses a maximal line.

Until now, this conjecture has been confirmed to be true for the degrees $n \leq 5$ (see [5], [19]). For a generalization of the Gasca-Maeztu conjecture to maximal curves see [20].

The following important result is due to Carnicer and Gasca.

Theorem 4.1.8 ([8], Thm. 4.1)**.** If the Gasca-Maeztu conjecture is true for all $k \leq n$, then any GC_n set possesses at least three maximal lines.

This yields, in view of Proposition 4.1.2, that each node of a GC_n set $\mathcal{X}$ uses at least one maximal line.

Denote by $\mu = \mu(\mathcal{X})$ the number of maximal lines of a node set $\mathcal{X}$. Thus, we have for any GC_n set $\mathcal{X}$:

$$3 \leq \mu(\mathcal{X}) \leq n + 2, \tag{4.1.2}$$

where for the first inequality it is assumed that GM conjecture is true.

In Section 4.2, we start consideration of the results of Carnicer, Gasca, and Godés, concerning the classification of GC_n sets according to the number of maximal lines the sets possess. Let us start with

Theorem 4.2.1 ([12])**.** Let $\mathcal{X}$ be a GC_n set with $\mu(\mathcal{X})$ maximal lines. Suppose also that GM conjecture is true for the degrees not exceeding n. Then $\mu(\mathcal{X}) \in \{3, n-1, n, n+1, n+2\}$.

In this chapter we consider the classification of GC_n sets for the following three cases:

1. Lattices with $n+2$ maximal lines - the Chung-Yao natural lattices.

Let a set $\mathcal{M}$ of $n+2$ lines be in general position, i.e., no two lines are parallel and no three lines are concurrent, $n \geq 0$. Then the Chung-Yao set is defined as the set $\mathcal{X}$ of all $\binom{n+2}{2}$ intersection points of these lines. We have that the $n+2$ lines of $\mathcal{M}$ are maximal for $\mathcal{X}$. Each fixed node here is lying in exactly 2 lines and does not belong to the remaining n lines. Observe that the product of the latter n lines gives the fundamental polynomial of the fixed node. Thus $\mathcal{X}$ is a GC_n set. Let us mention that any n-poised set $\mathcal{X}$, with $\mu(\mathcal{X}) = n+2$, clearly forms a Chung-Yao lattice. Recall that there are no n-poised sets with more maximal lines (Proposition 4.1.5, (iii)).

2. Lattices with $n+1$ maximal lines - the Carnicer-Gasca lattices.

Let a set $\mathcal{M}$ of $n+1$ lines be in general position, $n \geq 2$. Then the Carnicer-Gasca lattice $\mathcal{X}$ is defined as $\mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)}$, where $\mathcal{X}^{(2)}$ is the set of all intersection nodes of these $n+1$ lines, and $\mathcal{X}^{(1)}$ is a set of other $n+1$ non-collinear nodes, one in each line, to make the line maximal. We have that $|\mathcal{X}| = \binom{n+1}{2} + (n+1) = \binom{n+2}{2}$. It is easily seen that $\mathcal{X}$ is a GC_n set and has exactly $n+1$ maximal lines, i.e., the lines of $\mathcal{M}$. Let us mention that any n-poised set $\mathcal{X}$, with $\mu(\mathcal{X}) = n+1$, clearly forms a Carnicer-Gasca lattice (see [6], Proposition 2.4).

3. Lattices with n maximal lines.

Let a set $\mathcal{M}$ of n lines be in general position, $n \geq 3$. Then consider the lattice $\mathcal{X}$ defined as

$$\mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)} \cup \mathcal{X}^{(0)}, \tag{4.2.1}$$

where $\mathcal{X}^{(2)}$ is the set of all intersection nodes of these n lines, $\mathcal{X}^{(1)}$ is a set of other $2n$ nodes, two in each line, to make the line maximal and $\mathcal{X}^{(0)}$ consists of a single node, denoted by O, which does not belong to any line from $\mathcal{M}$. Correspondingly, we have that $|\mathcal{X}| = \binom{n}{2} + 2n + 1 = \binom{n+2}{2}$.

At the end of Section 4.2, we bring the following characterization of GC_n set $\mathcal{X}$, with $\mu(\mathcal{X}) = n$, due to Carnicer and Gasca.

Proposition 4.2.2 ([6], Prop. 2.5)**.** A node set $\mathcal{X}$ is a GC_n set with the set of maximal lines $\mathcal{M}$, $|\mathcal{M}| = n$, if and only if the representation (4.2.1) holds with the following additional properties:

(i) There are 3 lines $\ell_1^o, \ell_2^o, \ell_3^o$ concurrent at the node O: $O = \ell_1^o \cap \ell_2^o \cap \ell_3^o$ such that $\mathcal{X}^{(1)} \subset \ell_1^o \cup \ell_2^o \cup \ell_3^o$;

(ii) No line ℓ_i^o, $i = 1, 2, 3$, contains $n+1$ nodes of $\mathcal{X}$.

The remaining two cases, namely the cases $\mu(\mathcal{X}) = n-1$ and $\mu(\mathcal{X}) = 3$, are presented later in Chapter 6.

We define $\mathcal{N}_\ell$ and $\mathcal{X}_\ell$ sets and present some known results which will be used in the sequel.

Definition 4.3.1 ([7])**.** Given an n-poised set $\mathcal{X}$ and a line ℓ. Then

(i) $\mathcal{X}_\ell$ is the subset of nodes of $\mathcal{X}$ which use the line ℓ;

(ii) $\mathcal{N}_\ell$ is the subset of nodes of $\mathcal{X}$ which do not use the line ℓ and do not lie in ℓ.

It is easy to see that

$$\mathcal{X}_\ell \cup \mathcal{N}_\ell = \mathcal{X} \setminus \ell. \tag{4.3.1}$$

Note that the previously mentioned statement on maximal lines can be expressed as follows

$$\mathcal{X}_\ell = \mathcal{X} \setminus \ell, \text{ if } \ell \text{ is a maximal line.} \tag{4.3.2}$$

Suppose that λ is a maximal line of $\mathcal{X}$ and $\ell \neq \lambda$ is any line. Then we have that

$$\mathcal{X}_\ell \setminus \lambda = (\mathcal{X} \setminus \lambda)_\ell. \tag{4.3.3}$$

Let $\mathcal{X}$ be an n-poised set and ℓ be a line with $|\ell \cap \mathcal{X}| \leq n$. We call a maximal line λ *ℓ-disjoint* if

$$\lambda \cap \ell \cap \mathcal{X} = \emptyset. \tag{4.3.4}$$

Let $\mathcal{X}$ be an n-poised set and ℓ be a line with $|\ell \cap \mathcal{X}| \leq n$. We call two maximal lines λ', λ'' *ℓ-adjacent* if

$$\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}. \tag{4.3.6}$$

Next, we introduce the concept of an ℓ-reduction of a GC_n set and ℓ-proper GC_m subsets.

Definition 4.3.5. Let $\mathcal{X}$ be a GC_n set, ℓ be a k-node line, $k \geq 2$. We say that a set $\mathcal{Y} \subset \mathcal{X}$ is an ℓ-reduction of $\mathcal{X}$, and briefly denote this by $\mathcal{X} \searrow_\ell \mathcal{Y}$, if

$$\mathcal{Y} = \mathcal{X} \setminus (\mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_s),$$

where

(i) $\mathcal{C}_0$ is an ℓ-disjoint maximal line of $\mathcal{X}$, or $\mathcal{C}_0$ is the union of a pair of ℓ-adjacent maximal lines of $\mathcal{X}$;

(ii) $\mathcal{C}_i$ is an ℓ-disjoint maximal line of the GC set $\mathcal{Y}_i := \mathcal{X} \setminus (\mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{i-1})$, or $\mathcal{C}_i$ is the union of a pair of ℓ-adjacent maximal lines of $\mathcal{Y}_i,\ i = 1, \ldots s$;

(iii) ℓ passes through at least 2 nodes of $\mathcal{Y}$.

Definition 4.3.6. Let $\mathcal{X}$ be a GC_n set, ℓ be a k-node line, $k \geq 2$. We say that the set $\mathcal{X}_\ell$ is an ℓ-proper GC_m subset of $\mathcal{X}$ if there is a GC_{m+1} set $\mathcal{Y}$ such that

(i) $\mathcal{X} \searrow_\ell \mathcal{Y}$;

(ii) The line ℓ is a maximal line in $\mathcal{Y}$.

The following result immediately follows from Definitions 4.3.5 and 4.3.6.

Proposition 4.3.7. Suppose that $\mathcal{X}$ is a GC_n set. If $\mathcal{X} \searrow_\ell \mathcal{Y}$ and $\mathcal{Y}_\ell$ is an ℓ-proper GC_m subset of $\mathcal{Y}$ then $\mathcal{X}_\ell$ is an ℓ-proper GC_m subset of $\mathcal{X}$.

At the end of Chapter 4 we present a simple proof of the Gasca-Maeztu conjecture for the case $n = 4$. The Conjecture was proposed in 1981 by Gasca and Maeztu [16]. Until now, this has been confirmed only for the values $n \leq 5$. The case $n = 5$ was proven in 2014 by Hakopian, Jetter, and Zimmermann, in [19]. So far that is the only proof for $n = 5$. In addition, it is very long and complicated.

In Chapter 5 we consider the main result of the paper [1] by V. Bayramyan and H. Hakopian, stating that any n-node line of GC_n set is used either by exactly $\binom{n}{2}$ nodes or by exactly $\binom{n-1}{2}$ nodes, provided that the Gasca-Maeztu conjecture is true.

Here we show that this result is not correct in the case $n = 3$. Namely, we bring an example of a GC_3 set and a 3-node line there which is not used at all. The proof of the result in [1] is inductive and based on the case $n = 3$. For this reason we needed to consider a new proof of the result. Then we were able to establish that this is the only possible counterexample, i.e., the above mentioned result is true for all $n \geq 4$.

Theorem 5.1.1 ([1*]). Assume that Conjecture 4.1.7 holds for all degrees up to n. Let $\mathcal{X}$ be a GC_n set, $n \geq 4$, and ℓ be an n-node line. Then we have that

$$|\mathcal{X}_\ell| = \binom{n}{2} \quad \text{or} \quad \binom{n-1}{2}.$$

Moreover, the following hold:

(i) $|\mathcal{X}_\ell| = \binom{n}{2}$ if and only if there is a maximal line λ_0 such that $\lambda_0 \cap \ell \cap \mathcal{X} = \emptyset$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda_0)$. Hence it is a GC_{n-2} set;

(ii) $|\mathcal{X}_\ell| = \binom{n-1}{2}$ if and only if there are two maximal lines λ', λ'', such that $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda' \cup \lambda'')$. Hence it is a GC_{n-3} set.

We also characterize the exclusive case $n = 3$ and present some new results on the maximal lines and the usage of n-node lines in GC_n sets.

Proposition 5.1.3 ([1*]). Let $\mathcal{X}$ be a GC_3 set and ℓ be a 3-node line. Then we have that

$$|\mathcal{X}_\ell| = 3, \quad 1, \quad \text{or} \quad 0.$$

Moreover, the following hold:

(i) $|\mathcal{X}_\ell| = 3$ if and only if there is a maximal line λ_0 such that $\lambda_0 \cap \ell \cap \mathcal{X} = \emptyset$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda_0)$. Hence it is a GC_1 set.

(ii) $|\mathcal{X}_\ell| = 1$ if and only if there are two maximal lines λ', λ'', such that $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda' \cup \lambda'')$;

(iii) $|\mathcal{X}_\ell| = 0$ if and only if there are exactly three maximal lines in $\mathcal{X}$ and they intersect ℓ at three distinct nodes.

Furthermore, if the node set $\mathcal{X}$ possesses exactly three maximal lines then any 3-node line ℓ is either used by exactly three nodes or is not used at all:

$$|\mathcal{X}_\ell| = 3 \quad \text{or} \quad 0.$$

Next, we present the following simple but interesting by itself proposition.

Proposition 5.2.2 ([1*]). Let $\mathcal{X}$ be a GC_n set and ℓ be an n-node line, where $n \geq 4$. Suppose that there are n maximal lines passing through n distinct nodes in ℓ. Then there exists at least one more maximal line in $\mathcal{X}$.

At the end of Section 5.2 we present the following

Corollary 5.2.4 ([1*]). Assume that Conjecture 4.1.7 holds for all degrees up to n. Let $\mathcal{X}$ be a GC_n set with exactly three maximal lines, where $n \geq 4$. Then there are exactly three n-node lines in $\mathcal{X}$ and each of them is used by exactly $\binom{n}{2}$ nodes from $\mathcal{X}$.

Remark 5.2.5. It is worth mentioning that Corollary 5.2.4 is not valid in the case $n = 3$. Indeed, the GC_3 set $\mathcal{X}^\star$ and the 3-node lines ℓ_1, ℓ_2, ℓ_3 and $\ell^\star$ give us a counterexample for this.

In Chapter 6 we consider a conjecture proposed in the paper [1] by V. Bayramyan and H. Hakopian, concerning the usage of any k-node line in GC_n sets, $2 \leq k \leq n+1$. Here we make an adjustment in the formulation of the mentioned conjecture and then prove it. Namely, by assuming that the Gasca-Maeztu conjecture is true, we prove that for any GC_n set $\mathcal{X}$ and any k-node line ℓ the following statement holds:
The line ℓ is not used at all, or it is used by exactly $\binom{s}{2}$ nodes of $\mathcal{X}$, where s satisfies the condition $k-\delta \leq s \leq k, \ \delta = n+1-k$. If in addition $k-\delta \geq 3$ and $\mu(\mathcal{X}) > 3$ then the first case here is excluded, i.e., the line ℓ is necessarily a used line.

In Section 6.1, we present the characterization of GC_n sets according to the number of maximal lines, in the cases $\mu(\mathcal{X}) = n-1$ and $\mu(\mathcal{X}) = 3$.

1. Lattices with $n-1$ maximal lines.

Let a set $\mathcal{M} = \{\lambda_1, \ldots, \lambda_{n-1}\}$ of $n-1$ lines be in general position, $n \geq 4$. Then consider the lattice $\mathcal{X}$ defined as

$$\mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)} \cup \mathcal{X}^{(0)},$$

where $\mathcal{X}^{(2)}$ is the set of all intersection nodes of these $n-1$ lines, $\mathcal{X}^{(1)}$ is a set of other $3(n-1)$ nodes, three in each line, to make the line maximal and $\mathcal{X}^{(0)}$ consists of exactly three nodes, denoted by O_1, O_2, O_3, which do not belong to any line from $\mathcal{M}$. Correspondingly, we have that $|\mathcal{X}| = \binom{n-1}{2} + 3(n-1) + 3 = \binom{n+2}{2}$. Note that all the nodes of $\mathcal{X}^{(k)}$ belong to exactly k maximal lines and are called k_m-nodes, $k = 0, 1, 2$.

Denote by ℓ_i^{oo}, $1 \leq i \leq 3$, the line passing through the two 0_m-nodes $\{O_1, O_2, O_3\} \setminus \{O_i\}$. We call this lines OO lines. Suppose that $\mathcal{X}^{(1)} = \{A_i^1, A_i^2, A_i^3 \in \lambda_i : \ 1 \leq i \leq n-1\}$.

The following proposition presents the characterization of GC_n set $\mathcal{X}$, in the case $\mu(\mathcal{X}) = n-1$.

Proposition 6.1.1 ([11], Thm. 3.2)**.** A set $\mathcal{X}$ is a GC_n set with exactly $n-1$ maximal lines $\lambda_1, \ldots, \lambda_{n-1}$, where $n \geq 4$, if and only if, with some permutation of the indexes of the maximal lines and 1_m-nodes, the representation (6.1.1) holds with the following additional properties:

(i) $\mathcal{X}^{(1)} \setminus \{A_1^1, A_2^2, A_3^3\} \subset \ell_1^{oo} \cup \ell_2^{oo} \cup \ell_3^{oo}$;

(ii) Each line $\ell_i^{oo}, i = 1, 2, 3$, passes through exactly $(n-2)$ 1_m-nodes (and through two 0_m-nodes). Moreover, $\ell_i^{oo} \cap \lambda_i \notin \mathcal{X}, \ i = 1, 2, 3$;

(iii) The triples $\{O_1, A_2^2, A_3^3\}, \ \{O_2, A_1^1, A_3^3\}, \ \{O_3, A_1^1, A_2^2\}$ are collinear.

2. Lattices with 3 maximal lines - generalized principal lattices.

A principal lattice is defined as an affine image of the set

$$PL_n := \left\{(i,j) \in \mathbb{N}_0^2 : \quad i+j \leq n\right\}.$$

Let us set $I = \{0, 1, \ldots, n+1\}$. Observe that the following 3 set of $n+1$ lines, namely $\{x = i : \ i \in I\}$, $\{y = j : \ j \in I\}$, and $\{x+y = k : \ k \in I\}$, intersect at PL_n. We have that PL_n is a GC_n set. Moreover, the following is the fundamental polynomial of the node

$(i_0, j_0) \in PL_n$:

$$p^{\star}_{i_0 j_0}(x,y) = \prod_{0 \le i < i_0,\ 0 \le j < j_0,\ 0 \le k < k_0} (x-i)(y-j)(x+y-n+k),$$

where $k_0 = n - i_0 - j_0$.

Next, in Section 6.1, we bring the definition of the generalized principal lattice due to Carnicer, Gasca and Godés (see [9], [10]):

Definition 6.1.2 ([10])**.** A node set $\mathcal{X}$ is called a generalized principal lattice, briefly GPL_n, if there are 3 sets of lines each containing $n+1$ lines

$$\ell_i^j(\mathcal{X})_{i \in \{0,1,\ldots,n\}},\ j = 0,1,2,$$

such that the $3n+3$ lines are distinct,

$$\ell_i^0(\mathcal{X}) \cap \ell_j^1(\mathcal{X}) \cap \ell_k^2(\mathcal{X}) \cap \mathcal{X} \neq \emptyset \iff i+j+k = n$$

and

$$\mathcal{X} = \left\{ x_{ijk} \mid x_{ijk} := \ell_i^0(\mathcal{X}) \cap \ell_j^1(\mathcal{X}) \cap \ell_k^2(\mathcal{X}), 0 \le i,j,k \le n, i+j+k = n \right\}.$$

Following Theorem is a characterization for GPL_n set due to Carnicer and Godés:

Theorem 6.1.3 ([10], Thm. 3.6)**.** Assume that GM Conjecture holds for all degrees up to $n-3$. Then the following statements are equivalent:

(i) $\mathcal{X}$ is generalized principal lattice of degree n;

(ii) $\mathcal{X}$ is a GC_n set with $\mu(\mathcal{X}) = 3$.

In Section 6.2, we formulate the adjusted version of the conjecture proposed by V. Bayramyan and H. Hakopian in [1] (Conj. 3.7) as:

Theorem 6.2.1 ([6*], [8*])**.** Let $\mathcal{X}$ be a GC_n set, and ℓ be a k-node line, $k \ge 2$. Assume that GM Conjecture holds for all degrees up to n. Then we have that

$$\mathcal{X}_\ell = \emptyset, \text{ or} \tag{6.2.1}$$

$$\mathcal{X}_\ell \text{ is an } \ell\text{-proper } GC_{s-2} \text{ subset of } \mathcal{X}, \text{ hence } |\mathcal{X}_\ell| = \binom{s}{2}, \tag{6.2.2}$$

for some $k - \delta \le s \le k$ and $\delta = n + 1 - k$.
Moreover, if $k - \delta \ge 3$ and $\mu(\mathcal{X}) > 3$ then $\mathcal{X}_\ell \neq \emptyset$, i.e., (6.2.2) holds with $s \ge 2$. Furthermore, in the case $\mathcal{X}_\ell \neq \emptyset$ we have for any maximal line λ :
$|\lambda \cap \mathcal{X}_\ell| = 0$ or $|\lambda \cap \mathcal{X}_\ell| = s - 1$.

The following result shows that Theorem 6.2.1 is true for the node set $\mathcal{X}$ if $\mu(\mathcal{X}) = 3$.

Proposition 6.2.5 ([6*], [8*]). Let $\mathcal{X}$ be a GC_n set with $\mu(\mathcal{X}) = 3$ and ℓ be a k-node line, $k \geq 2$. Assume that GM Conjecture holds for all degrees up to $n-3$. Then we have that

$$\mathcal{X}_\ell = \emptyset, \text{ or } \mathcal{X}_\ell \text{ is an } \ell\text{-proper } GC_{k-2} \text{ subset of } \mathcal{X}, \text{ hence } |\mathcal{X}_\ell| = \binom{k}{2}.$$

Moreover, if $k \leq n$ and $\mathcal{X}_\ell \neq \emptyset$ then for a maximal line λ_1 of $\mathcal{X}$ we have that $\lambda_1 \cap \ell \notin \mathcal{X}$ and $|\lambda_1 \cap \mathcal{X}_\ell| = 0$.
For the remaining two maximal lines we have that $|\lambda \cap \mathcal{X}_\ell| = k-1$.
Furthermore, if the line ℓ intersects each maximal line at a node then $\mathcal{X}_\ell = \emptyset$.

We provide a result on the presence and usage of $(n-1)$-node lines in GC_n sets with $\mu(\mathcal{X}) = n-1$.

Proposition 6.2.6 ([6*], [8*]). Let $\mathcal{X}$ be a GC_n set with $\mu(\mathcal{X}) = n-1$, and ℓ be an $(n-1)$-node line, where $n \geq 4$. Assume also that through each node of ℓ there passes exactly one maximal line. Then we have that either $n = 4$ or $n = 5$. Moreover, in both these cases we have that $\mathcal{X}_\ell = \emptyset$.

We characterize the case $k - \delta = 2,\ \mu(\mathcal{X}) > 3$. For each n and k, with $k - \delta = 2k - n - 1 = 2$, we bring two constructions of GC_n sets and a non-used k-node line in each case. The following proposition shows that these are the only constructions with the mentioned property.

Proposition 6.4.1 ([6*], [8*]). Let $\mathcal{X}$ be a GC_n set and ℓ be a k-node line with $k - \delta := 2k - n - 1 = 2$ and $\mu(\mathcal{X}) > 3$. Suppose that the line ℓ is a non-used line. Then we have that either $\mathcal{X} = \bar{\mathcal{X}}^*,\ \ell = \bar{\ell}_4^*$, or $\mathcal{X} = \bar{\mathcal{Y}}^*,\ \ell = \bar{\ell}_3^*$.

List of Publications of the Author

[1*] H. Hakopian, V. Vardanyan, On a correction of a property of GC_n sets, Adv Comput Math (2019) 45: 311-325.

[2*] V. Vardanayan, On bivariate fundamental polynomials, British Journal of Mathematics and Computer Science 10(5): 1-17, 2015

[3*] V. Vardanyan, On Factorization of Fundamental Polynomials of Two and Three Variables Russian Journal of Mathematical Research. Series A, 2016, Vol.(4), Is. 2, pp. 77-84.

[4*] V. Vardanyan (2019). The Gasca-Maeztu Conjecture for n = 4. Asian Research Journal of Mathematics, 12(2), 1-7. https://doi.org/10.9734/arjom/2019/v12i230083

[5*] H. Hakopian, V. Vardanyan, On a correction of a property of GC_n sets, International Conference Dedicated to 90th Anniversary of Sergey Mergelyan, 20-25 May 2018, Yerevan, Armenia, Abstracts, pp. 37-38.

[6*] H. Hakopian, V. Vardanyan, On the usage of lines in GC_n sets, Emil Artin International Conference, Dedicated to 120th Anniversary Emil Artin, 29 May - 02 June, 2018, Yerevan, Armenia, Abstracts, p. 67.

[7*] H. Hakopian, V. Vardanyan, On the usage of lines in GC_n sets, International Conference Harmonic Analysis and Approximations VII, Dedicated to 90th Anniversary of Alexandr Talalyan, 16-22 September, 2018, Tsaghkadzor, Armenia, Abstracts, pp. 45-47.

[8*] H. Hakopian, V. Vardanyan, On the usage of lines in GC_n sets, accepted in Adv Comput Math (2019). https://doi.org/10.1007/s10444-019-09705-w

Bibliography

[1] V. Bayramyan, H. Hakopian, On a new property of n-poised and GC_n sets, Adv Comput Math, 43, (2017) 607–626.

[2] V. Bayramyan, H. Hakopian and S. Toroyan, A simple proof of the Gasca-Maeztu conjecture for n = 4, Jaén J. Approx. 7 (2015) 137–147.

[3] L. Berzolari, Sulla Determinazione di Una Curva o di Una Superficie Algebrica e Su Alcune Questioni di Postulazione, Rendiconti del R. Inst. Lombardo, Series (2), 47, 24, 1914.

[4] C. de Boor, Multivariate polynomial interpolation: conjectures concerning GC sets, Numer. Algorithms 45 (2007) 113–125.

[5] J. R. Busch, A note on Lagrange interpolation in $\mathbb{R}^2$, Rev. Un. Mat. Argentina 36 (1990) 33–38.

[6] J. M. Carnicer and M. Gasca, Planar configurations with simple Lagrange formula, in Mathematical Methods in CAGD: Oslo 2000, T. Lyche and L. L. Schumaker (eds.), 55—62. Vanderbilt University Press, Nashville, TN, 2001.

[7] J. M. Carnicer and M. Gasca, A conjecture on multivariate polynomial interpolation, Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.), Ser. A Mat. 95 (2001) 145–153.

[8] J. M. Carnicer and M. Gasca, On Chung and Yao's geometric characterization for bivariate polynomial interpolation, *in:* T. Lyche, M.-L. Mazure, and L. L. Schumaker (eds.), 21–30. Curve and Surface Design: Saint Malo 2002, Nashboro Press, Brentwood, 2003.

[9] J. M. Carnicer and M. Gasca, Generation of lattices of points for bivariate interpolation, Numer. Algorithms, 39 (2005) 69–79.

[10] J. M. Carnicer and C. Godés, Geometric characterization and generalized principal lattices, J. Approx. Theory 143 (2006) 2–14.

[11] J. M. Carnicer and C. Godés, Geometric characterization of configurations with defect three, in Albert Cohen, Jean Louis Merrien, Larry L. Schumaker (Eds.), Curve and Surface Fitting: Avignon 2006, Nashboro Press, Brentwood, TN, 2007, 61–70.

[12] J. M. Carnicer and C. Godés, Configurations of nodes with defects greater than three. J. Comput. Appl. Math. 233 (2010) 1640–1648.

[13] J. M. Carnicer and C. Godés, Extensions of planar GC sets and syzygy matrices. Adv Comput Math, https://doi.org/10.1007/s10444-018-9630-8

[14] K. C. Chung and T. H. Yao, On lattices admitting unique Lagrange interpolations, SIAM J. Numer. Anal. 14 (1977) 735–743.

[15] D. Eisenbud, M. Green and J. Harris (1996) Cayley-Bacharach theorems and conjectures, Bull. Amer. Math. Soc. (N.S.), 33(3), 295–324.

[16] M. Gasca and J. I. Maeztu, On Lagrange and Hermite interpolation in $\mathbb{R}^k$, Numer. Math. 39 (1982) 1–14.

[17] H. Hakopian and A. Malinyan, Characterization of n-independent sets with no more than $3n$ points, Jaén J. Approx. 4(2012), 119 – 134.

[18] H. Hakopian, K. Jetter, and G. Zimmermann, A new proof of the Gasca-Maeztu conjecture for $n = 4$, J. Approx. Theory 159 (2009) 224–242.

[19] H. Hakopian, K. Jetter and G. Zimmermann, The Gasca-Maeztu conjecture for $n = 5$, Numer. Math. 127 (2014) 685–713.

[20] H. Hakopian and L. Rafayelyan, On a generalization of Gasca-Maeztu conjecture, New York J. Math. 21 (2015) 351–367.

[21] H. Hakopian, S. Toroyan, On the uniqueness of algebraic curves passing through n-independent nodes, New York J. Math. 22 (2016) 441–452.

[22] A. Malinyan, Characterization of n -independent sets of $\leq 3n$ points in $\mathbb{R}^d$, Vestnik RAU. Seria Phys.-math. and natural sciences, N1, 3-15 (2013).

[23] J. Radon, Zur Mechanischen Kubatur, Monatshefte. Math. Physik, 52(4), 286-300, 1948.

[24] F. Severi, Vorlesungen Ëuber Algebraische Geometrie (Teubner, Berlin, 1921).

Ամփոփում

Այս թեզում մենք հետազոտվում են բազմաչափ միջարկմանը վերաբերվող որոշ խնդիրներ, մասնավորապես՝ ֆունդամենտալ բազմանդամների վերլուծումը արտադրիչների և ուղիղների օգտագործումը GC բազմություններում:

Աշխատանքի առաջին մասում բերվում են անհրաժեշտ և բավարար պայմաններ հարթության վրա տրված ոչ ավելի քան $2n+[n/2]+1$ կետերի բազմության ֆունդամենտալ բազմանդամների վերլուծության մասին: Ապացուցվում է, որ այս դեպքում բազմանդամները վեր են լուծվում առաջին և երկրորդ աստիճանի արտադրիչների արտադրյալի: Բերվում է նաև հակաօրինակ, որը ցույց է տալիս, որ ավելի շատ կետերի համար, ընդհանուր դեպքում, այդպիսի վերլուծություն չկա:

Այնուհետև ներկայացվում են անհրաժեշտ և բավարար պայմաններ տարածության մեջ տրված ոչ ավելի քան $3n+1$ կետերի բազմության n-անկախության և ֆունդամենտալ բազմանդամների գծային արտադրիչների վերլուծության վերաբերյալ:

Երկրորդ մասում դիտարկվում են GC_n բազմություններ, այսինքն n-ճշգրիտ բազմություններ, որտեղ ցանկացած կետ ունի ֆունդամենտալ բազմանդամ, որը գծային արտադրիչների (ուղիղների) արտադրյալ է: Ասում ենք, որ A կետը օգտագործում է l ուղիղը, եթե վերջինս A կետի ֆունդամենտալ բազմանդամի արտադրիչ է: Դիտարկվում է Գասքա-Մաեզթուի վարկածը ըստ որի՝ կամայական GC_n բազմությունում կա մաքսիմալ ուղիղ, այսինքն ուղիղ, որը անցնում է բազմության $n+1$ կետերով: Մինչ այժմ վարկածը ապացուցված է միայն $n\leq 5$-ի համար: Այս թեզում տրվում է Գասքա-Մաեզթուի վարկածի պարզ ապացույց $n=4$ դեպքի համար:

Այնուհետև քննարկվում է Վ. Բայրամյանի և Հ. Հակոբյանի ստացած արդյունքը, որը պնդում է, որ եթե ճիշտ է Գասքա-Մաեզթուի վարկածը, ապա GC_n բազմության n -կետանի ուղիղը օգտագործվում է կամ ճիշտ $\binom{n}{2}$, կամ ճիշտ $\binom{n-1}{2}$ կետերի կողմից: Ապացուցվում է, որ այս պնդումը ճիշտ չէ $n=3$-ի համար: Բերվում է GC_3 բազմության և նրանում 3-կետանոց ուղղի օրինակ, որը չի օգտագործվում որևէ կետի կողմից: Ապացուցվում է, որ սա միակ հնարավոր հակաօրինակն է և վերը նշված պնդումը ճիշտ է $n\geq 4$ դեպքում:

Այնուհետև, կատարելով որոշ ճշգրտում ձևակերպման մեջ, ապացուցվում է Վ. Բայրամյանի և Հ. Հակոբյանի կողմից առաջադրված վարկածը: Այն է՝ ենթադրելով, որ Գասքա-Մաեզթուի վարկածը ճիշտ է, ապացուցվում է, որ X GC_n բազմության մեջ k -կետանոց l ուղիղը կամ չի օգտագործվում ընդհանրապես կամ օգտագործվում է ճիշտ $\binom{s}{2}$ կետերի կողմից, որտեղ s-ը բավարարում է $k-\delta\leq s\leq k$ պայմանին և $\delta=n+1-k$: Ցույց է տրվում, որ X-ում l-ը օգտագործող կետերի ենթաբազմությունը կազմում է GC բազմություն, եթե այն դատարկ չէ: Ապացուցվում է նաև, որ վերը նշված առաջին դեպքը բացառվում է, այսինքն ուղիղը անպայման օգտագործվող է, երբ $k-\delta\geq 3$ և $\mu(X)>3$: Այստեղ $\mu(X)$-ը X-ի մաքսիմալ ուղիղների քանակն է:

Վերջում բնութագրվում են GC_n բազմությունների այն կոնստրուկցիաները, որտեղ կա k-կետանոց չօգտագործվող ուղիղ, երբ $k-\delta=2$ և $\mu(X)>3$:

Заключение

В диссертационной работе исследуются некоторые задачи, относящиеся к многомерной интерполяции, в частности, факторизации фундаментальных многочленов и использованию прямых в множествах GC.

В первой части работы приводятся необходимые и достаточные условия для факторизации фундаментальных многочленов для множества с не более, чем $2n+[n/2]+1$ точками на плоскости. Доказывается, что в этом случае многочлены могут быть представлены в виде произведения множителей первой и второй степени. Приводится контрпример, который показывает, что для множества с большим числом точек такого вида факторизации в общем случае не существует.

Затем даются необходимые и достаточные условия для n-независимости и факторизации фундаментальных многочленов для множества с не более, чем $3n+1$ точками, заданными в пространстве.

Во второй части работы рассматриваются GC_n множества, т. е. $n-$корректные множества, в которых каждая точка имеет фундаментальный многочлен, являющийся произведением линейных множителей. Скажем, что точка A использует прямую l, если последняя является множителем в фундаментальном многочлене точки A. Рассматривается гипотеза Гаска-Маэзту, согласно которой любое GC_n множество имеет максимальную прямую, т.е. прямую, проходящую через $n+1$ точку множества. До сих пор эта гипотеза доказана только для $n \leq 5$. В работе представляется простое доказательство гипотезы Гаска-Маэзту для случая $n=4$.

Далее обсуждается результат, установленный В. Байрамяном и А. Акопяном, в котором утверждается, что в случае, если гипотеза Гаска-Маэзту верна, то n-точечная прямая в множестве GC_n используется либо точно $\binom{n}{2}$ точками, либо точно $\binom{n-1}{2}$ точками. Доказыватся, что это утверждение неверно при $n=3$. Приводится пример множества GC_3 и 3-точечной прямой, которая не используется никакой точкой. Затем доказывается, что это единственный возможный контрпример, и что приведенное выше утверждение верно в случае $n \geq 4$.

Далее, внося некоторую корректировку в формулировку, доказывается гипотеза, поставленная В. Байрамяном и А. Акопяном. А именно, предполагая, что гипотеза Гаска-Маэзту верна, доказывается, что для GC_n множества X k-точечная прямая l не используется вообще или используется точно $\binom{s}{2}$ точками, где s удовлетворяет условию $k-\delta \leq s \leq k$ и $\delta = n+1-k$. Показывается, что вышеуказанный первый случай исключается, т. е. прямая l обязательно используется, если $k-\delta \geq 3$ и $\mu(X) > 3$. Здесь $\mu(X)$ означает количество максимальных прямых в X. Также доказывается, что подмножество точек в X, использующих прямую l, является множеством GC, если оно не пустое.

В конце работы характеризуются конструкции множеств GC_n, содержащие k-точечную прямую, которая не используется никакой точкой, где $k-\delta = 2$ и $\mu(X) > 3$.