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Integrable systems connected with black holes

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Defense takes place on 30th of May 2019 at 15:00 on meeting of the special council of the Higher Attestation Commission of RA, 048.
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PhD in ph-math sciences Elena Nikoghosyan

Abstract

This work is devoted to the study of some important questions in general relativity. They include topics related to astrophysical shock waves, impulsive signals, gravitational memory effect, black hole geometries and integrable systems connected with them.

We begin by studying the effects that an impulsive signal in a singular hypersurface can have on a particle which encounters it. We propose a new approach for studying the effect of null shells on null geodesic congruences. This is an exact method which allows one to easily calculate the change in the expansion, shear and rotation of the congruence upon crossing the shell and its evolution to the future of the shell. We find that the effect of the shell on the congruence is a discontinuity in the B-tensor (the gradient of the geodesic vector). We call this the B-memory effect, which is a more covariant way of describing the gravitational memory effect. Gravitational memory effect has deep connections with soft gravitons, which in turn is linked to the symmetries of null infinity of asymptotically flat spaces. We found the explicit relation of B-memory with the stress energy and gravitational wave components of the shell. We consider the simplest case of a null shell representing an outgoing gravitational wave and parametrized by a general soldering transformation (a subclass of which are the BMS supertranslations) in Minkowski space, but our method is applicable to any geodesic congruence that crosses a null shell localized on a killing horizon.

Furthermore, we study conformal mechanics associated with Near Horizon geometry of Extremal Myers-Perry (NHEMP) black hole. We propose a unified description of an arbitrary odd and even dimensional geometry and conformal mechanics. Then, the question of integrability of special cases of fully non-isotropic and fully isotropic cases is addressed in this description. We found a non-trivial transformation from non-isotropic NHEMP conformal mechanics to its isotropic case. The general case, when groups of equal and non-equal rotation parameters exist, is studied. It is shown that this problem reduces to its special cases of fully non-isotropic and fully isotropic NHEMP conformal mechanics. At the end we turn to the so-called Near Horizon Extremal Vanishing Horizon Myers-Perry black hole (NHEVHMP). The integrability properties of NHEVHMP in higher dimensions in fully isotropic, fully non-isotropic and general cases are discussed.

We have also studied quantum aspects of the superintegrable systems encountered in black hole backgrounds. We propose a geometrization procedure which associates to a non-relativistic quantum particle in a potential on a curved spacetime a purely geodesic motion in another geometry. In other words, we propose a correspondence between the solutions of Schroedinger equation and Klein-Gordon equation. We explain this procedure on the example of the Higgs oscillator and superintegrable Rosochatius system.

Relevance and motivation

Einstein equations have been discovered for over a century now and found many important applications in experimental and theoretical physics. Despite the long period they are being studied by the scientific community, there are just a few exact solutions known so far and one class of them is called black hole (BH) metrics. These solutions were one of the main discoveries of general relativity, first of all, due to their astrophysical importance. Black holes are assumed to be a final step in star evolution, are believed to make an important contribution in galaxy formation processes and are "blamed" to be responsible for a great amount of high energy radiation that we detect in the universe. Although black holes have not been observed directly, their indirect observations are overwhelming and from general considerations it is believed that they should be rotating objects with almost no electric charge. The space-time of such a black hole is best approximated by the Kerr metric [2], which is a four-dimensional stationary, asymptoticly flat vacuum solution of Einstein's equations. There exists higher dimensional generalization of Kerr space-time which is called Myers-Perry black hole [11]. Just like Kerr black hole, it is also a stationary, asymptoticly flat vacuum solution of Einstein's equations which describes a spinning black hole in an arbitrary dimension.

An important special case of Kerr black hole is the so called extremal Kerr solution which has the smallest possible mass for a given angular momentum or charge. Existence of astrophysical black holes is a matter of debate, but if in the future, the measurements of high angular momentum will be confirmed, extremal black holes will start to represent real astrophysical interest.

Black hole geometries are also important objects in mathematical physics. Many of them represent a background for integrable systems. Some of these integrable systems have been unknown prior to their discovery in black hole geometries. Particularly interesting is the integrability of Hamilton-Jacobi equation as it describes the geodesics of particles. Geodesics in the near horizon limit of Kerr black hole are associated with black hole accretions which might be a source of Very High Energy (VHE) gamma-ray bursts. Accretions around black holes can also be the key to the first direct observation of a black hole (e.g. with the Event Horizon Telescope).

As it is known, the Killing vectors of a geometry are associated with integrals of motion of a geodesic in that metric. In the case of the near horizon metric of an extremal rotating black hole, the killing vectors obey the structural relation of $SO(2,\ 1)$ algebra. It has been demonstrated (e.g. [10]) that the Casimir element of this $SO(2,\ 1)$ algebra gives rise to a reduced Hamiltonian system called spherical or angular mechanics, which contains all the specific information about the near horizon geometry. By reformulating this discussion one can say that a massive particle moving in the near horizon geometry of an extremal rotating black hole possesses dynamical conformal symmetry, i.e. defines "conformal mechanics" [4, 6–8], whose Casimir element can be viewed as a reduced Hamiltonian, which contains all the necessary information about the whole system.

On the other hand this reduced Hamiltonian or the spherical mechanics can be thought of as a separate system. Spherical mechanics associated with near horizon extremal black hole geometries are relatively unexplored. Latest works in this direction include [5], where the Hamiltonian of the spherical mechanics associated with Near Horizon Extreme Myers-Perry (NHEMP) geometry has been constructed for the special case when all rotation parameters of the black hole are equal. In [7, 8] the integrability of this system has been proven and the integrals of motion were presented. Extremal Myers-Perry black holes with nonequal nonvanishing rotation parameters in odd dimensions have been studied in [4] where the integrability of such systems was proven and separation of variables was carried out.

As we will see, the near horizon geometry of Myers-Perry black holes contains integrable and superintegrable systems like Rosochatius and Pöschl-Teller systems. Studies of these kind of systems is important as they appear in many topics of theoretical physics. Another approach that we have adopted here for investigating such systems is their geometrization procedure. Geometrical counterparts of classical systems have been studied extensively. They provide a new viewpoint to existing and well-known classical systems and spread some light on their underlying structure.

We will propose a geometrization procedure for quantum systems. We are mostly interested in problems which are superintegrable in higher dimensions. Particularly interesting are the Higgs oscillator [12,13], which is a particle on a d-sphere with a specific potential and the superintegrable Rosochatius system - a direct generalization of the Higgs oscillator. We will encounter the classical superintegrable Rosochatius system as the angular mechanics of near horizon limit of fully isotropic Myers-Perry black hole. Separation of variables in Rosochatius system results into a recursive family of one-dimensional Pöschl-Teller system.

Higgs oscillator, Rosochatius system and Pöschl-Teller system belong to a class of quantum quantum systems where energy are quadratic functions of the energy level number. After the geometrization procedure proposed in this work, these systems will result into Klein-Gordon equations with eigenmode frequencies linear in the frequency level number. In other words this means that the frequencies are highly resonant, which itself has important consequences in the AdS stability problem.

Another important class of solutions of Einstein's equations are gravitational waves. Compared to black holes, gravitational waves have been directly detected in 2015 by two LIGO (Laser Interferometer Gravitational-wave Observatory) detectors. Besides being a real physical phenomena and one of the most important predictions of the theory of general relativity, gravitational waves will take an important role in observational astronomy. Compared to other types of radiation, e.g. photons, neutrinos and cosmic rays, gravitational waves don't get refracted by gas clouds and don't get absorbed by cosmic bodies and can travel big distances, pointing directly back to the source. The importance of gravitational wave detectors will grow with their sensibility.

When a pair of inertial test particles encounter gravitational waves, their relative positions get shifted permanently. This phenomena is called the gravitational memory effect. It is known to be related to the theory of soft gravitons and symmetries of null infinity of asymptotically flat spaces and particularly black holes. We are going to discuss this effect and suggest its covariant formulation in frames of a model of impulsive gravitational waves. This model assumes that the space-time is divided into two domains by a hypersurface, which in general can contain a mixture of gravitational waves and other material sources. There are many examples of physical systems in nature which can be described in frames of this model. Such systems may appear after cataclysmic astrophysical events, such as a supernova or a collision of neutron stars. These systems are used to simulate an exploding white hole, to model an impulsive null signal from a system of neighboring test particles and have many other applications. In general one can choose these two metrics to be either continuous or discontinuous on the boundary hypersurface subject to the condition that the induced metric is unique, but either way the metric's transverse derivative will not be continuous. This always leads to a singularity in the form of a δ-function in the Riemann tensor.

There are two different approaches to describe singular hypersurfaces. The first one is called the distributional method. In this case a common set of coordinates is used for both sides of the hypersurface. The other method is a generalization of the "cut and paste" approach of Penrose. Here, the space-time coordinates on the two sides of the hypersurface can be chosen independently from each other, so in this sense it is a more general approach than the distributional algorithm. It was introduced by Israel to describe timelike hypersurfaces [1] but it was not suitable for the case of null hypersurfaces. In the timelike case, the Israel approach uses the extrinsic curvature of the hypersurface to describe the stress-energy tensor in it. When we move to the null case, the intrinsic metric of the hypersurface space-time becomes degenerate, because the normal vector becomes tangent and there is no distinguishable transverse vector defined. Hence, the extrinsic curvature, which is defined in terms of the metric, is no longer uniquely definable, so it cannot be used to study the hypersurface. This problem was solved and the approach was generalized for the lightlike case by Barrabès and Israel [3].

Aim of the dissertation

The purpose of this thesis is to address some important questions in general relativity, particularly related to gravitational waves, black holes and integrable systems connected with them. Our goals can be summarized in the following points:

- Develop a technique which would allow to analytically study the effects of impulsive signals on geodesic congruences. Give a general covariant definition of the gravitational memory effect of impulsive gravitational waves.

- Study the integrable systems connected with massive and massless particles in the background of Near Horizon Extremal Myers-Perry (NHEMP) black holes in arbitrary dimensions.

- Study the integrable systems connected with massive and massless particles in the background of Near Horizon Extremal Vanishing Horizon Myers-Perry (NHEVHMP) black holes in arbitrary odd dimensions.

- Map (super)integrable quantum systems to static spacetimes in arbitrary dimensions. In other words, relate solutions of Schroedinger equation to Klein-Gordon equation in different static background.

Novelty of the works

We suggested a new approach for studying geodesic congruences after they encounter impulsive gravitational waves. Analytic description of the congruences in the future allowed us to give a general covariant definition for the gravitational memory effect. We studied the geodesics of particles in the background of NHEMP and NHEVHMP black holes. Thus, we encountered integrable and superintegrable systems, identified some of them with previously known mechanics and successfully separated the variables in Hamilton-Jacobi equation and found the constants of motion. Furthermore, we suggested a new approach for geometrization of the quantum counterparts of these and other systems, this way relating them to Klein-Gordon equation in static background.

Practical value

We studied some important questions in general relativity and mathematical physics mainly related to the two most important solutions of the theory of relativity - gravitational waves and black holes. In particular, the work is related to astrophysical shock waves, gravitational waves, black holes, integrable systems associated with them as well as their quantum equivalents. We studied the effects of null shells on geodesic congruences and suggested a general covariant definition of the gravitational memory effect. Thus, we studied observable effects that astrophysical shock waves can have on test particles, after cataclysmic astrophysical events. We studied the geodesics of massive particles in NHEMP black hole geometries. This is the space-time in the vicinity of the horizon of higher dimensional rotating black holes. Thus, this work can have applications for studying accretions of black holes. The system is also important in mathematical physics as it describes integrable (in special cases superintegrable) system, where the constants of motion are fully studied. On the other hand, the quantum counterparts of this and other integrable systems are studied as well and a new technique is suggested for geometrization of these systems.

Summary

Here we outline of the main results of this thesis.

- A new approach has been suggested for studying the effects of impulsive gravitational waves on congruences encountering them. The technique has been applied on null congruences. It has been established that hypersurface orthogonal null congruences stay such after crossing the shell.
- A covariant definition of the gravitational memory effect has been suggested based on the B-tensor of the congruence. The relations between the components of the B-tensor and the stress-energy tensor of the shell have been derived. The B-tensor has been calculated and the approach has been demonstrated for BMS type soldering.
- A common description has been introduced for even and odd dimensional NHEMP geometries. This description was used to prove that the even dimensional fully non-isotropic NHEMP system is integrable.
- Integrals of motion, as well as the Killing vectors of the fully non-isotropic NHEMP in arbitrary dimensions have been presented in

initial coordinates. We found a non-trivial transformation between the integrals of motion of fully non-isotropic and fully isotropic NHEMP black hole geometries.

- We separated the variables of the most general partially isotropic NHEMP and showed its transformation to the special cases of fully non-isotropic and isotropic NHEMP.

- A new approach has been suggested for mapping Schrödinger equation on a curved background to a Klein-Gordon equation on the background of another geometry. We have shown that this procedure greatly simplifies for systems with quadratic spectra and applied it on the Higgs oscillator and the superintegrable Rosochatius system.

Content

In this thesis we discuss different problems related to asymptotic flat spaces, integrable systems and mathematical physics associated with black holes. **Chapter 1** is an introduction to the problems addressed in the thesis. In **Chapter 2**, we will study gravitational memory effect which is known to have deep connections with soft gravitons and symmetries of null infinity of asymptotically flat spaces. In **Chapter 3** and **Chapter 4**, we discuss Myers-Perry black holes, more particularly, the near horizon geometry and associated integrable systems. In **Chapter 5** we propose a geometrization procedure for a special class of quantum (super)integrable systems, which appear in many topics of mathematical physics (including in near horizon geometry of Myers-Perry black hole).

Chapter 1

This chapter is an introduction to the thesis. Here we discuss our motivations, the background of the problems and the main questions addressed. We briefly introduce the models and mathematical techniques used in solving the problems and the main results obtained.

Chapter 2

We have presented a new approach for studying congruences that cross a singular hypersurface. Our method is based on the physically justified assumption that the geodesic vector of a test particle is continuous across the hypersurface when using continuous coordinates. To obtain the geodesic flow to the future of the hypersurface one simply needs to do a coordinate transformation on the past coordinates to go to a continuous coordinate system. The resulting transformation on the geodesic congruence in $\mathcal{M}^-$ gives initial conditions on $\mathcal{N}$ to develop the geodesic vector field on $\mathcal{M}^+$ to the future.

We then proved that a parallel congruence upon crossing the shell gives rise to a hypersurface orthogonal congruence to the future of the shell, and in particular that the shell gives rise to a discontinuity in the B-tensor of the congruence. In general the jump in the expansion is determined by the energy density and currents on the shell while the jump in the shear is determined by the gravitational wave component together with the surface currents. Although we derived these results using a particular congruence, it is clear that the results are independent of the choice of congruence in the case of BMS supertranslations for which the surface currents are zero. We also provide a general argument that a hypersurface orthogonal congruence before the shell will give rise to a hypersurface orthogonal congruence to the future.

The change in the B-tensor after the passage of an outgoing gravitational wave leads to a covariant description of the gravitational memory effect - the B-memory effect.

Chapter 3

The isometry group of generic stationary extremal black holes in the near horizon region is known to have an $SO(2,1) = SL(2,\mathbb{R})$ part. Therefore, particle dynamics on the near horizon extreme geometries possesses dynamical $0+1$ dimensional conformal symmetry, i.e. it defines a "conformal mechanics" . This allows to reduce the problem to the study of system depending on latitudinal and azimuthal coordinates and their conjugate momenta with the effective Hamiltonian being Casimir of conformal algebra. Such associated systems are called "angular (or spherical) mechanics".

In this chapter, we continue our analysis of [4,6] and extend the analysis to Near Horizon Extremal Myers-Perry [11] (NHEMP) black holes [9]. We present the geometry of near-horizon extremal Myers-Perry black holes in generic even and odd dimensions, and construct the "angular mechanics" describing probe particle dynamics.

We analyze generic causal curve, massive or massless geodesic, in the NHEMP background in two special cases. First we analyze fully non-isotropic case. We show that this Hamiltonian system is separable in ellipsoidal coordinate system and establish that the system is integrable. We discuss constants of motion for "angular mechanics" associated with these systems. Moreover, we show how the Killing vectors and second rank Killing tensors are related to these constants of motion. Furthermore, we analyze the special case where all of the rotation parameters of the background NHEMP are equal to each other. In these case we have some extra Killing vectors and tensors and the system is superintegrable in even dimensions and maximally superintegrable in odd dimensions.

Chapter 4

When some of the rotation parameters are equal, the NHEMP geometry exhibits a bigger isometry group than $SL(2,\mathbb{R}) \times U(1)^N$; depending on the number of equal rotation parameters, the $U(1)^N$ part is enhanced to a rank N subgroup of $U(N)$. This larger isometry group brings larger number of Killing vectors and tensors and one hence expects the particle dynamics for these cases to become a superintegrable system. This is what we explore in this chapter and construct the corresponding conserved charges.

We also discussed the EVH case, which happens for odd dimensional extremal MP when one of the rotation parameters a_i vanishes. In the general NHEVHMP case, where the background isometry is $SO(2,2) \times U(1)^{\frac{d-3}{2}}$ the number of independent charges associated with Killing vectors is $\frac{d+1}{2}$. Despite enhancement of the isometry group compared to the generic NHEMP case, we found that this symmetry enhancement does not add independent constants of motion, the system in general does not poses extra constants of motion and remains just integrable.

By summarizing the results of **Chapter 3** and **Chapter 4** one can describe the rough picture as follows: We started with a system with $2N+1+\sigma$ variables with N isometries. Fixing the momenta associated with the isometries, we obtained and focused the $N-1+\sigma$ dimensional "angular mechanics" part. In this sector, whenever N number of rotation parameters m_i of the background metric are equal the $U(1)^N$ isometry is enhanced to $U(N)$ and this latter brings about other second rank Killing tensors. All in all, the fully isotropic case in odd dimensions with $U(\frac{d-1}{2})$ isometry, the $d-2$ dimensional spherical mechanics part is maximally superintegrable, it has $N+(N-2)=2N-2$ extra constants of motion. The fully isotropic case in even dimensions, however, is not maximally superintegrable; it has still $2N-1$ extra Killing tensors (one less than the N constants of motion to make the system fully superintegrable). We discussed the "special cases" in two different ways. First, we reanalyzed the system from the scratch and also took the equal rotation parameter limit of the generic case. As expected, these two cases matched. Our preliminary analysis, which we did not show here, indicate that the above statements is also true for the NH limit of extremal MP black holes in (A)dS backgrounds.

Here we explored second rank Killing tensors. One may suspect that the system has independent higher rank Killing tensors too, although it is unlikely. But if it does, the system for the generic rotation parameters becomes superintegrable. It is interesting to explore this question.

Chapter 5

Geometrization of dynamics is a recurrent theme in theoretical physics. While it has underlied such fundamental developments as the creation of General Relativity and search for unified theories of interactions, it also has a more modest (but often fruitful) aspect of reformulating conventional, well-established theories in more geometrical terms, in hope of elucidating their structure. One particular approach of the latter type is the Jacobi metric. This energy-dependent metric simply encodes as its geodesics the classical orbits of a nonrelativistic mechanical particle on a manifold moving in a potential.

We have presented a procedure associating to quantum systems a Klein-Gordon equation on a static spacetime. For systems with the quadratic energy spectrum, our procedure results in spacetimes with a resonant spectrum of evenly spaced frequencies. This correspondence generalizes the previously known relation between the Higgs oscillator and (global) Anti-de Sitter spacetime.

Implementing our procedure in practice requires solving a nonlinear elliptic equation. The latter form is closely reminiscent of elliptic equations extensively studied in relation to classic 'prescribed scalar curvature' problems of differential geometry (though the exact power appearing in the power-law nonlinearity is different). If one aims at constructing a massless Klein-Gordon (i.e., wave) equation corresponding to the original quantum-mechanical system, the nonlinearity drops out, resulting in a much simpler problem. In this case, known ground state wavefunctions for the original quantum system can be utilized for the conversion procedure. We have demonstrated how this approach works for superintegrable Rosochatius systems, resulting in a family of spacetimes resonant with respect to the massless wave equation.

The main practical target of our geometrization program are the following:

- to provide geometric counterparts for quantum systems with quadratic spectra (the resulting Klein-Gordon equation is set up on a highly special spacetime with a resonant spectrum of frequencies and the geometric properties of this spacetime are likely to yield insights into the algebraic properties of the original quantum system, including its high degree of degeneracy and hidden symmetries),

- to generate, starting from known quantum systems with quadratic spectra, highly resonant spacetimes (weakly nonlinear dynamics is likely to be very sophisticated, sharing the features of the extensively explored weakly nonlinear dynamics of AdS).

Conclusion

A new approach has been suggested to study the effects of an impulsive signal in a singular hypersurface on null geodesics in Minkowski space. It is based on the physically justified assumption that, in continuous coordinates, the geodesic vector of a test particle is continuous across the hypersurface.

Applying this technique on a parallel null congruence in flat space we obtained the initial conditions for the congruence to the future of the shell. In continues coordinates the geodesic vector flow does not suffer a jump upon crossing the shell. But a discontinuity arises in the gradient of the geodesic vector flow, the B-tensor. We have shown, for the parallel geodesic flow, that this discontinuity is related to different components of the stress-energy tensor. In particular the jump in the expansion is determined by the energy density and currents on the shell while the jump in the shear is determined by the gravitational wave component together with the surface currents. We proved that an impulsive signal does not effect the rotation of a congruence if it didn't rotate before crossing the shell. This change in the B-tensor after the passage of an outgoing gravitational wave leads to a covariant description of the gravitational memory effect - the B-memory effect. A particular example is given in the case of BMS type solderings.

Furthermore, the integrability problem of Hamilton-Jacobi equation in the near horizon geometry of Myers-Perry black hole in arbitrary (even or odd) d dimensions has been studied. We were able to introduce a convenient common description of the geometry in odd and even dimensions and unify these two cases into a single problem.

It was shown that integrals of motion of Hamilton-Jacobi equation in fully non-isotropic case can be expressed through inverse Vandermonde matrix in ellipsoidal coordinates. Solving this equation we found the hidden symmetries and expressed them through initial coordinates. We found the second rank Killing tensors generating these symmetries. We also found a non-trivial transformation relating the first integrals of fully isotropic NHEMP to the first integrals of fully non-isotropic NHEMP.

After finalizing the discussion of the special cases, the fully isotropic and fully non-isotropic, it is apparent that they are a part of a bigger picture, the most general case when rotation parameters are grouped in blocks of equal and non-equal values. Indeed, we found that when some of the rotation parameters are equal to each other and are different from the rest, the system becomes superintegrable. In short, the steps for obtaining the hidden symmetries is the following. We start from a system with $2N + 1 + \sigma$ variables with N isometries. Fixing the momenta associated with the isometries, we obtain and focus the $N-1+\sigma$ dimensional "angular mechanics" part. By introducing a special coordinate system, which is a mixture of spherical and ellipsoidal coordinates, we separate the variables in the angular mechanics thus introducing $N-1+\sigma$ independent constants, or the first integrals. This system reduced to its special cases of fully isotropic and fully non-isotropic NHEMP after appropriate assumptions.

The next step in our discussion was the extremal vanishing horizon geometry, which exists in odd dimensions, when one of the rotation parameters vanishes. The system remains integrable and no new independent constants of motion exist compared to the non-vanishing case.

We have suggested an approach for mapping quantum systems to a Klein-Gordon equation on a curved space-time. In general, the procedure is the following. We start from an equation which defines a scalar field and can be reduced to the Schrödinger equation after separating the time variable. The scalar field equation is not a Klein-Gordon type but can be transformed into such after an appropriate conformal rescaling of the metric and the scalar field. Such a conformal factor should satisfy a non-linear elliptic equation, which greatly simplifies with a further assumption of the Klein-Gordon equation being massless. We have shown that for quadratic spectra the ground state wavefunction of the initial Schrödinger equation satisfies the elliptic equation for the conformal factor.

We have demonstrated how the proposed mapping procedure can be applied on the Higgs oscillator and the superintegrable Rosochatius system. In the case of Higgs oscillator, the procedure results into a massive Klein-Gordon equation in the AdS background.

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Ամփոփում

Թեզում ուսումնասիրվել են հարաբերականության տեսության երկու կարևորագույն լուծումների՝ գրավիտացիոն ալիքների և սև խոռոչների հետ կապված հարցեր: Մասնավորապես, աշխատանքները վերաբերվում են աստղաֆիզիկական հարվածային ալիքներին, գրավիտացիոն ալիքներին, սև խոռոչներին, նրանց հետ կապված ինտեգրվող համակարգերին և որոշ ինտեգրվող համակարգերի քվանտային համարժեքներին:

Քննարկված հարցերը խմբավորվել են երեք խնդիրների շուրջ, որոնց կարելի է տալ հետևյալ ընդհանուր ձևակերպումները.

- իմպուլսիվ գրավիտացիոն ալիքների հետևանքով առաջացող գրավիտացիոն հիշողության երևույթ,
- ինտեգրվող համակարգեր կապված NHEMP սև խոռոչների հետ,
- որոշ քվանտային համակարգերի կապը Կլայն-Գորդոնի լուծումների հետ:

Թեզի հիմանակն արդյունքները ամփոփված են հետևյալ կետերում:

1. Նոր մոտեցում է առաջարկվել իմպուլսիվ գրավիտացիոն ալիքները հատող գեոդեզիկ փնջերի վրա նրանց ազդեցությունը ուսումնասիրելու համար: Այդ տեխնիկան կիրառվել է լուսանման փնջերի վրա: Հաստատվել է, որ հիպերմակերևույթի ուղղահայց լուսանման փնջերը մնում են այդպիսին թաղանթը հատելուց հետո:

2. Գրավիտացիոն հիշողության երևույթի կովարիանտ սահմանում է տրվել հիմնվելով փնջի B-թենզորի վրա: Դուրս է բերվել կապ թաղանթի B-թենզորի և լարվածություն էներգիայի թենզորի բաղադրիչների միջև: B-թենզորը հաշվվել է և մոտեցումը ցուցադրվել է BMS տիպի կապերի համար:

3. Ընդհանուր նկարագրություն է ներկայացվել զույգ և կենտ չափանի NHEMP երկրաչափությունների համար: Այս նկարագրությունը օգտագործվել է ապացուցելու համար, որ զույգ չափանի լրիվ ոչ իզոտրոպային NHEMP համակարգը ինտեգրելի է:

4. Կամայական չափանի լրիվ ոչ-իզոտոպային NHEMP-ի շարժման ինտեգրալները, ինչպես նաև Կիլլինգի վեկտորները ներկայացվել են

նախնական կոորդինատներում: Մենք գտել ենք ոչ-տրիվիալ ձևափոխություն լրիվ ոչ-իզոտրոպիկ և լրիվ իզոտրոպիկ NHEMP սև խոռոչների երկրաչափությունների շարժման ինտեգրալների միջև:

5. Մենք առանձնացրել ենք ամենաընդհանուր մասամբ իզոտրոպիկ NHEMP համակարգի փոփոխականները և ցույց ենք տվել այդ համակարգի անցումը լրիվ ոչ-իզոտրոպիկ և իզոտրոպիկ NHEMP հատուկ դեպքերի:

6. Նոր մոտեցում է առաջարկվել կոր տարածությունների վրա Շրյոդինգերի հավասարումը մեկ այլ երկրաչափության վրա Կլայն-Գորդոնի հավասարման հետ կապելու համար: Մենք ցույց ենք տվել, որ այս ընթացակարգը մեծապես պարզենում է քառակուսային սպեկտրներով համակարգերի համար և այն կիրառել ենք Հիգսի օսցիլյատորի և սուպերինտեգրվող Ռոսոխատիուսի համակարգի նկատմամբ:

Резюме

В диссертации были исследованы вопросы, связанные с двумя важнейшими решениям теории относительности - гравитационными волнами и черными дырами. Она основана на работах, относяшихся, в частности, к астрофизическим ударным волнам и гравитационным волнам, а также к черным дырам, и связанных с ними классическим и квантовым интегрируемым системам.

Обсуждаемые вопросы сгруппированы в три класса задач, которые можно сформулировать следующим образом:

- эффект гравитационной памяти, вызванный импульсными гравитационными волнами,

- интегрируемые системы, связанные с черными дырами NHEMP,

- связь некоторых квантовомеханических систем систем с решениями уравнения Клейна-Гордона.

Результаты диссертации собраны в следующих пунктах.

1. Предложен новый подход для изучения влияния импульсных гравитационных волн на пересекающие их геодезические пучки. Этот метод был применен к светоподобным пучкам. Было подтверждено, что пучки, ортогональные к гиперповерхности, остаются такими же после пересечения мембраны.

2. Дано ковариантное определение явлению гравитационной памяти на основе B-тензора пучка. Выявлена связь между B-тензором и компонентами тензора энергии-импульса. Этот подход был применен к соединениям типа BMS. В частности, вычислен соответствующий B-тензор.

3. Дано единообразное описание четных и нечетных геометрий NHEMP. Оно использовано для доказательства интегрируемости четных, полностью неизотропних систем NHEMP.

4. Построены интегралы движения и векторы Киллинга для полностью неизотропной NHEMP произвольной размедности, приведен

их вид в исходных координатах. Найдено нетривиальное соответствие между интегралами движения полностью неизотропных и полностью изотропных черных дыр NHEMP.

5. Разделены переменные наиболее общей частично изотропной системы NHEMP, проведена контракция переход к частным случаям полностью неизотропной и полностью изотропной NHEMP.

6. Предложен новый подход, связывающий уравнение Шредингера на искривленных пространствах с уравнением Клейна-Гордона в другой геометрии. Показано, что эта процедура значительно упрощается для систем с квадратичным спектром. Детально рассмотрены два частных случая суперинтегририемых систем на сфере: осциллятор Хигса и система Росохатиуса.