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Greedy algorithms in multidimensional Banach spaces

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Contents

INTRODUCTION

One of the most important challenges in mathematics is to explain the nature by the means of the available tools. We can only explain such complex structures by the simpler ones, and this approach is called an approximation.

Approximation theory is a relatively young branch of mathematical analysis, for it needed the concept of a function. As is well-known, the first approach to defining a function based on this dependence and to abstract from formulae was developed by Euler. Euler also solved one the first known problems in approximation theory. He tried to solve the problem of drawing a map of the Russian empire with exact latitudes, giving the best possible approximation of the relationship between latitudes and altitudes.

The main challenge of the theory of approximations is to split complex functions into simpler ones - approximants. The algorithms of finding these approximants have a purpose to increase the effectiveness, that is, to minimize the error. It is also important to construct algorithms that are suitable for practical problems and applications in various fields.

In a Banach space with Schauder basis it is natural to approximate a function by the terms of its series expansion. For example, we can approximate a function by partial sums of its series. In early studies, some linear approximations were obtained. But in recent years, with the rise of interest in problems related to Big Data and high resolution images, the usefulness and effectiveness of non-linear approximations became apparent. In late 1990s, greedy algorithms found popularity. These algorithms are used in applications in image/signal/data processing and in designing neural networks. Important results have been obtained by R. DeVore, S. Konyagin, V. Temlyakov, P. Wojtaszczyk,

M. Grigoryan, S. Gogyan, among others.

In this dissertation, we study some interesting problems related to the Thresholding Greedy Algorithm. Namely, we investigate the value of the quasi-greedy and democratic constants for the subsystems of the Haar system in $L_1[0,1]^d$. We also study the behavior of the greedy algorithm, after modifing the function on a small set.

For any normalized basis $\Psi = \{\psi_k\}_{k=1}^{+\infty}$ in the Banach space X and for any $f \in X$, we have the following expansion:

$$f = \sum_{k=1}^{+\infty} c_k(f, \Psi)\psi_k, \tag{1}$$

where $\lim c_k(f, \Psi) = 0$. Let $\Lambda_0 = \emptyset$, and by induction, define the sets of positive integers Λ_m such that $\mid \Lambda_m \mid = m$, $\Lambda_{m-1} \subset \Lambda_m$, and

$$\min_{k\in\Lambda_m} \mid c_k(f, \Psi) \mid \geq \max_{k\notin\Lambda_m} \mid c_k(f, \Psi) \mid . \tag{2}$$

Note that the sets Λ_m are not uniquely determined, but for the purposes of our paper, it is not essential. We remark that the theorems presented below hold for any choice of Λ_m satisfying condition (2). We call the function

$$G_m(f) = \sum_{k\in\Lambda_m} c_k(f, \Psi)\psi_k,$$

the m-term greedy approximant of the function f with respect to the system Ψ, see [1] and [2] for more details. The approximation method which uses greedy approximants is called a Thresholding Greedy Algorithm (TGA). To unterstand the effectiveness of the approximaion with a greedy algorithm we compare its accurancy with the best m-term approximation error using the basis vectors. We denote the best m-term approximation as follows,

$$\sigma_m(f, \Psi) = \inf_{\substack{\alpha_1,\alpha_2,\ldots,\alpha_m \\ k_1,k_2,\ldots k_m}} \|f - \sum_{i=1}^{m} \alpha_i \Psi_{k_i}\|. \tag{3}$$

Definition. (S. Dilworth, N. Kalton, D. Kutzarova, V. Temlyakov) The basis Ψ is called a greedy basis in X, if there exists $C \geq 1$ such that for any $m \in \mathbb{N}$ and $f \in X$ the following relation holds:

$$\|f - G_m(f, \Psi)\|_X \leq C\sigma_m(f). \tag{4}$$

It is also natural to compare the error of the TGA with the best m-term approximation using the function expansion terms. We denote by $\tilde{\sigma}_m(f, \Psi)$ the following quantity:

$$\tilde{\sigma}_m(f, \Psi) = \inf_{k_1, k_2, \ldots k_m} \|f - \sum_{i=1}^{m} c_{k_i}(f)\Psi_{k_i}\|. \tag{5}$$

Definition. (S. Dilworth, N. Kalton, D. Kutzarova, V. Temlyakov) The basis Ψ is called an almost greedy basis in X, if there exists $C \geq 1$ such that for any $m \in \mathbb{N}$ and $f \in X$ the following relation holds:

$$\|f - G_m(f, \Psi)\|_X \leq C\tilde{\sigma}_m(f). \tag{6}$$

The smallest value of C for which the inequality holds is called the almost greedy constant for the system Ψ.

We can also compare the error with the approximation using partial sums.

Definition. The basis Ψ is called a partially greedy basis in X if there exists $C \geq 1$ such that or any $m \in \mathbb{N}$ and $f \in X$ the following relation holds:

$$\|f - G_m(f, \Psi)\|_X \leq C\|f - S_m(f, \Psi)\|_X. \tag{7}$$

The situation is simple when we replace the Banach space X by the Hilbert space H. If Ψ is an orthonormal basis for the Hilbert space H with an inner product $(\cdot, \cdot)$, then we have

$$c_n(f) = (f, \psi_n).$$

We can estimate the norm $\|f\|$ using Parseval's identity,

$$\|f\|^2 = \sum_{n=1}^{\infty} |(f, \psi_n)|^2. \tag{8}$$

It is clear that, in this case, $G_m(f, \Psi)$ is the best m-term approximation with respect to Ψ:

$$\|f - G_m(f, \Psi)\| = \sigma_m(f, \Psi). \tag{9}$$

So, the orthonormal basis in the Hilbert space is a greedy basis.

The authors of [2] show that any greedy basis can be characterized as an unconditional basis with the additional property of being democratic. Here, we give a slightly more general definition of democratic systems.

Definition. (V. Temlyakov, S. Konyagin) Let $\Psi = \{\psi_k\}_{k=1}^{+\infty}$ be a normalized system in X. Then Ψ is called democratic in X if and only if there exists a constant C such that for any two finite subsets A, B of positive integers with equal number of elements, i.e. $\mid A \mid = \mid B \mid$, the following relation holds:

$$\| \sum_{i \in A} \psi_i \|_X \leq C \| \sum_{i \in B} \psi_i \|_X. \tag{10}$$

The smallest C for which this inequality holds is called the democratic constant for Ψ.

Theorem A (V. Temlyakov, S. Konyagin, [2])**.** *The basis is a greedy basis in X if and only if it is unconditional and democratic in X.*

In [1], the author shows that unconditionality does not imply democracy. It is known that the two dimensional Haar system $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$ defined as the tensor product of the one dimensional Haar system $\mathcal{H}$ with itself is an unconditional basis in $L_p[0,1]$ for $1 < p < \infty$. On the other hand, $\mathcal{H}^2$ is not a greedy basis in $L_p[0,1]$ for $1 < p < \infty, p \neq 2$ which means that is not a democratic basis. So we can conclude that $\mathcal{H}^2$ is an unconditional basis in

$L_p[0,1], 1 < p < \infty, p \neq 2$, but it is not democratic. Furthermore, democracy does not imply unconditionality. Indeed, let X be the set of all real sequences $x = (x_1, x_2, \dots)$ such that

$$\|x\|_X = \sup_{N \in \mathbb{N}} |\sum_{n=1}^{N} x_n| \tag{11}$$

is finite. It is easy to check that X is a Banach space endowed with the norm $\|\cdot\|_X$. Next, let $\psi_k = (\psi_1^k, \psi_2^k, \dots, \psi_n^k, \dots) \in X$ for $k = 1, 2, \dots,$ be defined in the following way

$$\psi_n^k = \begin{cases} 1, & \text{for } n = k, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Let X_0 denote the subspace of X generated by the elements of ψ_k. It is easy to see that for any two finite subsets A, B of positive integers with number of elements m we have

$$\|\sum_{k \in A} \psi_k\| = \|\sum_{k \in B} \psi_k\| = m. \tag{13}$$

Which means that ψ_k is democratic system. On the other hand

$$\|\sum_{k \in A} (-1)^k \psi_k\| = 1. \tag{14}$$

Finally, from 13 and 14 follows that ψ_k is not unconditional.

In order to give the characterization of almost greedy bases, we remind the following definition of quasi-greedy bases.

Definition. (V. Temlyakov, S. Konyangin) A basis $\Psi = \{\psi_k\}_{k=1}^{+\infty}$ is called a quasi-greedy basis of X if there exists a constant C such that for any $f \in X$ and for any $m \in \mathbb{N}$, we have

$$\|G_m(f)\| \leq C \cdot \|f\|. \tag{15}$$

In [3], Temlyakov defines the quasi-greedy constant G as the smallest number such that $\|G_m(f)\| \leq G\|f\|$ and $\|f - G_m(f)\| \leq G\|f\|$. He also shows that the quasi-greedy basis $\Psi = \{\psi_k\}_{k=1}^{+\infty}$ with a quasi-greedy constant G satisfies the following inequalities, for any real numbers c_j and any finite set of indexes A,

$$(4G^2)^{-1} \min_{i\in A} |c_j| \| \sum_{k\in A} \psi_k \| \leq \| \sum_{k\in A} c_k \psi_k \| \leq 2G \max_{k\in A} |c_k| \| \sum_{k\in A} \psi_k \|. \tag{16}$$

It will be convenient for us to define the quasi-greedy constant as the smallest constant such that $\|f - G_m(f)\| \leq G\|f\|$.

Theorem B (P. Wojtaszczyk, [4]). *The basis Ψ in the Banach space X is a quasi-greedy basis if and only if for any element $f \in X$, we have*

$$\lim_{m\to\infty} \|f - G_m(f, \Psi, \Lambda)\|_X = 0. \tag{17}$$

If Ψ is an unconditional basis, then for every permutation of natural numbers π we have that the $\sum_{k=1}^{+\infty} c_{\pi(k)}(f, \Psi)\psi_{\pi(k)}$ converges, so we can conclude that the unconditional bases are quasi-greedy. On the other hand, any quasi-greedy basis is not necessarily an unconditional basis. Morever, in [2], the authors constructed the following basis which is quasi-greedy and democratic, while it is not an unconditional basis. Let X be the set of all real sequences $x = (x_1, x_2, \dots) \in l_2$ such that

$$\|x\|_1 = \sup_{N\in\mathbb{N}} | \sum_{n=1}^{N} \frac{x_n}{\sqrt{n}} | \tag{18}$$

is finite. Note, that $(X, \|\cdot\|)$ is a Banach space, where $\|\cdot\| = \max(\|\cdot\|_{l_2}, \|\cdot\|_1)$. Let $\psi_k = (\psi_1^k, \psi_2^k, \dots, \psi_n^k, \dots) \in X$ for $k = 1, 2, \dots$, be defined as follows:

$$\psi_n^k = \begin{cases} 1, & \text{for } n = k, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Next, let X_0 denote the subspace of X generated by the elements of ψ_k. $\{\psi_k\}$ is a democratic and unconditional basis in X_0 but it is not quasi-greedy.

The following theorem characterizes almost greedy bases.

Theorem C (S. Konyagin, V. Temlyakov, [2]). *The basis is an almost greedy basis in X if and only if it is quasi- greedy and democratic in X.*

The estimate of the almost greedy constant in terms of democratic and quasi greedy constants is obtained in [5]. Let K be the almost greedy constant of the basis, and G and D be the quasi-greedy and democratic constants, respectively, then

$$K \leq 8G^4 \cdot D + G + 1. \tag{20}$$

In order to characterize partially greedy bases, we introduce the following definition of conservative bases.

Definition. A basis Ψ is conservative if there is a constant C such that

$$\|\sum_{k\in A} \psi_k\| \leq C \|\sum_{k\in B} \psi_k\|, \tag{21}$$

where $|A| \leq |B|$ and for every $m \in A$ and $n \in B$ we have $m < n$.

The following theorem characterizes partially greedy bases:

Theorem D (S. Dilworth, N. Kalton, D. Kutzarova and V. Temlyakov , [5]). *The basis Ψ is a partially greedy basis in X if and only if it is a quasi-greedy and conservative in X.*

The definition implies that partially greedy bases are always 'almost greedy' which themselves are greedy.

In CAHPTER 1, we study results concerning only the Haar system. Let $\mathcal{H} = \{h_{\mathcal{I}}\}$ be the Haar system (see CHAPTER 1 for the definition). We denote by $\mathcal{H}_i = \{h_i^{(1)}, h_i^{(2)}, \ldots, h_i^{(2^i)}\}$ the i-th pocket of the Haar system. In [6], the authors show that the Haar system is not a quasi-greedy basis in $L_1[0,1]$. They also prove that if $\{n_i\}$ is an increasing sequence of positive integers such that $\frac{n_{i+1}}{n_i} > 2$, then the system $\bigcup_i H_{n_i}$ is a quasi-greedy system in $L_1[0,1]$. In order to characterize all quasi-greedy $\bigcup_i H_{n_i}$ systems, Gogyan [7] proves the following Theorem.

Theorem E (S. Gogyan, [7]). *The system* $\bigcup_i H_{n_i}$ *is a quasi-greedy system in* $L_1[0,1]$ *if and only if* $\sup\{j-i : n_j - n_i = j-i\} < \infty$.

Moreover, [8] characterizes all the quasi-greedy subsystems of the Haar system in $L_1[0,1]$. We denote by $\mathcal{S}$ the set of the dyadic intervals, and by $H(\mathcal{S})$ the length of the longest chain of $\mathcal{S}$.

Theorem F (S. Gogyan, [8]). *The system* $\{h_{\mathcal{I}}\}_{\mathcal{I}\in\mathcal{S}}$ *is a quasi-greedy system in* $L_1[0,1]^d$ *if and only if* $H(S) < \infty$.

Furthermore, an estimate $G(\mathcal{S}) \leq 2^{(H+2)}$ is obtained for a quasi-greedy constant in [8].

Democratic subsystems of the one-dimensional and multidimensional Haar systems in $L_1[0,1]$ and $L_1[0,1]^d$ are characterized respectively in [8] and [9]. The definition of the multidimensional Haar system whose elements have cubic supports is introduced in [9], see also CHAPTER 1. The proof of [9, Theorem 1] implies that the democratic constant satisfies the condition $D(\mathcal{S}) \leq 2^{(H-1)d}$, where S is the set of dyadic cubes and H is the length of the longest full chain of the subsystem, see also CHAPTER 1.

As we can see, the greedy algorithm is rearranging the terms of the expansion in a decreasing order. It has been shown in [6] that the Haar system is not a quasi-greedy one in $L_1[0,1]$. There is a more general theorem which states that for every measurable set $E \subset [0,1]$ with $0 < |E| < 1$ there exists a function $f \in L_1[0,1]$ such that

$$\lim_{m\to\infty} \|G_m(f,\mathcal{H})\| = +\infty. \tag{22}$$

To overcome such kind of problems, one can consider modifying a function in a small set. In particular, N. Luzin presented the following very important Theorem.

Theorem G (N. Luzin, [10]). *For any measurable, almost everywhere finite in* $[0,1]$ *function* f *and any* $\epsilon > 0$ *there exists a measurable set* E *with* $|E| > 1-\epsilon$ *and a continuous function* g *in* $[0,1]$, *coinciding with* f *in* E.

Luzin's idea of changing functions in order to improve its properties has been studied and developed by many authors, e.g. Menshov, Talalyan, Arutjunjan, Oskolov, Grigoryan, Gogyan, among others. Below we present some of the latter mentioned results regarding the convergence of Fourier series and greedy algorithms.

In [2], the authors prove that there exists a function $f_0(x) \in \bigcap\limits_{1\le p<2} L_p$ for which the greedy algorithm with respect to the trigonometric system does not converge in measure. Grigoryan shows that there exists a set of an arbitrary small measure such that after by modifying the function in that set, the greedy algorithm with respect to the trigonometric system of the modified function converges in measure. Denoting by $\{a_n\}_{n=-\infty}^{\infty}$ the complex Fourier coefficients of the function f;

$$a_k(f) = \int_0^1 f(t)e^{-i2\pi kt}\mathrm{d}x, \tag{23}$$

Grigoryan [11] proves the following theorems.

Theorem H (M. Grigoryan, [11]). *For every* $\epsilon > 0$, *there exists a measurable set* $E \subset [0,1]$ *with* $|E| > 1-\epsilon$ *such that for every* $p \in [1,2)$ *and* $f \in L_p[0,1]$ *there is a function* $g \in L_1[0,1]$ *with* $g = f$ *on* E *such that the greedy algorithm of the function* g *with respect to the trigonometric system is norm-convergent in* $L_1[0,1]$, *and converges to* f *on* E *in metric* $L_p(E)$.

Theorem I (M. Grigoryan, [11]). *For every* $\epsilon > 0$ *there exists a measurable set* $E \subset [0,1]$ *with* $|E| > 1-\epsilon$ *that for every* $p \in [1,2)$ *and* $f \in L_p[0,1]$ *there is a function* $g \in L_1[0,1]$ *with* $g = f$ *on* E, *and a permutation of integer numbers* $\{\sigma(k)\}$ *such that*

1) $\lim_{n\to\infty}\int_0^1 |\sum_{|k|\le n} a_{\sigma(k)}(g)e^{i2\pi\sigma(k)x} - g(x)|\mathrm{d}x = 0,$

2) $\lim_{n\to\infty}\int_0^1 |\sum_{|k|\le n} a_{\sigma(k)}(g)e^{i2\pi\sigma(k)x} - f(x)|^p\mathrm{d}x = 0$

3) $|a_{\sigma(k)}(g)| > |a_{\sigma(k+1)}(g)|,$, *for every* $k \le 0$,

4) $\int_0^1 |f(x) - g(x)|\mathrm{d}x < \epsilon.$

The following two theorems indicate that in the case of the Haar system, we can modify the function in a small set such that its non-zero terms will be in a decreasing order, while we are not able to order all the terms monotonically.

Theorem J (S. Gogyan, M. Grigoryan, [12])**.** *For any mea surable set* $E, E \subset [0,1]$ *with* $0 < |E| < 1$ *there exists a function* $f_0 \in L_1[0,1]$ *such that if* $f \in L_1[0,1]$ *coinciding with* f_0 *on* E*, then the sequence* $\{|c_n(f)|\}_{n=1}^{\infty}$ *cannot be monotonically decreasing, where* $c_n(f)$ *is the* n*-th coefficient of the Haar system normalized in* $L_1[0,1]$.

Theorem K (S. Gogyan, M. Grigoryan, [12])**.** *For any* $0 < \epsilon < 1$*, there exists a measurable set* $E \subset [0,1]$ *with* $|E| > 1 - \epsilon$*, such that for any function* $f \in L_1[0,1]$ *there is some* $\tilde{f} \in L_1[0,1]$ *coinciding with* f *on* E *and all non-zero terms of the sequence* $\{c_n(\tilde{f})\}$ *are arranged in a decreasing order.*

CHAPTER 1

ON QUASI-GREEDY CONSTANT

ON QUASI-GREEDY CONSTANT

It is well known from [6] that the Haar system is not quasi-greedy, so it is natural to investigate the subsystems of the Haar system. The authors of [6] also give an example of a quasi-greedy Haar subsystem. Later, Gogyan [13] characterizes all the quasi-greedy subsystems of the Haar system in $L_1[0,1]$. To formulate this result we first recall the definition of the Haar system.

We denote $\Delta_1 = \Delta_0^{(0)} = [0,1]$. For $j = 1, 2, \ldots, 2^i$ and $i = 0, 1, 2, \ldots,$ the intervals

$$\Delta_{2^i+j} = \Delta_i^{(j)} = \left[\frac{j-1}{2^i}, \frac{j}{2^i}\right),$$

are called dyadic intervals. Let us donate the set of all dyadic intervals by $\mathcal{D}$. Every dyadic interval is a union of two dyadic intervals, namely $\Delta_n = \Delta_{2n-1} \cup \Delta_{2n}$. These two intervals are called the left and right halves of the interval Δ_n, respectively. To each dyadic interval corresponds exactly one function of the Haar system, which is given by

$$h_n(t) = h_i^{(j)}(t) = h_{\Delta_n(t)} = \begin{cases} 2^i, & t \in \Delta_{2n-1}, \\ -2^i, & t \in \Delta_{2n}, \\ 0, & \text{otherwise.} \end{cases}$$

We also assume $h_1 = h_{\Delta_1} \equiv 1$. The set of the functions $\mathcal{H} = \{h_n\}_{n=1}^{\infty}$ is called a Haar system. Below we present the plots of a few terms of the Haar system.

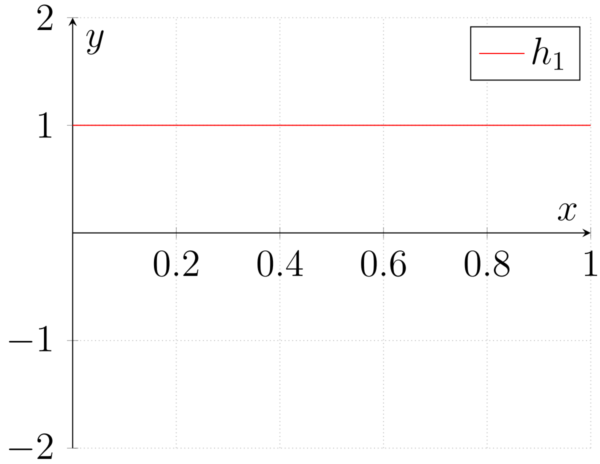

Figure 1.1

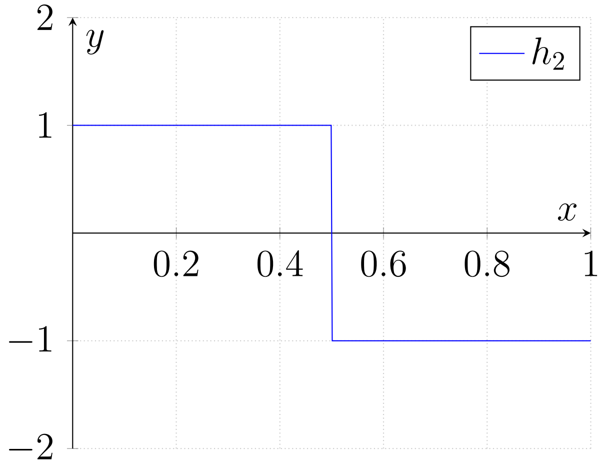

Figure 1.2

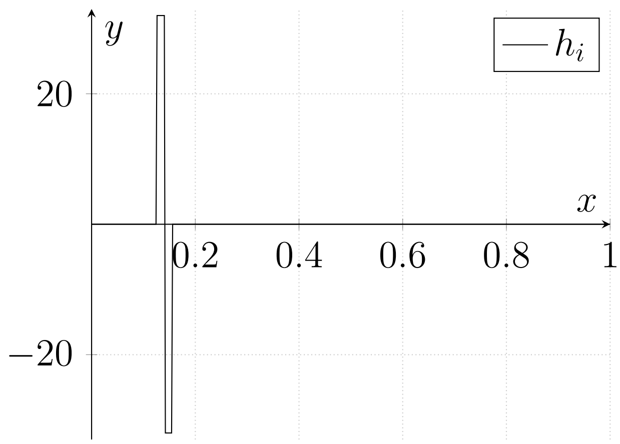

Figure 1.3

For any $f \in L_1[0,1]$, the coefficients of the expansion are determined as follows,

$$c_1(f,\mathcal{H}) = c_{\Delta_1}(f,\mathcal{H}) = \int_{[0,1]} f, \tag{1.1}$$

and

$$c_n(f,\mathcal{H}) = c_{\Delta_1}(f,\mathcal{H}) = \int_{\Delta_{2n-1}} f - \int_{\Delta_{2n}} f, \qquad n \geq 2, \tag{1.2}$$

For the set $\mathcal{A}$, $\mathcal{A} \subset \mathcal{D}$ we denote by $\mathcal{H}_{\mathcal{A}} = \{h_{\mathcal{I}}\}_{\mathcal{I}\in\mathcal{A}}$ and $L_{\mathcal{A}} = \overline{span(\mathcal{H}_{\mathcal{A}})}$, where the closure is taken with the $L_1[0,1]$-norm. For any $\mathcal{I},\mathcal{J} \in \mathcal{D}$ with $\mathcal{J} \subset \mathcal{I}$, denote

$$C(\mathcal{I},\mathcal{J}) = \{\Delta \in \mathcal{D} : \mathcal{J} \subseteq \Delta \subseteq \mathcal{I}\} \tag{1.3}$$

which is called a chain [14]. The length of the chain $C(\mathcal{I},\mathcal{J})$ is the number of elements in the chain. Figure 1.4 represents the graph of dyadic intervals. For example the nods $1,2,3,4$ are compose a chain, while the nods $5,6,7$ do not.

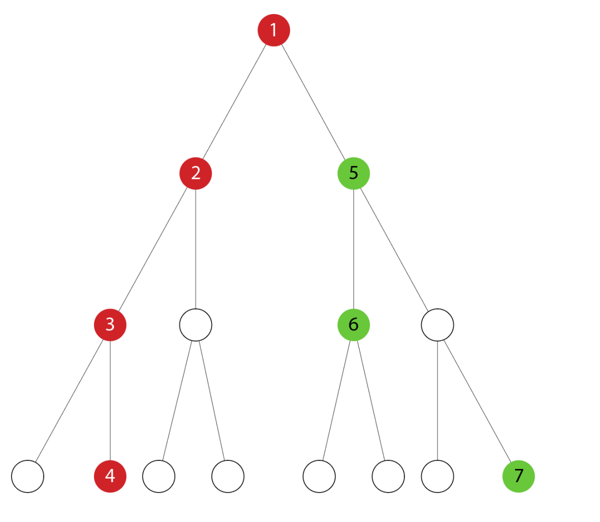

Figure 1.4

Denote $H(\mathcal{A})$ the length of the longest chain in subsystem $\mathcal{A}$, see [14]. The following theorem was proved in [13].

Theorem 1 (S. Gogyan, [13]). *For the set $\mathcal{A}$, $\mathcal{A} \subset \mathcal{D}$, the Haar subsystem $\mathcal{H}_\mathcal{A}$ is a quasi-greedy basis in $L_\mathcal{A}$ if and only if $H(\mathcal{A}) < +\infty$. Moreover, for every $m \in \mathbb{N}$ and $f \in L_\mathcal{A}$,*

$$\|G_m(f)\| \leq 2^{H(\mathcal{A})}\|f\|.$$

In this chapter, we improve the estimate for the quasi-greedy constant $\mathcal{H}_\mathcal{A}$.

Theorem 2. *Assume there exists a subsystem of a Haar system $\mathcal{H}_\mathcal{A}$ which is a quasi-greedy subsystem in $L_1[0,1]$. Then, for the quasi-greedy constant $G_\mathcal{A}$, we have the following estimate,*

$$\frac{H(\mathcal{A})}{16} \leq G_\mathcal{A} \leq 2H(\mathcal{A}) + 1.$$

For sake of convenience of the presentation of the proof of Theorem 2, below we introduce some notations.

We write, for $f \in L_1[0,1]$,

$$\|f\|_\Delta = \int_\Delta | f |,$$

and we write $\|f\|$ if $\Delta = [0,1]$. We also make the following notations,

$$sp(f) = \{\Delta \ : \quad c_\Delta(f) \neq 0\},$$

$$P_\mathcal{I}(f) = f - \sum_{\mathcal{J} \subseteq \mathcal{I}} c_\mathcal{J}(f) h_\mathcal{J}, \text{ for any } \quad \mathcal{I} \in \mathcal{D}.$$

Note that the function $P_\mathcal{I}(f)$ is constant on the set $\mathcal{I}$, and coincides with f outside $\mathcal{I}$. For a finite set of dyadic intervals $\mathcal{S}$, $\mathcal{S} = \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_k\}$, we denote

$$P_\mathcal{S}(f) = P_{\mathcal{I}_1}(P_{\mathcal{I}_2}(\ldots(P_{\mathcal{I}_k}(f))\ldots).$$

We also remark that the function $P_\mathcal{S}(f)$ does not depend in which order the operators $P_{\mathcal{I}_i}$, $1 \leq i \leq k$ are applied.

We will use the monotonicity property of the Haar system in $L_1[0,1]$, that is, for any positive integers m and n, with $m \geq n$ we have the relation

$$\|S_m(f)\| \geq \|S_n(f)\|, \tag{1.4}$$

where

$$S_k(f) = \sum_{i=1}^{k} c_i(f) h_i.$$

For any $f \in L_1[0,1]$ and a dyadic interval $\Delta_n = [a,b)$, we define

$$C_1(f,\Delta_n)(t) = \begin{cases} f(t) & : \quad t \notin \Delta_{2n}, \\ f(t - \frac{b-a}{2}) & : \quad t \in \Delta_{2n}, \end{cases}$$

and

$$C_2(f,\Delta_n)(t) = \begin{cases} f(x) & : \quad x \notin [a, \frac{a+b}{2},) \\ f(x + \frac{b-a}{2}) & : \quad x \in [a, \frac{a+b}{2}). \end{cases}$$

In fact, the operators $C_1(f,\Delta_n)$ and $(C_2(f,\Delta_n))$, introduced in [7], copy the function f from the left (right) half of the interval Δ_n to the right (left) half of it.

Below, we recall several lemmas from [14] which will be used to prove Theorem 2.

Lemma 1. *([14, Lemma 1]) For any $\mathcal{I} \in \mathcal{D}$ and $f \in L_1[0,1]$, we have*

$$\|f\|_{\mathcal{I}} \geq | c_{\mathcal{I}}(f) | . \tag{1.5}$$

Lemma 2. *([14, (8)]) Let $f \in L_1[0,1]$ and $| c_\Delta(f) | \leq 1$ for any $\Delta \in \mathcal{D}$. Then, for every $\mathcal{I} \in \mathcal{D}$,*

$$\|P_{\mathcal{I}}(f)\|_{\mathcal{I}} \leq 1. \tag{1.6}$$

Lemma 3. *Let $f, g \in L_1[0,1]$ such that $\|f+g\| > 0$, and let*

$$\|f\| = B\|f+g\|.$$

Then, one of the following inequalities holds

$$\|C_1(f,\Delta_n)\| \leq B\|C_1(f+g,\Delta_n)\|, \tag{1.7}$$

$$\|C_2(f,\Delta_n)\| \leq B\|C_2(f+g,\Delta_n)\|, \tag{1.8}$$

for any interval $\Delta_n = [a,b), n > 1$. *Moreover, equalities in* (1.7) *and* (1.8) *hold at the same time.*

Next, for any functions $f, g \in L_1[0,1], \|f+g\| > 0$ and Δ_n, we define

$$C(f,g) = \begin{cases} (C_1(f,\Delta_n), C_1(f,\Delta_n)) & : \quad \text{if} \quad (1.7) \quad \text{is true and} \quad C_1(f+g,\Delta_n) \neq 0 \\ (C_2(f,\Delta_n, C_2(g,\Delta_n))) & : \quad \text{otherwise.} \end{cases} \tag{1.9}$$

Lemma 4. *Let* $f, g \in L_1[0,1]$ *and* $\Delta_n \in \mathcal{D}$ *be such that*

- *i)* $\Delta_n \notin sp(f)$, $\Delta_n \notin sp(g)$,
- *ii)* $sp(f) \cap sp(g) = \emptyset$,
- *iii)* $\|f\| > 0$, $\|f+g\| > 0$,
- *iv)* $H(sp(f+g)) < \infty$.

Then, for the functions $(f', g') = C((f,g),\Delta_n)$, *the following conditions hold:*

- *a)* $\Delta_n \notin sp(f')$, $\Delta_n \notin sp(g')$,
- *b)* $c_k(f') = c_k(f)$ *and* $c_k(g') = c_k(g)$ *for any* k *with* $\Delta_k \not\subset \Delta_s$, *where* Δ_s *is the half of the interval of* Δ_n *where the values of* f *and* g *are copied.*
- *c)* $c_{\mathcal{I}}(f') = c_{\mathcal{I}+\frac{\mu(\Delta_n)}{2}}(f')$ *and* $c_{\mathcal{I}}(g') = c_{\mathcal{I}+\frac{\mu(\Delta_n)}{2}}(g')$, *for all* $\mathcal{I} \subset \Delta_{2n-1}$
- *d)* $sp(f') \cap sp(g') = \emptyset$,
- *e)* $H(sp(f'+g')) \leq H(sp(f+g))$,
- *f)* $\|f'\| > 0$, $\|f'+g'\| > 0$,
- *g)* $\frac{\|f'\|}{\|f'+g'\|} \geq \frac{\|f\|}{\|f+g\|}$.

Now, we are ready to present the main Lemma which will be useful to prove Theorem 2.

Lemma 5. *Let p and q be polynomials with respect to the Haar system satisfying the following relations:*

i) $sp(p) \cap sp(q) = \emptyset$,

ii) the function q is constant on all dyadic intervals where the function p is constant,

iii) $\|p\| > 0$ *and* $\|p+q\| > 0$,

iv) $H := H(sp(p+q)) < \infty$,

v) $| c_{\mathcal{I}}(p) | \geq 1$ *for any* $\mathcal{I} \in sp(p)$,

vi) $| c_{\mathcal{I}}(q) | \leq 1$ *for any* $\mathcal{I} \in sp(q)$,

vii) $p\big|_{\mathcal{I}_+} = p\big|_{\mathcal{I}_-}$ *and* $q\big|_{\mathcal{I}_+} = q\big|_{\mathcal{I}_-}$ *for any* $\mathcal{I}$, $\mathcal{I} \notin sp(p+q)$. *Then, for any* $\mathcal{I} \notin sp(p+q)$,

$$\|p\|_{\mathcal{I}} \leq (2H+1) \cdot \|p+q\|_{\mathcal{I}} + 1, p\big|_{\mathcal{I}} = const, \tag{1.10}$$

$$\|p\|_{\mathcal{I}} \leq (2H+1) \cdot \|p+q\|_{\mathcal{I}} - (2H-2), p\big|_{\mathcal{I}} \neq const. \tag{1.11}$$

Proof Let $\mathcal{S}$ be the set of all dyadic intervals $\mathcal{I}$ for which $c_{\mathcal{I}}(p+q) = 0$ and the function $p+q$ is not constant on the dyadic interval whose left or right half is $\mathcal{I}$. We prove the statement of the lemma by an induction on $ord(\mathcal{I}, \mathcal{S})$, $\mathcal{I} \in \mathcal{S}$, where we refer for the definition of ord to [14]. If $ord(\mathcal{I}, \mathcal{S}) = 0$, then $p|_{\mathcal{I}} = Const$. Next, using Lemma 2 for q, we obtain

$$\|p\|_{\mathcal{I}} \leq \|p+q\|_{\mathcal{I}} + \|q\|_{\mathcal{I}} \leq \|p+q\|_{\mathcal{I}} + 1 \leq (2H+1) \cdot \|p+q\| + 1. \tag{1.12}$$

Suppose that statement holds for $\mathcal{I}$ with $ord(\mathcal{I}, S) < i$, and let us show that it is true for $\mathcal{I}$ with $ord(\mathcal{I}, S) = i$. Note that $p|_{\mathcal{I}} \neq Const$, and it follows from condition vii) that $p\big|_{\mathcal{I}_+} \neq Const$ and $p\big|_{\mathcal{I}_-} \neq Const$. Denote by $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_k$ the maximal elements of $\mathcal{S}$ which are contained in $\mathcal{I}_+$ and $p\big|_{\mathcal{I}_j} \neq Const, j = 1, 2, \ldots, k$. Similarly, denote by $\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_m$ the maximal elements of $\mathcal{S}$ which are contained in $\mathcal{I}_+$ and $p\big|_{\mathcal{J}_j} = Const, j = 1, 2, \ldots, m$. Then, note that

$$\mathcal{I}_+ = \cup_{j=1}^{k} \mathcal{I}_j \cup \cup_{j=1}^{m} \mathcal{J}_j. \tag{1.13}$$

Let us consider the case when $\mathcal{J}_j$ and $\mathcal{J}_s$ are the left or right halves of some dyadic interval $\mathcal{J}$. Then, by the assumption of the Lemma, we have $c_{\mathcal{J}}(p) \neq 0$, and therefore $\|p+q\|_{\mathcal{J}} \geq 1$. Next, using Lemma 2 for q, we obtain that $\|q\|_{\mathcal{J}} \leq 1$. Hence,

$$(2H+1)\|p+q\|_{\tilde{\mathcal{J}}} - \|p\|_{\tilde{\mathcal{J}}} > 2H\|p+q\|_{\tilde{\mathcal{J}}} - 1 > 2H - 1 > 2H - 2. \tag{1.14}$$

Thus, relation (1.11) holds on the interval $\mathcal{J}$. Denote by $\mathcal{I}_{k+1}$, $\mathcal{I}_{k+2}$, ..., $\mathcal{I}_s$ the set of the unions of all pairs in the sequence $\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_m$ whose unions are dyadic intervals. Next, denote by $\mathcal{K}_1$, $\mathcal{K}_2$, ..., $\mathcal{K}_l$ the set of the remaining intervals. For the intervals $\mathcal{I}_i$ we obtain the inequality (1.11), and for intervals $\mathcal{J}_i$ we obtain the inequality (1.10). Note, that from the construction it follows that $l \leq s(H-1)$, and we deduce that

$$\begin{gathered} \|p\|_{\mathcal{I}_+} = \sum_{i=1}^{s} \|p\|_{\mathcal{I}_i} + \sum_{i=1}^{l} \|p\|_{\mathcal{J}_i} < \\ (2H+1)\sum_{i=1}^{s} \|p+q\|_{\mathcal{I}_i} - s(2H-2) + (2H+1)\sum_{i=1}^{l} \|p+q\|_{\mathcal{J}_i} + l \leq \\ (2H+1)\|p+q\|_{\mathcal{I}_+} - (H-1). \end{gathered} \tag{1.15}$$

Condition vii) of the Lemma implies that $\|p\|_{\mathcal{I}} = 2\|p\|_{\mathcal{I}_+}$, $\|p+q\|_{\mathcal{I}} = 2\|p+q\|_{\mathcal{I}_+}$, and therefore,

$$\|p\|_{\mathcal{I}} \leq (2H+1)\|p+q\|_{\mathcal{I}} - 2(H-1), \tag{1.16}$$

which concludes the proof.

Proof Theorem 2 First, let us prove the right hand-side of the inequality, i.e. $G_{\mathcal{A}} \leq 2H(\mathcal{A}) + 1$. Assume $f \neq 0$. Note that it is enough to consider the case when $G_m(f) \neq f$. Denote

$$p = \frac{G_m(f)}{\max\ \{|\ c_{\Delta_n}(f - G_m(f))\ |: \Delta_n \in sp(f - G_m(f))\}}, \tag{1.17}$$

$$q = \frac{f - G_m(f)}{\max\ (\{|\ c_{\Delta_n}(f - G_m(f))\ |: \Delta_n \in sp(f - G_m(f))\}}, \tag{1.18}$$

We need to find an estimate for the value of $\frac{\|p\|}{\|p+q\|}$. Since p is a polynomial and a Haar system has the property of monotonicty, then, without a loss of generality, we can assume that q is also a polynomial. We have that $H(sp(p+q)) < \infty$. Consider the dyadic intervals which do not belong to $sp(p+q)$ and on which the function $p+q$ is not constant. We arrange these intervals in an increasing order (in the sense of measure), and denote them by $\mathcal{J}_1, \mathcal{J}_2, \cdots, \mathcal{J}_k$. Let

$$(p', q') = C(C(\cdots(C((p, q, \mathcal{J}_1), \mathcal{J}_2), \cdots \mathcal{J}_k). \tag{1.19}$$

Since the functions p', q' satisfy all the conditions of the main Lemma, we obtain that

$$\frac{\|G_m(f)\|}{\|f\|} \leq \frac{\|p'\|}{\|p' + q'\|} \leq 2H + 1. \tag{1.20}$$

Now, we proceed to prove $G_{\mathcal{A}} \geq \frac{H}{16}$. Assume that there exists a chain with a length H in the subsystem $\mathcal{S}$. For the sake of simplicity assume that $\{h_n^{(1)}, h_{n+1}^{(1)}, \ldots, h_{n+H-1}^{(1)}\} \subset \mathcal{S}$. Consider the function $f = \sum_{i=n}^{n+H-1} h_i^{(1)}$. Below is the graph of function f:

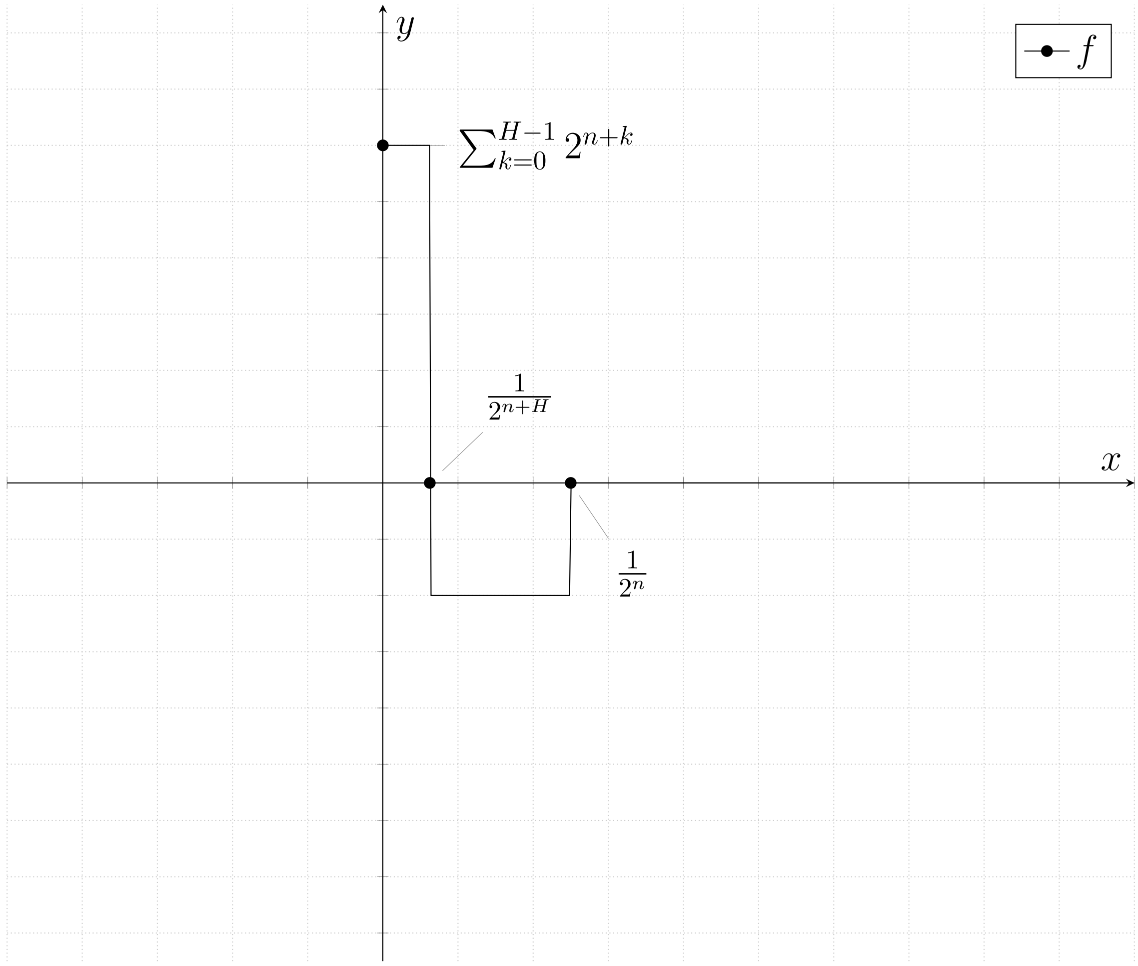

Figure 1.5

Note that

$$\|f\| = 2 - \frac{1}{2^{H-1}} \leq 2. \tag{1.21}$$

Let $m = \left[\frac{H}{2}\right]$, and consider the following realization of the greedy algorithm $G_m(f) = \sum_{i=0}^{m-1} h_{n+2i}^{(1)}$. For $G_m(f)$, one has

$$\|G_m(f)\| \geq \frac{m}{4}. \tag{1.22}$$

Therefore,

$$\|G_m(f)\| \geq \frac{H}{16}\|f\|, \tag{1.23}$$

which yields that $\|G_m\| \geq \frac{H}{16}$.

This result can be generalized for multidimensional case by using techniques of the proof of Theorem 2. There are two generalizations of the Haar system in

$L_1[0,1]^d$. We consider the multidimensional Haar system whose elements have cubic supports, and we first recall the definition of this system. The dyadic interval is the interval of type $\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)$, with $1 \le j \le 2^n$, $n \ge 0$. For a dyadic interval $\mathcal{I} = \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right) \subset [0,1)$, we write

$$r_{\mathcal{I}}^{(0)}(t) = \begin{cases} \frac{1}{|\mathcal{I}|} & : \ t \in \mathcal{I} \\ 0 & : \ t \notin \mathcal{I} \end{cases}, \qquad r_{\mathcal{I}}^{(1)}(t) = \begin{cases} \frac{1}{|\mathcal{I}|} & : \ t \in \left[\frac{j-1}{2^n}, \frac{2j-1}{2^{n+1}}\right) \\ -\frac{1}{|\mathcal{I}|} & : \ t \in \left[\frac{2j-1}{2^{n+1}}, \frac{j}{2^n}\right) \\ 0 & : \ t \notin \mathcal{I} \end{cases}, \tag{1.24}$$

For dyadic intervals $\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_d$ of the same length, the cube

$$\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2 \times \dots \mathcal{I}_d \tag{1.25}$$

is called a dyadic cube. By $\mathcal{D}^d$ we denote the set of all dyadic cubes of dimension d. To remind definition of multidimensional Haar system we need one more notation. Denote

$$\mathbb{M} = \mathcal{D}^d \times \{1, 2, \dots, 2^d - 1\}. \tag{1.26}$$

To each element $(\mathcal{I}, j) \in \mathbb{M}$ corresponds one element of the multidimensional Haar function $h_{\mathcal{I}}^{(j)}$, which is defined in the following way

$$h_{\mathcal{I}}^{(j)}(x) = \prod_{i=1}^{d} r_{\mathcal{I}_k}^{(\epsilon_k)}(x_k), \tag{1.27}$$

where $x = (x_1, x_2, \dots, x_d) \in [0,1]^d$ and numbers $\epsilon_k \in \{0,1\}$ are defined by the representation $j = \sum_{k=1}^{d} \epsilon_k 2^{d-k}$. The set of functions $h_{\mathcal{I}}^{(j)}$ together with $h_{[0,1)^d}^{(0)} \equiv 1$ is a multidimensional Haar system.

Now, for any $\mathcal{I}, \mathcal{J} \in \mathcal{D}^d$ with $\mathcal{J} \subset \mathcal{I}$, denote

$$C(\mathcal{I}, \mathcal{J}) = \{\Delta \in \mathcal{D}^d : \mathcal{J} \subseteq \Delta \subseteq \mathcal{I}\} \tag{1.28}$$

which is called a chain [14]. The length of the chain $C(\mathcal{I}, \mathcal{J})$ is the number of elements in the chain. Also we will say that $\mathcal{J}$ is a son of $\mathcal{I}$ iff $\mathcal{J} \subset \mathcal{I}$ and

$\mu(\mathcal{J}) = 2^{-d}\mu(\mathcal{I})$. By the term *complete chain* $C^d(\mathcal{I},\mathcal{J})$ we mean the following set

$$C^d(\mathcal{I},\mathcal{J}) = \{(\Delta, k) \in \mathbb{M} \ : \ \mathcal{J} \subseteq \Delta \subseteq \mathcal{I}, \ 1 \leq k \leq 2^d - 1\}. \tag{1.29}$$

By length of chain $C^d(\mathcal{I},\mathcal{J})$ we mean the length of the chain $C(\mathcal{I},\mathcal{J})$. Also, for $\mathcal{S} \subset \mathbb{M}$ denote by $G_{\mathcal{S}}$ the quasi-greedy constant constant for the system $\{h_{\mathcal{I}}^{(j)}\}_{(\mathcal{I},j)\in\mathcal{S}}$ in $L_1[0,1]^d$. We also put $H(\mathcal{S})$ the length of the longest chain in subsystem $\mathcal{S}$.

In this work, we improve the estimate for the quasi-greedy constant $G_{\mathcal{S}}$ in $L_1[0,1]^d$ as follows.

Theorem 3. *Assume there exists a subsystem of the Haar system* $\{\{h_{\mathcal{I}}^{(j)}\}_{\mathcal{I}\in\mathcal{S}}\}_{j=1}^{2^d-1}$*, which is a quasi-greedy subsystem in* $L_1[0,1]^d$*. Then, for the quasi-greedy constant* $G_{\mathcal{S}}$*, we have the following estimate:*

$$\|G_{\mathcal{S}}\| \leq 2^d(H-1)+1. \tag{1.30}$$

For $\mathcal{I} \in \mathcal{D}^d$, we denote

$$P_{\mathcal{I}}(f) = f - \sum_{\mathcal{J}\in\mathcal{D}^d, \mathcal{J}\subseteq\mathcal{I}} \sum_{j=1}^{2^d-1} c_{\mathcal{J}}^{(j)}(f) h_{\mathcal{J}}^{(j)}.$$

We emphasize that when $\mathcal{J} = [0,1)^d$, we allow j to take the value 0 to ensure that $h_{[0,1)^d}^{(0)}$ is counted. Note that the function $P_{\mathcal{I}}(f)$ is constant on the set $\mathcal{I}$ and coincides with f outside $\mathcal{I}$. Before presenting the main lemma in the multidimensional case, we proceed by recalling a few lemmas from [8] which are generalization for the multidimensional case.

This lemma we will use in next section too.

Lemma 6. *([8, Lemma 1]) For any* $f \in L_1[0,1]^d$ *and* $\mathcal{I},\mathcal{J} \in \mathcal{D}^d$ *with* $\mathcal{J} \subseteq \mathcal{I}$*, we have*

$$\|f\| \geq | \ c_{\mathcal{J}}^{(i)} \ | \ \text{ for all } 1 \leq j \leq 2^d - 1. \tag{1.31}$$

Lemma 7. *([8, Lemma 2]) Let $f \in L_1[0,1]^d$ and $\mid c_\Delta^{(i)} \mid \leq 1$ for any $\Delta \in \mathcal{D}^d$, and $1 \leq j \leq 2^d - 1$. Then, for every $\mathcal{I} \in \mathcal{D}^d$,*

$$\|P_{\mathcal{I}}(f)\|_{\mathcal{I}} \leq 1$$

Next, let $\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_{2^d}$ be the sons of $\mathcal{I}, \mathcal{I} \in \mathcal{D}^d$, and define the $d-$dimensional vectors t_{ik} from the conditions $\mathcal{I}_k = \mathcal{I}_i + t_{ik}$. Then, for any function f we define

$$C_i(f, \mathcal{I}) = \begin{cases} f(t); & \text{if } t \notin \mathcal{I}, \\ f(t); & \text{if } t \in \mathcal{I}, \\ f(x + t_{ik}); & \text{for } x \in \mathcal{I}_i \text{ for some } 1 \leq i \neq k \leq 2^d. \end{cases} \tag{1.32}$$

In fact, the operator C_i are copy operators for multidimensional case.

Lemma 8. *Let $f, g \in L_1[0,1]^d$ such that $SP(f+g) \neq \emptyset$, and let*

$$\|f\| = B\|f+g\| \tag{1.33}$$

Then,

$$\|C_i(f,\mathcal{I})\| \geq B\|C_i(f+g,\mathcal{I})\|, \text{ for some } 1 \leq i \leq 2^d, \tag{1.34}$$

or

$$\|C_i(f,\mathcal{I})\| = B\|C_i(f+g,\mathcal{I})\|, \text{ for all } 1 \leq i \leq 2^d. \tag{1.35}$$

Lemma 9. *Let $f, g \in L_1[0,1]^d$ and $\mathcal{I} \in \mathcal{D}^d$ be such that*

i) $\mathcal{I} \notin SP(f)$ and $\mathcal{I} \notin SP(g)$,

ii) $sp(f) \cap sp(g) = \emptyset$,

iii) $sp(f+q) \neq \emptyset$.

Then, for the functions

$$(f', g') = C((f,g), \Delta) = C_i((f,g), \Delta) \tag{1.36}$$

the following conditions hold:

1) $\Delta_n \notin SP(f')$, $\Delta_n \notin SP(g')$,

2) $c^{(j)}_{\mathcal{J}}(f') = c^{(j)}_{\mathcal{J}}(f)$ *and* $c^{(j)}_{\mathcal{J}}(g') = c^{(j)}_{\mathcal{J}}(g)$ *for all* $\mathcal{J}, \mathcal{J} \not\subset \mathcal{I} \setminus \mathcal{I}_i$,

3) *for any* $\mathcal{J} \subset \mathcal{I} \setminus \mathcal{I}_i$ *there exist* $\tilde{\mathcal{J}} \in \mathcal{I}_i$, *that* $c^{(j)}_{\mathcal{J}}(f') = c^{(j)}_{\tilde{\mathcal{J}}}(f)$ *and* $c^{(j)}_{\mathcal{J}}(g') = c^{(j)}_{\tilde{\mathcal{J}}}(g)$ *for all* $1 \leq j \leq 2^d - 1$,

4) $sp(f') \cap sp(g') = \emptyset$,

5) $sp(f' + g') \neq \emptyset$,

6) $\frac{\|f'\|}{\|f'+g'\|} \geq \frac{\|f\|}{\|f+g\|}$,

7) $H(SP(f' + q')) \leq H(SP(f + g))$.

Here we present the main lemma.

Lemma 10. *Let p and q be polynomials with respect to the multidimensional Haar system satisfying the following relations:*

i) $sp(p) \cap sp(q) = \emptyset$,

ii) *the function q is constant on all dyadic intervals where the function p is constant,*

iii) $[0.1)^d \notin SP(p + q)$,

iv) $\|p\| > 0$ *and* $\|p + q\| > 0$,

v) $H := H(SP(p + q)) < \infty$,

vi) $| c^{(j)}_{\mathcal{I}}(p) | \geq 1$ *for any* $\mathcal{I} \in SP(p)$ *and* $1 \leq j \leq 2^d - 1$,

vii) $| c^{(j)}_{\mathcal{I}}(q) | \leq 1$ *for any* $\mathcal{I} \in SP(q)$ *and* $1 \leq j \leq 2^d - 1$,

viii) *For any* $\mathcal{I} \notin SP(p + q)$ *and* $1 \leq k \leq 2^d$ *we have* $C_k(p, \mathcal{I}) = p$ *and* $C_k(q, \mathcal{I}) = q$.

Then,

$$\frac{\|p\|}{\|p+q\|} < 2^d(H-1)+1 \tag{1.37}$$

We skip the proof of Lemma 10 since it follows the same pattern as the proof of Lemma 5. Then, the proof of Theorem 3 directly follows from Lemma 10, applying the same techniques of the proof of Theorem 2.

ON DEMOCRATIC CONSTANT

Democratic bases have important role on classification of greedy type bases. Democratic subsystems of the 1-dimensional and multidimensional Haar systems in $L_1[0,1]$ and $L_1[0,1]^d$ are characterized respectively in [8] and [9]. In this section we use the definition of the multidimensional Haar system presented in Section 1.

For $\mathcal{S} \subset \mathbb{M}$ denote by $D(\mathcal{S})$ the democratic constant for the system $\{h_{\mathcal{I}}^{(j)}\}_{(\mathcal{I},j)\in\mathcal{S}}$ in $L_1[0,1]^d$.

Theorem 4 ([9])**.** *Let $\mathcal{S} \subset \mathbb{M}$ be given and let H be the length of longest complete chain in $\mathcal{S}$ (we assume it's equal to $+\infty$ if there are arbitrary long complete chains). Then $\{h_{\mathcal{I}}^{(j)}\}_{(\mathcal{I},j)\in\mathbb{M}}$ is democratic in $L_1[0,1]^d$ if and only if $H < +\infty$.*

From the proof of the theorem one may conclude that democratic constant satisfies to the condition $D(\mathcal{S}) \leq 2^{Hd}$. In this section we improve this result by proving following theorem.

Theorem 5. *Let $\mathcal{S} \subset \mathbb{M}$ and $\mathcal{S}$ contains complete chains having maximal length H. Then $D(\mathcal{S}) < 2^d(2^d-1)(H+1)$.*

Below, we recall several lemmas which will be used to prove Theorem 5.

Lemma 11 ([9], Lemma 2)**.** *Let $f \in L_1[0,1]^d$, and $\mathcal{I},\mathcal{J} \in \mathcal{D}^d$ be such that:*

1) $|c_{\mathcal{I}}^{(i)}(f)| \leq 1$ *for any* $(\mathcal{I}, i) \in \mathbb{M}$*,*

2) $\mathcal{I}$ *is a son of* $\mathcal{J}$*,*

3) $c_{\mathcal{J}}^{(i_0)}(f) = 0$ *for some* $1 \leq i_0 \leq 2^d - 1$*.*

Then,

$$\|P_{\mathcal{I}}(f)\|_{\mathcal{I}} \leq 1 - 2^{-d}. \tag{1.38}$$

We proceed with presenting the main Lemma which will be useful in the proof of Theorem 5.

Lemma 12. *Let* $\Lambda \subset \mathbb{M}$ *and let* H *be the length of the longest complete chain in* Λ*. Then,*

$$\| \sum_{(\mathcal{I},i)\in\Lambda} h_{\mathcal{I}}^{(i)} \| \geq \frac{|\ \Lambda\ |}{2^d(2^d-1)(H+1)}. \tag{1.39}$$

Proof Let

$$\{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_k\} = \{\mathcal{I}\ :\ \exists 1 \leq i \leq 2^d - 1,\ \ (\mathcal{I}, i) \in \Lambda\}.$$

According to the definition we have $k \geq \frac{|\Lambda|}{2^d-1}$. Without loss of generality, we may assume that

if for $1 \leq i < j \leq k$ one has $\mathcal{I}_j \cap \mathcal{I}_i \neq \emptyset$, then $\mathcal{I}_j \subset \mathcal{I}_i$ and for all $i < s < j$ one has $\mathcal{I}_s \subset \mathcal{I}_i$.

Put $f_0 = 0$ and for $1 \leq s \leq k$ denote

$$f_s = \sum_{i \leq s, j,\ (\mathcal{I}_i, j)\in\Lambda} h_{\mathcal{I}_i}^{(j)}.$$

Again, without loss of generality, we may assume that if $\mathcal{I}_i$ and $\mathcal{I}_j$ are sons of $\mathcal{I}_t$ with respect to $\{\mathcal{I}_i\}$ (with $i < j$) then

$$\|f_t\|_{\mathcal{I}_i} \geq \|f_t\|_{\mathcal{I}_j}.$$

Note, that

$$\| \sum_{(\mathcal{I},i)\in\Lambda} h_{\mathcal{I}}^{(i)} \| = \|f_k\| = \sum_{i=1}^{k} \Big(\|f_i\| - \|f_{i-1}\| \Big).$$

From the monotonicity of the Haar system and definitions of f_i it follows that all terms in the brackets are non-negative. To complete the proof of the Lemma it remains to show, that at least $\frac{1}{H+1}$ of them have value at least 2^{-d}. Indeed, consider an arbitrary sequence $\mathcal{I}_i, \mathcal{I}_{i+1}, \ldots, \mathcal{I}_{i+H}$. By taking into account the definition of H, we can state that they do not form a complete chain in Λ. It follows from construction that for at least one $j, i < j \leq i + H$ we have one of following cases:

i) $\mathcal{I}_j \subset \mathcal{I}_{j-1}$ and $\mathcal{I}_j$ is not a son of $\mathcal{I}_{j-1}$. Consider a dyadic cube $\mathcal{J}$ whose son is $\mathcal{I}_j$. According to Lemma 6, we have

$$\|f_j\|_{\mathcal{I}_j} \geq 1$$

and according to Lemma 11, we have (since $(\mathcal{J}, 1) \notin \Lambda$)

$$\|f_{j-1}\|_{\mathcal{I}_j} \leq 1 - 2^{-d}.$$

Since f_{j-1} and f_j coincide outside $\mathcal{I}_j$ we conclude

$$\|f_j\| - \|f_{j-1}\| \geq 2^{-d}.$$

ii) $\mathcal{I}_j$ is a son of $\mathcal{I}_{j-1}$ and for some $t, 1 \leq t \leq 2^d - 1$ we have $(\mathcal{I}_{j-1}, t) \notin \Lambda$. This case is similar to the previous case. According to Lemma 6, we have

$$\|f_j\|_{\mathcal{I}_j} \geq 1$$

and according to Lemma 11 we have

$$\|f_{j-1}\|_{\mathcal{I}_j} \leq 1 - 2^{-d}.$$

Since f_{j-1} and f_j coincide outside $\mathcal{I}_j$, we conclude

$$\|f_j\| - \|f_{j-1}\| \geq 2^{-d}.$$

iii) $\mathcal{I}_j \cap \mathcal{I}_{j-1} = \emptyset$. In this case we have

$$\|f_{j-1}\|_{\mathcal{I}_j} \leq \frac{1}{2},$$

therefore,

$$\|f_j\|_{\mathcal{I}_j} - \|f_{j-1}\|_{\mathcal{I}_j} \geq \frac{1}{2}.$$

Lemma is proved.

The proof of the Theorem then easily follows from the lemma.

Proof Let $A \subset \mathcal{S}$ and $|A| = n$. Note that it is enough to estimate the norm $\| \sum_{(\mathcal{I}_i, j_i) \in A} h_{\mathcal{I}_i}^{(j_i)} \|$.

We have, by Lemma 12, that

$$\| \sum_{(\mathcal{I}_i, j_i) \in A} h_{\mathcal{I}_i}^{(j_i)} \| \geq \frac{n}{2^d(2^d - 1)(H+1)}. \tag{1.40}$$

Also, by the triangle inequality, we get

$$\| \sum_{(\mathcal{I}_i, j_i) \in A} h_{\mathcal{I}_i}^{(j_i)} \| \leq n. \tag{1.41}$$

Finally, by (1.40) and (1.41), we have that

$$D(S) < 2^d(2^d - 1)(H+1), \tag{1.42}$$

and this concludes the proof.

CHAPTER 2

MODIFYING FUNCTION

It is an old idea to improve the property of the function, by changing its values on some small set. The first fundamental result in this direction was obtained by N. Luzin in [10]. He proved that any finite function can be changed on the set of given positive measure such, that it becomes continuous. Later, D. Menshov and others proved several fundamental theorem that any measurable function can be changed ('corrected') on the set of given small measure such, that its Fourier series by trigonometric function converges uniformly (see [15]).

Later, many results in this direction are obtained by P. Ulyanov, A. Talalyan, M. Grigoryan and others. Here we continue the research developed by M. Grigoryan in his series of articles. In [12] the following two theorems were proved.

Theorem L (Theorem 3, [12]). *For any measurable set $E, E \subset [0,1]$ with a measure $0 < |E| < 1$, there exists a function $f_0 \in L_1[0,1]$ such that if some function $f \in L_1[0,1]$ coincides with f_0 on E, then the sequence $\{|c_n(f)|\}_{n=1}^{\infty}$ cannot be monotonically decreasing, where $c_n(f)$ is the n-th coefficient of a Haar system normalized in $L_1[0,1]$.*

[12] also presents the following Theorem.

Theorem M (Theorem 4, [12]). *For any $0 < \epsilon < 1$ there exists a measurable set $E \subset [0,1]$ with $|E| > 1 - \epsilon$ such that for any function $f \in L_1[0,1]$ there* *exists some $\tilde{f} \in L_1[0,1]$ coinciding with f on E and such that all non-zero terms of the sequence $\{c_n(\tilde{f})\}$ are arranged in a decreasing order.*

Note that it follows from these theorems that one can modify a function such that all the non-zero terms of the series of coefficients with respect to the Haar system are arranged in a decreasing order, even though we are not able to order all of the terms monotonically.

The analogous questions were considered for other bases and in other spaces. For the space $C(0,1)$ of continuous on $[0,1]$ functions and for Faber-Schauder system some interesting results are obtained by M. Grigoryan and A. Sargsyan in [16]. Let's remind the definition of Faber-Schauder system. Definition and notations are due [16].

The Faber-Schauder system is the sequence of functions $\Phi = \{\phi_n\}_{n=0}^{+\infty}$, defined on segment $[0,1]$, in which $\phi_0 \equiv 1$, $\phi_1(x) = x$ on $[0,1]$ and for $n = 2^k + i$, $k = 0, 1, 2, \ldots$, $i = 1, 2, \ldots, 2^k$ one has

$$\phi_n(x) = \phi_k^{(i)}(x) = \begin{cases} 0, & x \notin [\frac{i-1}{2^k}, \frac{i}{2^k}], \\ 1, & x = \frac{2i-1}{2^{k+1}}, \\ \text{is} & \text{linear and continious on} [\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}}] \text{ and } [\frac{2i-1}{2^{k+1}}, \frac{i}{2^k}] \end{cases}$$

The following definition is also due [16].

Definition. Let Ψ is a basis in a Banach space X and let $0 < t \leq 1$. We say that coefficients of $f \in X$ are t-monotone with respect to Ψ if and only if for any integer $1 \leq n < m$ one has either

$$c_m(f) = 0,$$

or

$$\mid c_n(f) \mid \geq t \cdot \mid c_m(f) \mid .$$

The following theorem is proved in [16].

Theorem N (M. Grigoryan, A. Sargsyan, [16]). *For every $\epsilon \in (0,1)$ there exists a measurable set $E \subset [0,1]$ with measure $\mu(E) < 1-\epsilon$, such that to each function $f \in C[0,1]$ one can find a function $g \in C[0,1]$ that coincides with f on E and coefficients of whichâĂŹs expansion by the Faber-Schauder system are t-monotone for all $t \in (0, \frac{1}{2})$].*

Also, in 2013 the following theorem was proved.

Theorem O (M. Grigoryan, V. Krotov, [17]). *Let $\{a_n\}$ is a decreasing sequence of real numbers such, that*

$$\lim_{n\to\infty} a_n = 0,$$

and

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = +\infty.$$

Then for any ϵ with $0 < \epsilon < 1$ and for any measurable and almost everywhere finite (on $[0,1]$) function f there exists a function $\tilde{f} \in C[0,1]$ with the following properties:

i) $\mu\{\tilde{f} \neq f\} < \epsilon$,

ii) $a_n = c_n(\tilde{f})$ *for all n with $c_n(\tilde{f}) \neq 0$, where $c_n(\tilde{f})$ is the n-th Faber-Schauder series coefficient of function $\tilde{f}$.*

In [12] authors were asking a question either if exists a basis such, that after correcting the function the sequence of Fourier-series coefficients are ordered in decreasing order. In this chapter we show, that this question has positive answer. The example of such an basis is constructed in [18]. In this chapter we prove the following theorem.

Theorem 6. *There exists a normalized basis $\Psi = \{\psi_n\}_{n=1}^{\infty}$ in $L_1[0,1]$ such that for any $\epsilon, 0 < \epsilon < 1$ there exists a measurable set $E \subset [0,1]$ with a measure*

$|E| > 1 - \epsilon$ *such that for every function* $f \in L_1[0,1]$ *there exists a function* $\tilde{f}, \tilde{f} \in L_1[0,1]$ *which coincides with* f *on* E *and all of the terms of the sequence* $\{c_n(\tilde{f})\}$ *are arranged in a decreasing order.*

To prove Theorem 6 we construct the basis with required property and adopt ideas from [12] The basis with required property is constructed in [18] when the sequence $\{M_i\}_{i=1}^{\infty}$ is increasing very fast, e.g. $M_i = 2^{2^i}$. In this chapter we assume that $M_i = 2^{2^i}$.

Let us split the Haar system into two subsequences, $\{b_i\}$ and $\{\phi_i\}$, where $b_i = h_i^{(2)}, i = 1, 2, \ldots$. Next, denote $N_0 = 0$ and $N_i = \sum_{k=1}^{i} M_i = \sum_{k=1}^{i} 2^{2^k}$. [12] investigates a system of functions $F = \left\{ \{f_{(i,j)}\}_{j=0}^{M_i} \right\}_{i=1}^{\infty}$ which is defined in the following way:

$$f_{(i,0)} = \phi_i - \frac{1}{M_i + 1} \sum_{N_{i-1}+1 \le k \le N_i} b_k,$$

$$f_{(i,j)} = f_{(i,0)} + b_{N_{i-1}+j}, j = 1, 2, \ldots, M_i.$$

It is noted in [12] that the system $\{f_{(i,j)}\}$ is a basis in $L_1[0,1]$. Here we remark some properties of the system F and numbers M_i, which will be useful in the further discussions.

$$2 \le \|f_{(i,j)}\| \le 3, \tag{2.1}$$

$$\phi_i = \frac{1}{M_i + 1} \sum_{j=0}^{M_i} f_{(i,j)}, \tag{2.2}$$

$$\sum_{j=1}^{i} \frac{M_j}{M_{i+1}} < \frac{2}{M_i} < 2^{-i}. \tag{2.3}$$

Lemma 13. *Let the dyadic interval* $\Delta = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)$, *real numbers* $0 < \epsilon < 1, \gamma \neq 0$, $\delta_0 > 0, \delta_1 > 0$, $\delta_2 > 0$ *and positive integer* A *are given such, that* $\Delta \cap (0, \epsilon) = \emptyset$.

Then there exists a polynomial

$$P = \sum_{i=A}^{B} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)} \tag{2.4}$$

such that

i) *the sequence* $\{c_{(i,j)}\}$ *is positive and monotonically decreasing, also all of them are less than* δ_0,

ii) $\mu\big(\Delta \cap \{\mid P - \gamma \mid < \delta_1\}\big) = \mu\big(\Delta \cap \{\mid P + \gamma \mid < \delta_1\}\big) = \frac{\mu(\Delta)}{2}$,

iii) $\|P\|_{L^\infty([0,1]\setminus\Delta)} < \delta_1$.

iv) *Partial sums of* (2.4) *are increasing on* $[\epsilon, 1]$ *by* L^1 *norm,*

Proof Let's choose enough big α and divide Δ into 2^α equal dyadic intervals. Each of them corresponds to some Haar function.

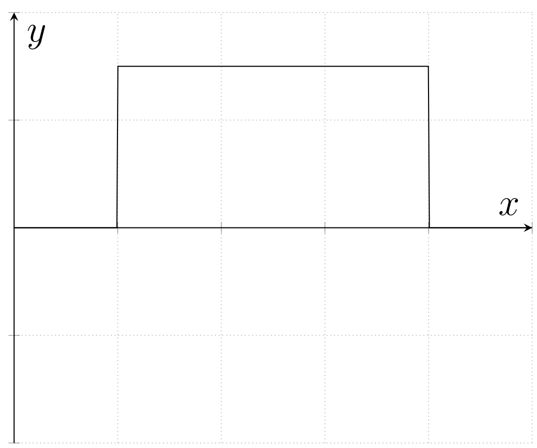

Figure 2.1

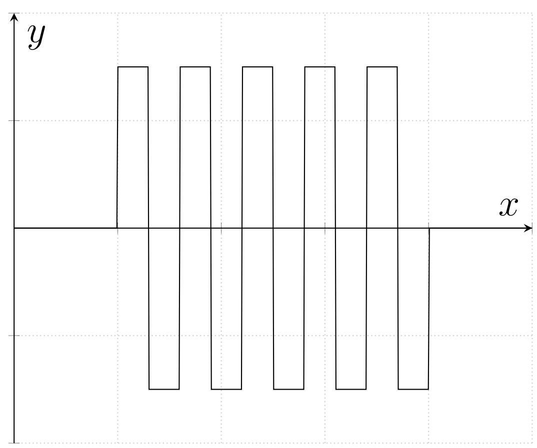

Figure 2.2

Assume these Haar functions are ϕ_{s+1}, ϕ_{s+2}, ..., ϕ_{s+2^α}. Each of them has maximal absolute value $2^{n+\alpha}$. It's obvious, that

$$p_1 = \sum_{i=1}^{2^\alpha} \frac{\gamma\phi_{s+i}}{2^{n+\alpha}}$$

is equal to γ on one half of Δ and is equal to $-\gamma$ on other half. Consider the function

$$P = \frac{\gamma}{2^{n+\alpha}}\left(\sum_{i=A}^{s} \frac{\phi_i}{M_s - 1} + \sum_{i=1}^{2^\alpha} \phi_{s+i}\right) = p_2 + p_1. \tag{2.5}$$

Let's estimate p_2. Since $\phi_i = h_t$ for some positive integer t, where $i < t < 2i$, and h_t has maximal absolute value between t and $2t$, we can state, that

$$\|\phi_i\|_\infty \leq 4i,$$

therefore

$$\|p_2\|_\infty \leq \frac{\gamma}{2^{n+\alpha}} \sum_{i=A}^{s} \frac{\|\phi_i\|_\infty}{M_s - 1} \leq \frac{2\gamma s^2}{2^{n+\alpha}(M_s-1)} \leq \gamma 2^{-\alpha}.$$

Since α can be chosen arbitrary big, we may guarantee, that $\|p_2\|_\infty$ is as small as we want. We conclude, that $P = p_1 + p_2$ is mainly focused on interval Δ and satisfies to the requirements ii) and iii) of current lemma. Note, that in above inequality we estimated also the partial sum's absolute value, but with j index taking all values. Note, that in the sum by j all coefficients are equal and since the Haar system is monotone basis, then the estimation holds also for all partial sums.

Now, let's investigate the coefficients of (2.5). By putting formulas for ϕ_i into (2.5) we get

$$P = \sum_{i=A}^{s}\sum_{j=0}^{M_i} \frac{\gamma}{(M_s-1)2^{n+\alpha}(M_i+1)} f_{(i,j)} + \sum_{i=1}^{2^\alpha}\sum_{j=0}^{M_i} \frac{\gamma f_{(s+i,j)}}{2^{n+\alpha}(M_{s+i}+1)}. \tag{2.6}$$

The sequence of coefficients in each sum is decreasing, since M_i is increasing sequence. It remains to compare the last coefficient in the first sum and the first coefficient in the second sum. Their division is equal

$$\frac{\gamma}{(M_s-1)2^{n+\alpha}(M_s+1)} : \frac{\gamma}{2^{n+\alpha}(M_{s+1}+1)} > \frac{M_{s+1}}{M_s^2} = 1.$$

Lemma 14. *Let the dyadic interval* $\Delta = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)$*, real numbers* $0 < \epsilon < 1, \gamma \neq 0$*,* $\delta_0 > 0, \delta_1 > 0$*, and positive integer* A *are given such, that* $\Delta \cap (0, \epsilon) = \emptyset$*.*

Then there exists a polynomial

$$P = \sum_{i=A}^{B} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)} \tag{2.7}$$

such that

i) the sequence $\{c_{(i,j)}\}$ *is positive and monotonically decreasing, also all of them are less than* δ_0*,*

ii) $\mu\big(\Delta \cap \{\mid P - \gamma \mid < \delta_1\}\big) > (1-\epsilon)\mu\Delta\mu$*,*

iii) $\|P\|_{L^\infty([0,1]\setminus\Delta)} < \delta_1$*.*

iv) Partial sums of (2.4) *are increasing on* $[\epsilon, 1]$ *by* L^1 *norm,*

Proof Let's apply Lemma 13 with the parameters given in this lemma. Then we get a polynomial

$$P_0 = \sum_{i=A}^{A_1} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)}, \tag{2.8}$$

for which

i) the sequence $\{c_{(i,j)}\}$ is positive and monotonically decreasing, also all of them are less than δ_0,

ii) $\mu\big(\Delta \cap \{\mid P_0 - \gamma \mid < \frac{\delta_1}{2}\}\big) = \mu\big(\Delta \cap \{\mid P + \gamma \mid < \frac{\delta_1}{2}\}\big) = \frac{\mu(\Delta)}{2}$,

iii) $\|P_0\|_{L^\infty([0,1]\setminus\Delta)} < \frac{\delta_1}{2}$.

iv) Partial sums of (2.4) are increasing on $[\epsilon, 1]$ by L^1 norm,

Now let's consider the set $\{\Delta \cap \{\mid P + \gamma \mid < \delta_1\}$. It has the half measure of Δ and is a finite union of dyadic intervals (their number is 2^α, where α is the parameter used in the proof of previous Lemma). Let's denote these intervals by Δ_1, Δ_2, ..., Δ_{2^α}. Now let's apply Lemma 13 with the following parameters 2γ as value of γ, the smallest non-zero coefficient of P_0 as δ_0, $\frac{\delta_1}{4}$ as δ_1, $\frac{\delta_2}{4}$ as δ_2 and $A_1 + 1$ as A. Then we get a polynomial

$$P_1 = \sum_{i=A_1+1}^{A_2} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)}, \tag{2.9}$$

for which

i) the sequence $\{c_{(i,j)}\}$ is positive and monotonically decreasing, also all of them are less than δ_0,

ii) $\mu\big(\Delta_1 \cap \{\mid P_1 - 2\gamma \mid < \frac{\delta_1}{2^2}\}\big) = \mu\big(\Delta_1 \cap \{\mid P_1 + 2\gamma \mid < \frac{\delta_1}{2^2}\}\big) = \frac{\mu(\Delta_1)}{2}$,

iii) $\|P_1\|_{L^\infty([0,1]\setminus\Delta_1)} < \frac{\delta_1}{2^2}$.

iv) Partial sums of (2.4) are increasing on $[\epsilon, 1]$ by L^1 norm,

Then we apply Lemma 13 for Δ_2 by choosing the smallest non-zero coefficient of P_1 as δ_0, 2γ as γ, $\frac{\delta_1}{2^3}$ as δ_1, and $A_2 + 1$ as A. Then we get a polynomial

$$P_2 = \sum_{i=A_2+1}^{A_3} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)},$$

for which

i) the sequence $\{c_{(i,j)}\}$ is positive and monotonically decreasing, also all of them are less than δ_0,

ii) $\mu\big(\Delta_2 \cap \{| \; P_2 - 2\gamma \; |< \frac{\delta_1}{2^3}\}\big) = \mu\big(\Delta_2 \cap \{| \; P_2 + 2\gamma \; |< \frac{\delta_1}{2^3}\}\big) = \frac{\mu(\Delta_2)}{2}$,

iii) $\|P_2\|_{L^\infty([0,1]\setminus\Delta_2)} < \frac{\delta_1}{2^3}$.

iv) Partial sums of (2.4) are increasing on $[\epsilon, 1]$ by L^1 norm,

Inductively, let's apply Lemma 13 for Δ_i by choosing the smallest non-zero coefficient of P_{i-1} as δ_0, 2γ as γ, $\frac{\delta_1}{2^{i+1}}$ as δ_1 and $A_i + 1$ as A. Then we get a polynomial

$$P_i = \sum_{i=A_i+1}^{A_{i+1}} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)},$$

for which

i) the sequence $\{c_{(i,j)}\}$ is positive and monotonically decreasing, also all of them are less than δ_0,

ii) $\mu\big(\Delta_i \cap \{| \; P_i - \gamma_i \; |< \frac{\delta_1}{2^{i+1}}\}\big) = \mu\big(\Delta_i \cap \{| \; P_i + 2\gamma \; |< \frac{\delta_1}{2^{i+1}}\}\big) = \frac{\mu(\Delta_i)}{2}$,

iii) $\|P_i\|_{L^\infty([0,1]\setminus\Delta_i)} < \frac{\delta_1}{2^{i+1}}$.

iv) Partial sums of (2.4) are increasing on $[\epsilon, 1]$ by L^1 norm,

Now, denote

$$\tilde{\Delta} = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_{2^\alpha},$$

and

$$\tilde{P} = P_0 + P_1 + \ldots + P_{2^\alpha}.$$

Note, that $\tilde{P}$ is 'close' to γ on half part of $\tilde{\Delta}$ and is 'close' to -3γ on other half part of $\tilde{\Delta}$. Also $\tilde{P}$ is 'close' to γ on the set $\Delta \setminus \tilde{\Delta}$.

Now, let's consider the half part of $\tilde{\Delta}$ where $\tilde{P}$ is close to -3γ (denote this set $\overline{\Delta}$. This set is finite union of dyadic intervals. For each such a dyadic interval apply lemma 13 with 4γ as γ. The choice of A, δ_0 and δ_1 is done as above. So we get a set of functions, such, that their sum is 'close' to 4γ on half part side of $\overline{\Delta}$ and -4γ on other half part. Outside of $\overline{\Delta}$ their sums is 'close' to 0. By doing this process in total ν times we get a sequence of functions. Denote by P their sum. According to the construction this function has all properties, that are required in the Lemma. Just we need to ensure, that $2^{-\nu} < \epsilon$.

Lemma 15. *Let the function*

$$Q = \begin{cases} \gamma_i & on\ \Delta_i \\ 0 & otherwise \end{cases}$$

and $0 < \epsilon < 1$ *are given, where* $\{\Delta_i\}_{i=1}^n$ *is a set of disjoint dyadic intervals, such that* $Q = 0$ *on* $[0, \epsilon]$. *Then for every real numbers* $\delta_0 > 0, \delta_1 > 0$, *and positive integer* A *there exists a polynomial*

$$P = \sum_{i=A}^{B} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)} \tag{2.10}$$

and a set $E \subset [0,1]$ *with* $\mu(E) > 1 - \epsilon$, *such that*

i) *the sequence* $\{c_{(i,j)}\}$ *is positive and monotonically decreasing, also all of them are less than* δ_0,

ii) $\{| P - Q |> \delta_1\} \bigcap E = \emptyset$,

iii) $\|P\|_{L^\infty([0,1]\setminus\bigcup_i \Delta_i)} < \delta_1$.

iv) *Partial sums of* (2.10) *are increasing on* $[\epsilon, 1]$ *by* L^1 *norm.*

Proof By applying the Lemma 14 for the function which is equal to γ_1 on Δ_1 and is equal to 0 outside of it we get a polynomial P_1 of the form

$$P_1 = \sum_{i=A}^{A_1} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)}$$

which satisfies to the conditions

i) the sequence $\{c_{(i,j)}\}$ is positive and monotonically decreasing, also all of them are less than δ_0,

ii) $\mu\big(\Delta_1 \cap \{\mid P_1 - \gamma_1 \mid < \frac{\delta_1}{2}\}\big) > (1-\epsilon)\mu(\Delta_1)$,

iii) $\|P\|_{L^\infty([0,1]\setminus\Delta_1)} < \frac{\delta_1}{2}$,

iv) partial sums of are increasing on $[\epsilon, 1]$ by L^1 norm,

Denote

$$E_1 = \Delta_1 \setminus \{t \in \Delta_1 \quad : \quad \mid P(t) - \gamma_1 \mid > \frac{\delta_1}{2}\}.$$

From the above estimation follows, that $\mu(E_1) \leq \epsilon\mu(\Delta_1)$.

Denote by $\tilde{\delta}_1$ the smallest non-zero coefficient in the expansion of P_1. Then, by applying the Lemma 14 for the function which is equal to γ_2 on Δ_2 and is equal to 0 outside of it we get a polynomial P_2 of the form

$$P_2 = \sum_{i=A_1+1}^{A_2} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)}$$

which satisfies to the conditions

i) the sequence $\{c_{(i,j)}\}$ is positive and monotonically decreasing, also all of them are less than $\tilde{\delta}_1$,

ii) $\mu\big(\Delta_2 \cap \{\mid P_2 - \gamma_2 \mid < \frac{\delta_1}{2^2}\}\big) > (1-\epsilon)\mu(\Delta_2)$,

iii) $\|P_2\|_{L^\infty([0,1]\setminus\Delta_2)} < \frac{\delta_1}{2^2}$,

iv) partial sums of are increasing on $[\epsilon, 1]$ by L^1 norm, Denote

$$E_2 = \Delta_2 \setminus \{t \in \Delta_2 \quad : \quad | P(t) - \gamma_2 | > \frac{\delta_2}{2}\}.$$

By iterating, after i-th step $(2 \leq i \leq n-1)$ denote by $\tilde{\delta}_i$ the smallest non-zero coefficient in the expansion of P_i. Then, by applying the Lemma 14 for the function which is equal to γ_{i+1} on Δ_{i+1} and is equal to 0 outside of it we get a polynomial P_{i+1} of the form

$$P_{i+1} = \sum_{i=A_i+1}^{A_{i+1}} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)}$$

which satisfies to the conditions

i) the sequence $\{c_{(i,j)}\}$ is positive and monotonically decreasing, also all of them are less than $\tilde{\delta}_i$,

ii) $\mu\big(\Delta_{i+1} \cap \{| P_{i+1} - \gamma_{i+1} | < \frac{\delta_1}{2^{i+1}}\}\big) > (1-\epsilon)\mu(\Delta_{i+1})$,

iii) $\|P_{i+1}\|_{L^\infty([0,1]\setminus\Delta_2)} < \frac{\delta_1}{2^{i+1}}$,

iv) partial sums of are increasing on $[\epsilon, 1]$ by L^1 norm,

Denote

$$E_{i+1} = \Delta_{i+1} \setminus \{t \in \Delta_{i+1} \quad : \quad | P(t) - \gamma_{i+1} | > \frac{\delta_1}{2}\}.$$

Finally, denote

$$P = \sum_{i=1}^{n} P_i$$

and

$$E = [0,1] \setminus \left(\bigcup_i \Delta_i \setminus E_i \right).$$

It is obvious that the sequence $\{c_{(i,j)}\}$ is positive and monotonically decreasing, also all of them are less than $\tilde{\delta}_i$. Also we have

$$\mu(E) = 1 - \sum_{i=1}^{n} \mu(\Delta_i \setminus E_i) \geq 1 - \epsilon \sum_{i=1}^{n} \mu(\Delta_i) \geq 1 - \epsilon.$$

Now let's estimate $\mid P - Q \mid$. For $t \in \Delta_i \setminus E_i$, $1 \leq i \leq n$ we have

$$\mid P(t) - Q(t) \mid \leq \sum_{j=1, j \neq i}^{n} \|P_i\|_{L^\infty(\Delta_i)} + \mid P_i(t) - Q(t) \mid \leq$$

$$\sum_{j=1, j \neq i}^{n} \frac{\delta_1}{2^i} + \frac{\delta_1}{2^i} < \delta_1.$$

And for $t \notin \bigcup_i \Delta_i$ we have $P(t) = 0$.

Proof of the Theorem. Let $\mathbb{Q}$ is the set of all functions that are step-by-step constant, all values are rational, constant intervals are dyadic and the number of constant intervals is finite. Consider the set $Q \times N_+$. Order it in some way. For any $(q_n, A_n) \in Q \times N_+$ there exists a polynomial p_n with respect to $\{f_{(i,j)}\}$ such that

$$p_n = \sum_{i=A_n}^{B} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)},$$

and a set $E_n, E_n \subset [0,1]$ with $\mu(E_n) > 1 - \epsilon 2^{-n}$ is such that

i) the sequence $\{c_{(i,j)}\}$ is positive and monotonically decreasing, also all of them are less than the smallest non-zero coefficient of p_{n-1},

ii) $\mu\big(\mid q_n - p_n \mid < \frac{\epsilon}{2^{n+1}}\}\big) > 1 - \epsilon 2^{-n}$,

iii) partial sums of are increasing on $[\epsilon, 1]$ by L^1 norm.

Finally denote

$$E = \bigcap_n E_n.$$

Note that

$$\mu(E) = 1 - \mu\Big(\bigcup_n([0,1] \setminus E_n)\Big) > 1 - \sum_n \epsilon 2^{-n} = 1 - \epsilon.$$

Now we prove that the set E satisfies to the requirements of the Theorem. Let $f_0 \in L^1(0,1)$ is given. Choose small enough δ and choose a function $\tilde{q}_1 \in \mathbb{Q}$ such that

$$\|f_0 - \tilde{q}_1\| < \frac{\delta}{2}.$$

Then, consider the function g_1 which is defined from the condition $g_1 = p_{k_1}$, where $(q_{k_1}, 1) = (\tilde{q}_1, 1)$. According to the construction we have

$$g_1 = \sum_{i=1}^{B_1} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)}$$

where $c_{(i,j)}$ is decreasing, partial sums are increasing as well as

$$\mu(|g_1 - \tilde{q}_1| > \frac{\delta}{2}) \bigcap E = \emptyset.$$

Then we have

$$\|f_0 - g_1\|_{L^1(E)} \leq \|f_0 - \tilde{q}_1\| + \|g_1 - \tilde{q}_1\|_{L^\infty(E)} \leq \delta.$$

Now, denote $f_1 = f_0 - g_1$. Choose a function $\tilde{q}_2 \in \mathbb{Q}$ such that

$$\|f_1 - \tilde{q}_2\| < \frac{\delta}{2^2}.$$

Let's chose g_2 which is defined from the condition $g_2 = p_{k_2}$, where $(q_{k_2}, B_1 + 1) = (\tilde{q}_2, B_1 + 1)$. According to the construction we have

$$g_2 = \sum_{i=B_1+1}^{B_2} \sum_{j=0}^{M_i} c_{(i,j)} f_{(i,j)}$$

where $c_{(i,j)}$ is decreasing and all of them are smaller than the smallest non-zero coefficient of g_1, partial sums are increasing as well as

$$\mu(|g_2 - \tilde{q}_2| > \frac{\delta}{2^2}) \bigcap E = \emptyset.$$

Then we have

$$\|f_1 - g_2\|_{L^1(E)} \leq \|f_1 - \tilde{q}_2\| + \|g_2 - \tilde{q}_2\|_{L^\infty(E)} \leq \frac{\delta}{2}.$$

By continuing this process infinitely many times we get a series $g_1+g_2+\ldots$ which converges to f_0 on E. Also the series is convergent itself, since its monotonic and bounded by $2\|f_0\|$.

CONCLUSION

The main results of the current thesis are:

1. The quasi-greedy constant of the quasi-greedy subsystems of the Haar system in $L_1[0,1]$ is estimated. The previously known exponential $G < 2^H$ estimate has been improved linearly:

$$\frac{H}{16} \leq G \leq 2(H+1).$$

2. The democratic constant of the democratic subsystems of the multi-dimensional Haar system in $L_1[0,1]^d$ is also estimated. The previously known exponential $D < 2^{Hd}$ estimate has been improved linearly:

$$D < 2^d(2^d-1)(H+1).$$

3. There exists a basis such that for any function $f \in L_1[0,1]$, after modifying it on a set of a small measure, its expansion coefficients with respect to that basis form a monotonically decreasing sequence.

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3. **On the Democratic Constant of Haar Subsystems in** $L_1(0,1)^d$.

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4. **On the quasi-greedy constant of the Haat subsytems in** $L_1(0,1)$.

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