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Boundary Value Problems for Elliptic and Parabolic Equations and Applications in Mathematical Modeling of Two-Phase Substance

Thesis

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Chapter 0.

Preliminary

0.1. Advances and Historical Trends

We initial the work by reviewing the advances and the historical trends of melting and freezing process and their mathematical modeling which they were prepared by different researchers in this area. Conservation laws were applied by physicians to construct the mathematical formulation of melting and freezing process for different types of materials. Stefan problem is famous between researches that it describes the behavior of ice block and its temperature distribution during the phase transition process.

0.2. Stefan Problem

When the boundary between solid and melted phase is moving in time we call them free boundary problems, hence the Stefan problem is free boundary problem. Suppose the semi-infinite ice block in one-dimension where its initial temperature is $u(x) = 0$ for $x \in [0, +\infty)$. If the heat flux $f(t)$ at the left part of ice block causes it to melt where $x = s(t)$ is the boundary between two phases, we will have the Stefan problem as finding u and s such that

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial t^2} \quad in \quad \{(x,t): 0 < x < s(t), t > 0\}$$
$$-\frac{\partial \theta}{\partial t}(0,t) = f(t), \ t > 0$$
$$\theta(s(t),t) = 0$$
$$s'(t) = -\frac{\partial \theta}{\partial x}(s(t),t), \ t > 0$$
$$\theta(x,0) = 0, \ x \geq 0$$
$$s(0) = 0$$

0.3. Obstacle problem

Obstacle problem is another important problem in the free and moving boundary problems topic. The aim of this problem is to look for the equilibrium of an elastic membrane whose boundary is constrained to hold above a given obstacle. Obstacle problem is applied in the fluid filtration, constrained heating, optimal control, financial mathematics and some other scopes. To introduce the mathematical definition of the Obstacle problem assume that

1. $D \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary ∂D
2. a smooth function $f(X)$ on ∂D
3. a smooth function $\varphi(X)$ defined on D such that $\varphi(X)\big|_{\partial D} < f(X)$ for every $X \in D$,

and also define K to be the closed convex set

$$K = \left\{u(X) \in H^1(D): u(X)\big|_{\partial D} = f(X), u \geq \varphi\right\}$$

The solution to the obstacle problem is the unique function u_0 that minimizes the Dirichlet integral

$$D(u) = \int_D (\nabla u)^2 dX,$$

for every $u \in K$.

Shoshana Kamin in 1958 prepared the first proof of the existence and uniqueness of the generalized solution of the Stefan problem in three-dimension [1], [2]. Her work was followed by Oleink, O. A. and he generalized the proof in the work a method of solution of the general Stefan problem in 1960 [3]. Meirmanov prepared comprehensive mathematical information about the Stefan problem in the text book of the Stefan Problem in 1992, and also for existence and uniqueness of solution of Stefan problem and its regularity it is recommended to refer to Daniele Andreucci's recently works.

The study of free boundary problems consists of two separated parts:

1) the study of the topological structure
2) the study of differentiable structure

The first techniques were constructed on analytic continuation of a conformal mapping and requiring topological knowledge of the free boundary and analyticity of the data. Also Convexity of the boundary and concavity of the obstacle is imposed to obtain the necessary topological structure conditions. The techniques based on differentiable structure were developed by H. Lewy in the works [4], [5], and [6], then later in the linear case those works were applied by H. Lewy and G. Stampacchia [7], [8] and D. Kinderlehrer continued the topic in the minimal surface case. Also D. Kinderlehrer investigated the application of Quasi-conformal extensions where he supposed a prior topological behavior, then earned the $C^{(1,\alpha)}$ character of the free boundary [10], [11], [12].

Luis Caffarelli has had great role in developing the theory of free boundary problems, in particular he researched the regularity of free boundaries in the article [13]. When we consider the energy minimizing function over the set of functions bigger than an obstacle we confront the regularity of the free boundary problems. Different researchers have produced mathematical works in this scope. Caffarelli in [14] proved that in a three-dimensional filtration problem, the free surface is the class of C^1, and all the second derivatives of the variational solution are continuous up to the free boundary, on the non-coincidence set, also he proved that the variational solution is a classical one, and he prepared necessary hypothesis which the free boundary is as smooth as the obstacle. He recommended the strong use of the geometry of the problem where it implied that the free boundary was Lipschitz, and the Laplacian of the obstacle was constant.

In the article [14] he treated the general nonlinear free boundary problem, then he proved that if X_0 is a point of density for the coincidence set, in a neighborhood of X_0, the free boundary is a C^1 surface and all the second derivatives of the solution are continuous up to it. Also he considered one phase Stefan problem and proved that for a fixed time t_0, the point X_0 is a density point for the coincidence set then in a neighborhood in space and time of (X_0, t_0) the free boundary is a surface of class C^1in space and time and all the second derivatives of the solutions are continuous up to the free boundary, hence the solution is a classical one in that neighborhood.

Caffarelli in the work the obstacle problem revisited [15] reviewed some basic properties of obstacle problem and now we mention some vital parts of his work. He proved that u_0 stays between

$$\lambda_1 = \min f(X), \lambda_2 = \max\big(f(X), \varphi(X)\big)$$

Also u_0 is superharmonic, and support $\Delta u_0 \subset \{u_0 = \varphi\}$, and u_0 is continuous. Mean value theorem leads to some properties of harmonic and superharmonic functions and the article refer them:

1) Harnack inequality
   If v is harmonic and non-negative in B_1, then for $R < 1$

$$\sup_{B_R} v \leq C(R) \inf_{B_R} v$$

2) Derivative estimate
   If v is harmonic in $B_1(0)$, then

$$|\nabla v(0)| \leq \operatorname*{osc}_{B_1(0)} v$$

3) Other derivative estimate

$$\left|D^{(k)} v(0)\right| \leq C(k) \frac{1}{r^k} \operatorname*{osc}_{B_r(0)} v$$

4) If v is a superharmonic in D, then v cannot have a local minimum in D, unless v is constant.

5) If v is continuous in ∂B_1, then there exists a unique harmonic function v, such that

$$v\big|_{\partial B_1} = f$$

Then he proved that u is as regular as φ in $C^{1,1}$, in the other words if φ has a modulus of continuity $\sigma(r)$, then u has modulus of continuity $C\sigma(2r)$. The same result holds between $\nabla\varphi$ and ∇v.

H. Brezis and D. Kinderlehrer in [16] investigate the following problem within the assumptions:

1) $\Omega \subset \mathbb{R}^n$ is a bounded, connected, and open

2) $a_i(P)$ is a locally coercive C^2 −vector field, where $P = (p_1, \dots, p_n)$

3) ψ is the obstacle function of class $C^2(\Omega)$, where $\psi \leq 0$ on $\partial\Omega$,

then they prove that the problem

$$u \in K,$$
$$\int_{\Omega} a_j(Du)D_j(v-u)dX \geq 0 \;\; ; \; \forall v \in K,$$

has the compact solution of class $C^{1,1}(M)$ for any $M \subset \Omega$, where

$$K = \{v: v \; is \; Lipschits, v \geq \varphi, v|_{\partial\Omega} = 0\}$$

Also they distinguished the subsets $D = \{X: u = \varphi\}$ and $\Omega \setminus D$, therefore

1) $u = \varphi$ on $\partial D \cap \Omega$

2) $\nabla u = \nabla\varphi$ on $\partial D \cap \Omega$

3) $A(u) = -\partial_i\big(a_i(\nabla u)\big) = \sum a_{ij}(\nabla u)u_{ij} = 0$ in $W = \Omega \setminus D$, where A is elliptic.

Caffarelli and Riviere in the paper smoothness and analyticity of free boundaries in variational inequalities [14] in 1976 assumed that

1) $\varphi \in C^4(\Omega)$

2) $A(\varphi)$, $\nabla\big(A(\varphi)\big)$ do not vanish simultaneously

3) $X_0 \in \partial D \cap \Omega$,

then $A(\varphi) < 0$ in a neighborhood of X_0. Thus they concluded that locally there are

1) An open set W, and a ball $B_l(X_0)$

2) $\varphi \in C^4\big(B_l(X_0)\big)$, and $A(\varphi) < \lambda_0 < 0$ on $B_l(X_0)$

3) $u \in C^{1,1}\big(B_l(X_0)\big)$, and $u|_{B_l(X_0) \setminus w} = \varphi$

4) $u \geq \varphi$ on $B_l(X_0)$, and $Au = 0$ on $W \cap B_l(X_0)$

They defined $v = u - \varphi$ that v satisfies $B_{l'}(X_0)$,

$$a_{ij}(\nabla\varphi)D_{ij}(u-\varphi) = f > 0 \quad \text{on} \quad W \cap B_{l'}(X_0)$$

They showed that near $(\partial W) \cap B_{l'}(X_0)$

$$a_{ij}(\nabla\varphi)\, D_{ij}(u) = a_{ij}(\nabla u)\, D_{ij}(u) + O\big(d(X,\partial W)\big) = O\big(d(X,\partial W)\big),$$

then

$$a_{ij}(\nabla\varphi)\, D_{ij}(u) \in C^{\frac{1}{2}}\big(B_{l'}(X_0)\big)$$

0.4. Heat Transfer

In the scope of heat transfer and its mathematical simulation in three-dimension Raymond Viskanta and Aydin Ungan in 1986 prepared a paper which they presented a numerical approach to derive the modeling of circulation and heat transfer in an electrically boosted glass tank [22]. The simulation includes the equations

$$\nabla.(\rho\vec{u}) = 0$$

$$\nabla.(\rho\vec{u}u) = \nabla.(\mu\nabla u) - \frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)$$

$$\nabla.(\rho\vec{u}v) = \nabla.(\mu\nabla v) - \frac{\partial P}{\partial y} + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial y}\right)$$

$$\nabla.(\rho\vec{u}w) = \nabla.(\mu\nabla w) - \frac{\partial P}{\partial z} - \beta\rho_0 g(T - T_0) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial z}\right)$$

$$\nabla.(\rho\vec{u}T) = \nabla.\left(\frac{k_{eff}}{c}\nabla T\right)$$

To investigate the influence of electric field on convection of electrically conducting fluids they enforced the Maxwellian equations into their modeling within the momentum and energy conservation equations. The Maxwellian equation is

$$\nabla \times \vec{E}(t) = 0$$

The electric potential $\psi(t)$ were used to achieve

$$\vec{E}(t) = -\nabla\psi(t),$$

where

$$\psi(t) = \psi_m(\cos(\omega t + \theta) + j.\sin(\omega t + \theta))$$

One essential part of their modeling included the boundary conditions of

$$\frac{\partial}{\partial n} Re\,\psi = \frac{\partial}{\partial n} Im\,\psi = 0,$$

where

$$Re\,\psi = \psi_m \cos\theta_i$$

$$Im\,\psi = \psi_m \sin\theta_i$$

Finally they presented that electric boosting has convenient influence on the melting circulation and heat transfer. Also they create an efficient iterative method to deriving numerical solution which it has high speed to get the result, that is, they reduced the computer running time.

0.5. Navier-Stokes Equations

The classical model of fluids was applied in physics based on a set of partial differential equations, where it known as the Navier-Stokes equations. Navier in 1822 derived the equations for homogeneous incompressible fluids from a molecular argument [24]. Inviscid fluids are fluids with zero viscosity, and the physicians didn't understand this concept before 1822 when it was introduced by Navier.

$$\rho\left(\frac{Du}{Dt} - X\right) + \frac{dP_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dy} = 0,$$

$$P_1 = p - 2\mu\left(\frac{du}{dx} - \delta\right),$$

$$T_1 = -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right),$$

$$3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}.$$

Poisson assumed the same arguments in 1829 and got the equations for compressible fluid [25]. These well-known equations were originally obtained in the 1843 by Saint Venant and Stokes in 1845 on the basis of conservation laws (momentum conservation law) [26], [27].

$$\rho\left(\frac{d\boldsymbol{u}}{dt} - \boldsymbol{X}\right) = div(\boldsymbol{T}),$$

$$\boldsymbol{T} = \left(-p - \frac{2}{3}\mu\, div(\boldsymbol{u})\right)\boldsymbol{1} + 2\mu\boldsymbol{D}[\boldsymbol{u}],$$

$$\frac{d\boldsymbol{u}}{dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}.\nabla)\boldsymbol{u},$$

$$\boldsymbol{D}[\boldsymbol{u}] = \frac{1}{2}[\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}}],$$

The theory of potential flow for a viscous fluid was discovered by Stokes in 1850. The problem considered by Stokes was solved exactly by using the linearized Navier-Stokes equations by Lamb in 1932, but his work didn't include potential flow assumption [28].

By the assumption of sufficient randomness in microscopic molecular processes they could be earned from molecular dynamics as well as from cellular automata of the kind. Since 1960s computers have been strong enough to prepare large computations, then it has become increasingly common to see numerical results given far into the turbulent regime - leading sometimes to the assumption that turbulence has somehow been derived from the Navier-Stokes equations. In the mathematical analysis of the Navier-Stokes has never established the formal uniqueness and existence of solutions. Indeed, there is even some evidence that singularities might almost inevitably form, which would imply a breakdown of the equations, and perhaps a need to account for underlying molecular processes.

0.6. Finite Element Method

Finite element method is considered as one of the convenient mathematical techniques for the computer solution of complex problems. It is applied in wide different areas like civil engineering, mechanics, heat conduction, geo-mechanics, etc. Finite element method is a numerical approach to obtain an approximate solution to elliptic partial differential equations, that it converts the boundary value problem into a set of easily solvable algebraic equations. There are three basic steps during the procedure, first the problem must be formulated in the variational form, then by discretization of the domain the system of algebraic equations is obtained and at last the classical technique would be applied to get the numerical solution of the system.

In 1909 Ritz developed a method for the approximate solution of problems [44]. His work consists of minimization of functional where every unknown produces the system of equations within new unknown coefficients that must be determined. In 1943 Courant improved the Ritz method by introduction of the specific linear functions that they are defined over triangular regions [45]. He eliminate the main restriction of the Ritz function (the functions are used by Ritz must satisfied to the boundary conditions). The Ritz procedure with Courant functions is very similar to the finite element method. Because of the absence of the strong computers they didn't happen essential develops in the Courant time, since large-scale computations were necessary for the essential progress in this scope.

Finite element method has gotten its amazing developing when Argyris [46], Turner [47], and Hrennikov [48] published their articles. Zienkiewicz and Cheung in 1967 published the first textbook about FEM, and they called it "The Finite Element Method in Structural and Continuum Mechanics", and it includes the theory and applications of FEM [49]. Also the method has been applied in the field of structural mechanics by researchers. Now FEM is applied in wide different area like heat conduction problems, fluid dynamics, electric and magnetic fields, and other branches.

0.7. Properly Elliptic Equation

Many important problems of the elasticity theory, hydrodynamics, signal processing and other fields of science and engineering are reduced to the investigation of elliptic partial differential equations. Solutions of elliptic equations are also the steady solutions of diffusion equations, and describe the stable condition of the phenomenon when time tends to infinity. Such equations, as all other partial differential equations, have many solutions, therefore, we must consider these equations with some kinds of initial or boundary conditions providing uniqueness (or finite dimension) of the solution of the corresponding boundary value problem. Physical conditions in technical problems reduced to Laplace equation (for example the potential equation of electrostatics) define the natural boundary conditions (Dirichlet, Newman, etc.) provided correctness of the boundary value problem for one differential equation with real constant coefficients.

For a long time it was supposed, that the same kind of boundary conditions may be posed for the elliptic systems of partial differential equations. But in 1948 A.V. Bitsadze presented an example of an elliptic equation with complex coefficients (or second order elliptic system of PDE with real coefficients) for which the Dirichlet problem is not correct [70]. After that, in works of N. E. Tovmasyan [71-75], E.V. Zolotareva, A.I.Volpert [76-78] and others it was shown, that all classical boundary value problems are correct only for properly elliptic equations. After that, in works of I. A. Bikchantaev [79], A. P. Soldatov [80], W. I. Wendland, N. E. Tovmasyan, M. M. Sirajudinov and others, many correct boundary value problems for improperly elliptic equations were formulated and investigated.

Chapter 1.

Basic Definitions

1.1. Introduction

Mathematical modeling of heat transfer (Transport Phenomena) is applied by researchers to investigate the environment of the furnaces during the melting process and turbulent conditions inside them, because it has the advantages of reasonable cost and applicable exactness. In the present work we will earn the mathematical modeling of transport phenomena in three-dimension for the especial furnaces with cylindrical shapes, that in the business area, they are called Garnissage furnace. According to their physical shapes we will demonstrate the mathematical equations of heat transfer in three-dimension, and then we will search for its numerical solution.

In this stage we will perform some rules of physics, that is, the mass, momentum, and energy conservation laws to gain three essential partial differential equations; that is, the continuity, Navier-Stokes, and heat conduction equations. The melting process in the Garnissage furnace starts when we impose enough heat to the solid ingredient materials by electrical boosters inside the furnace, but the heat transferring by electrical boosters couldn’t melt the whole materials and whenever the central parts are melted the parts far from the boosters still is solid, then the boundary between the solid and liquid parts are moving during the melting process. One basic work is to achieve the mathematical modeling of free boundary between the solid and liquid materials, and therefore more exactly determine the physical shape of furnace.

In the first chapter we review some basic definitions of physics that they are necessary in the modeling process, then we prepare the details of the mathematical modeling that includes the performing of the conservation laws to construct the continuity, Navier-Stokes, and heat conduction equations. Also the Stefan condition is convenient tool to describe the behavior of the free boundary and the details of the construction in one dimension is available in the remainder of this chapter, but we will handle three-dimensional version of the Stefan condition in this research.

The space of functions have key role in the theory of partial differential equations, in particular Sobolev spaces are the most important in this topic. We finish the first chapter by introducing some necessary definitions and properties of Sobolev spaces in special order. In the second chapter we define stream function in two dimension and then we gain the mathematical modeling of heat transfer according to stream function. The modeling based on stream function has its sufficient advantages and there is complete description about the modeling process, variational approach and weak formulation, and numerical solution of the transport system.

Mathematical modeling in the cylindrical coordinate system has been shown in third chapter. According to Garnissage tank shape and its symmetric properties this coordinate system would be valuable to apply these properties and then we earn the conservation equations in the cylindrical coordinate system, so the Stefan condition. After modeling process we follow the variational approach to convert the system of equations into the weak formulation. For expressing the system in the variational formulation we will exert sufficient smooth test functions with small support.

Finally in the fourth chapter we will discrete the domain to special mesh cubes that they are divided into 24 tetrahedrons. To follow the finite element method we compute the piecewise continuous test function according to 24 tetrahedrons, and then we try to determine values of coefficients in the linear and nonlinear final system of equations that is prepared in the approximation process. In this stage there are different integrals and we will divide the integrals based on tetrahedrons. At last we will derive the system of equations that they are combination of linear and nonlinear, thus to solve the system numerically the Newton's method would be recommended.

1.2. Mathematical Modeling

In this chapter we will prepare the mathematical modeling of flow and heat transfer in three-dimension. To achieve the mathematical modeling we invoke the conservation laws of physics, that is, the mass, momentum and energy conservation laws to drive the transport phenomena system. Transport system includes the continuity equation, Navier-Stokes equations, and energy equation. We will apply some thermal characteristics to make the modeling more realistic, and at the end we earn the free boundary conditions that they are another important part of our work to introduce the complete numerical approach for solving the transport system of partial differential equations. Before any mathematical work we need some basic physical definitions.

1.3. Basic Definitions

In this part we state some basic definitions of physics that they were applied during the modeling process, then we start by pressure.

Pressure
Pressure p is defined as the force/area acting normal to a surface. If ΔA is an infinitesimally small surface and ΔF_n is given normal force, then

$$p = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A}$$

$$p = \frac{dF_n}{dA} \tag{1.3.1}$$

The pressure $p(x, y, z, t)$ can vary in space and time, so the pressure is a time-varying scalar field.

Density
Density ρ is defined as the mass/volume, for an infinitesimally small volume.

$$\rho = \lim_{\Delta \nu \to 0} \frac{\Delta m}{\Delta \nu}$$

$$\rho = \frac{dm}{dv} \tag{1.3.2}$$

Like the pressure, density $\rho(x, y, z, t)$ is also a scalar field.

Velocity
Consider a fluid element as it moves along. As it passes some point, its instantaneous velocity is defined as the velocity at that point. This velocity is a vector with three separate components, and will in general vary between different points and different times.

$$\boldsymbol{V}(x, y, z, t) = u(x, y, z, t)i + v(x, y, z, t)j + w(x, y, z, t)k, \tag{1.3.3}$$

so $\boldsymbol{V}$ is a time-varying vector field, whose components are three separate time-varying scalar fields $u, v,$ and w.

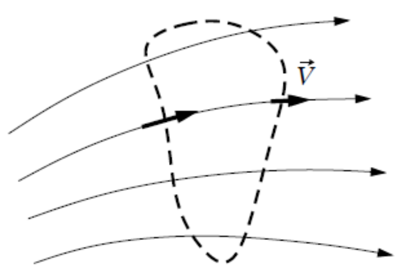
Speed
Speed is the magnitude of the velocity vector.

$$V(x, y, z, t) = |\boldsymbol{V}| = \sqrt{u^2 + v^2 + w^2} \tag{1.3.4}$$

In general the speed is a time-varying scalar field.

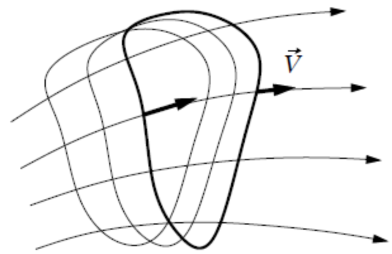
Steady and Unsteady Flows
If the flow is steady, then p, ρ, and $\boldsymbol{V}$ don't change in time for any point, and hence can be given as $p(x, y, z)$, $\rho(x, y, z)$, and $\boldsymbol{V}(x, y, z)$. If the flow is unsteady, then these quantities do change in time at some or all points.

Eulerian type Control Volume
In this case volume is fixed in space, and fluid can freely pass through the volume's boundary.



(1.3.5)

Lagrangian type Control Volume
Volume is attached to the fluid, and it is freely carried along with the fluid, and no fluid passes through its boundary.



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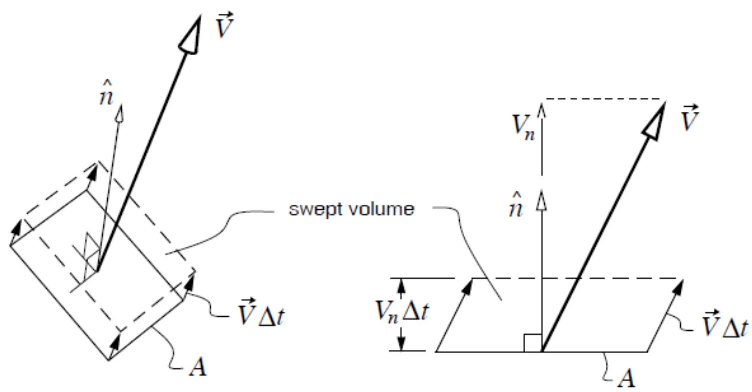
Mass of Fluid

Consider a small patch of the surface of the fixed control volume. The patch has area *A*, and normal unit vector $\hat{\boldsymbol{n}}$. The plane of fluid particles on *A* at time *t* given by

$$\Delta v = (\boldsymbol{V}.\hat{\boldsymbol{n}})At, \quad (1.3.7)$$

$$\Delta v = V_n At, \quad (1.3.8)$$

where $V_n = \boldsymbol{V}.\hat{\boldsymbol{n}}$ is the component of the velocity vector normal to the area.



(1.3.9)

The mass of fluid in swept volume, which passed through the area during the Δt interval, is

$$\Delta m = \rho \Delta v \quad (1.3.10)$$

$$\Delta m = \rho V_n A \Delta t \quad (1.3.11)$$

Mass Flow

The mass flow is defined as the time rate of mass fluid passing through the area.

$$m' = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t}$$

$$m' = \rho(\boldsymbol{V}.\hat{\boldsymbol{n}})A \quad (1.3.12)$$

$$m' = \rho V_n A \quad (1.3.13)$$

Mass Flux

The mass flux is defined as the mass flow per area.

$$\frac{m'}{A} = \rho V_n \tag{1.3.14}$$

Viscosity

Viscosity is a measure of a fluid's resistance to flow. It describes the internal friction of a moving fluid.

Stress

The force per unit area applied to an object. Objects subject to stress tend to become distorted or deformed.

Dynamic Viscosity

Dynamic viscosity μ is a measure of the ratio of the stress on a region of a fluid to the rate of change of strain it undergoes.

Kinematic Viscosity

Kinematic viscosity ϑ is a measure of the rate at which momentum is transferred through a fluid. It is the dynamic viscosity divided by the density.

$$\vartheta = \frac{\mu}{\rho} \tag{1.3.15}$$

Shear Stress

Shear stress $\boldsymbol{\tau}$ is the form of stress that subjects an object to which force is applied to skew, tending to cause shear strain. It is the component of stress coplanar with a material cross section. Shear stress arises from the force vector component parallel to the cross section.

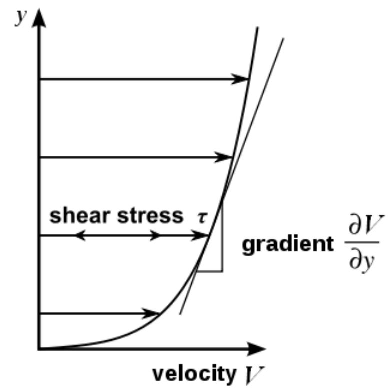
$$\boldsymbol{\tau} = \frac{F}{A} \tag{1.3.16}$$

The applied force is proportional to the area and velocity gradient in the fluid

$$F = \mu A \frac{\partial \boldsymbol{V}}{\partial y} \tag{1.3.17}$$

Let μ is the dynamic viscosity of the fluid, and $\boldsymbol{V}$ is the velocity of the fluid along the boundary, and y is the height above the boundary. Then the shear stress, for a Newtonian fluid, at a surface element parallel to a flat plate is

$$\boldsymbol{\tau} = \mu \frac{\partial \boldsymbol{V}}{\partial y} \tag{1.3.18}$$



(1.3.19)

If

$$\boldsymbol{\tau} = (\tau_{ij})_{3\times3} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \qquad (1.3.20)$$

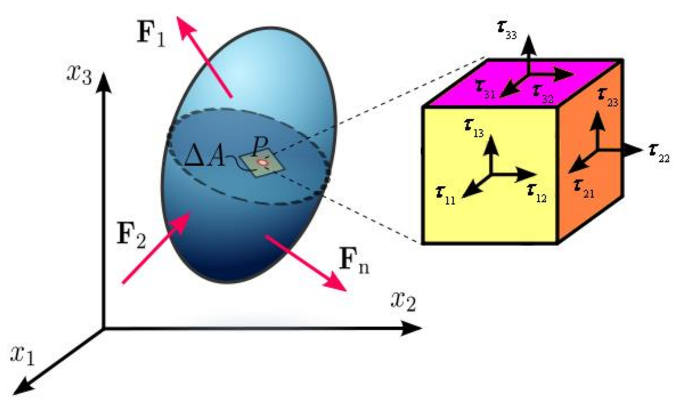
are the stress components on an infinitesimal cubic fluid element, and $\hat{\boldsymbol{n}}$ is the unit outward normal vector to the surface, then surface stress is

$$\boldsymbol{\tau}.\hat{\boldsymbol{n}} \qquad (1.3.21)$$

Incompressible Flow

Incompressible flow refers to a flow in which the material density is constant in fluid. If the fluid is incompressible and viscosity is constant across the fluid, the equation of stress can be written in terms of an arbitrary coordinate system as

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (1.3.22)$$



(1.3.23)

where u_i is the fluid's velocity in the direction of axis i, and x_j is jth spatial coordinate. Let $\boldsymbol{I}$ is the unit 3×3 matrix, and p is the pressure. Total stress tensor $\boldsymbol{\sigma}$ can be defined as

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\tau} \qquad (1.3.24)$$

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.3.25}$$

where

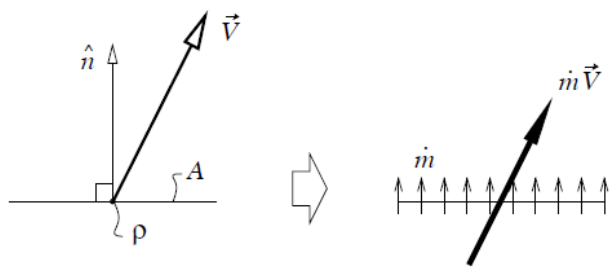
$$\delta_{ij} = \begin{cases} 0 & ; i \neq j \\ 1 & ; i = j \end{cases}$$

Momentum Flow
The momentum flow as defined

Momentum Flow = (Mass Flow) × (Momentum/Mass),

where the Momentum/Mass is the velocity vector $\boldsymbol{V}$, then

$$\text{Momentum Flow} = m'\boldsymbol{V} = \rho(\boldsymbol{V}.\hat{\boldsymbol{n}})A\boldsymbol{V} = \rho V_n A\boldsymbol{V} \tag{1.3.26}$$



(1.3.27)

Momentum Flux
The momentum flux is defined as the momentum flow per area.

$$\text{Momentum Flux} = \rho(\boldsymbol{V}.\hat{\boldsymbol{n}})\boldsymbol{V} = \rho V_n\boldsymbol{V} \tag{1.3.28}$$

Material Derivative
In continuum mechanics, the material derivative (or substantial derivative) describes the time rate of change of some physical quantity for a material element subjected to a space and time dependent velocity field. Suppose that $\varphi(x, y, z, t)$ is time-varying scalar field, the material derivative of φ is defined as

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + \boldsymbol{V}.(\nabla\varphi), \tag{1.3.29}$$

where ∇ is gradient.

For time-varying vector field $\boldsymbol{U}(x, y, z, t)$ it is defined as

$$\frac{D\boldsymbol{U}}{Dt} = \frac{\partial\boldsymbol{U}}{\partial t} + \boldsymbol{V}.(\nabla\boldsymbol{U}), \tag{1.3.30}$$

where ∇ is covariant.

Enthalpy
The heat absorbed by a material under constant pressure is a thermodynamic quantity called the enthalpy.

$$de = cd\theta, \tag{1.3.31}$$

where θ is temperature, and c is specific heat.

Heat Flux
The amount of heat crossing a unit area per unit time is called the heat flux.

$$\boldsymbol{q} = -k_{eff} \nabla\theta, \tag{1.3.32}$$

where k_{eff} is the thermal conductivity of the material, and it is varying with temperature.

Heat Flow Rate
The heat flow across a surface of area A, represents the heat crossing the area A in the direction normal to the surface per unit time.

$$(\boldsymbol{q}.\hat{\boldsymbol{n}})A = -k_{eff} \nabla\theta.\hat{\boldsymbol{n}}A \tag{1.3.33}$$

1.4. Equation of Continuity

For earning the continuity equation we exert the mass conservation law to the finite fixed control volume as the following relation

$$\frac{d}{dt}(\textit{Mass in volume}) = \textit{Mass flow into volume}, \tag{1.4.1}$$

then by using the relations (1.3.10) and (1.3.12) we gain

$$\frac{d}{dt}\iiint \rho dv = \oiint \rho \boldsymbol{V}.(-\hat{\boldsymbol{n}})dA \tag{1.4.2}$$

The negative sign is necessary because $\hat{\boldsymbol{n}}$ is outward. According to Gauss's theorem we rewrite the integral equality (1.4.2) as

$$\oiint \rho \boldsymbol{V}.\hat{\boldsymbol{n}}dA = \iiint \nabla.(\rho\boldsymbol{V})dv, \tag{1.4.3}$$

now we insert the value of (1.4.3) into (1.4.2), and thus

$$\iiint \left(\frac{\partial\rho}{\partial t} + \nabla.(\rho\boldsymbol{V})\right)dv = 0 \tag{1.4.4}$$

The relation (1.4.4) must hold for any control volume and in particular for every infinitesimal control volume at every point in the flow, then

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \boldsymbol{V}) = 0 \qquad (1.4.5)$$

The equation (1.4.5) is called the Continuity Equation. Since for the steady flow density doesn't vary in time, then the equality (1.4.5) is stated as

$$\nabla.(\rho \boldsymbol{V}) = 0, \qquad (1.4.6)$$

and also for low-speed flow (incompressible flow), steady or unsteady, the density ρ is constant and we express the equation (1.4.6) as

$$\nabla.\boldsymbol{V} = 0, \qquad (1.4.7)$$

which the equality states that the velocity field has zero divergence for low-speed flow.

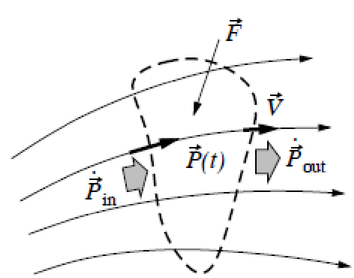
1.5. Equation of Momentum

Based on Newton's second law, and by applying the impulse of a force $\boldsymbol{F}$ to some mass, during a short time interval dt, a momentum change $d\boldsymbol{P}_a$ in that affected mass is produced. Now we apply this rule for a fixed control volume.

$$\frac{d\boldsymbol{P}_a}{dt} = \boldsymbol{F}, \qquad (1.5.1)$$

then

$$\frac{d\boldsymbol{P}_a}{dt} + \boldsymbol{P}'_{out} - \boldsymbol{P}'_{in} = \boldsymbol{F} \qquad (1.5.2)$$



(1.5.3)

In the equation (1.5.2), $\boldsymbol{P}$ is defined as the instantaneous momentum inside the control volume.

$$\boldsymbol{P}(t) = \iiint \rho \boldsymbol{V} dv \qquad (1.5.4)$$

By evaluating a surface integral of the momentum flux over the entire boundary we have

$$\boldsymbol{P}'_{\boldsymbol{out}} - \boldsymbol{P}'_{\boldsymbol{in}} = \oiint \rho(\boldsymbol{V}.\hat{\boldsymbol{n}})\boldsymbol{V}dA \qquad (1.5.5)$$

Applied Forces

The force $\boldsymbol{F}$ consists of two types:

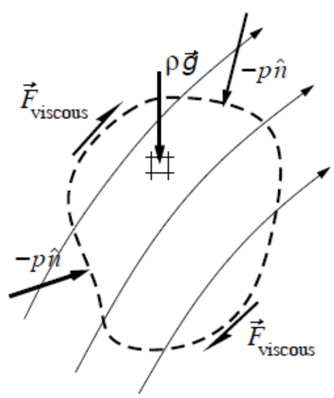
1. **Body forces**. These act on fluid inside the volume. Gravity is the most important body force.

$$\boldsymbol{F}_{gravity} = \iiint \rho\boldsymbol{g}dv \qquad (1.5.6)$$

2. **Surface forces**. These act on the surface of the volume. Pressure and Viscous are two important surface forces.

$$\boldsymbol{F}_{pressure} = \oiint p(-\hat{\boldsymbol{n}})dA \qquad (1.5.7)$$

$$\boldsymbol{F}_{viscous} = \oiint \boldsymbol{\tau}.\hat{\boldsymbol{n}}dA \qquad (1.5.8)$$



(1.5.9)

Integral Momentum Equation

Substituting all the momentum (1.5.4), momentum flow (1.5.5), and all forces (1.5.6), (1.5.7), and (1.5.8) into Newton's second law (1.5.2) we gain the integral momentum equation as

$$\frac{d}{dt}\iiint \rho\boldsymbol{V}dv + \oiint \rho(\boldsymbol{V}.\hat{\boldsymbol{n}})\boldsymbol{V}dA = -\oiint p\hat{\boldsymbol{n}}dA + \oiint \boldsymbol{\tau}.\hat{\boldsymbol{n}}dA + \iiint \rho\boldsymbol{g}dv, \qquad (1.5.10)$$

then by using Gauss's theorem and gradient theorem we earn

$$\iiint \left(\frac{\partial(\rho\boldsymbol{V})}{\partial t} + \boldsymbol{V}.\nabla(\rho\boldsymbol{V}) + \nabla p - \nabla.\boldsymbol{\tau} - \rho\boldsymbol{g}\right)dv = 0 \qquad (1.5.11)$$

The integral equation (1.5.11) must hold for any control volume and in particular for every infinitesimal control volume at every point in the flow, then

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t}+\boldsymbol{V}.\nabla(\rho \boldsymbol{V})=-\nabla p+\nabla.\boldsymbol{\tau}+\rho \boldsymbol{g} \tag{1.5.12}$$

For low-speed flow the density is constant and we get the momentum equation (1.5.12) as the following

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t}+\boldsymbol{V}.\nabla \boldsymbol{V}\right)=-\nabla p+\nabla.\boldsymbol{\tau}+\rho \boldsymbol{g}, \tag{1.5.13}$$

$$\rho\frac{D\boldsymbol{V}}{Dt}=-\nabla p+\nabla.\boldsymbol{\tau}+\rho \boldsymbol{g}, \tag{1.5.14}$$

where $\frac{D}{Dt}$ is the material derivative. Also by inserting total stress tensor (1.3.24) we achieve

$$\rho\frac{D\boldsymbol{V}}{Dt}=\nabla.\boldsymbol{\sigma}+\rho \boldsymbol{g} \tag{1.5.15}$$

Let the velocity vector

$$\boldsymbol{V}(x,y,z,t)=u(x,y,z,t)i+v(x,y,z,t)j+w(x,y,z,t)k, \tag{1.5.16}$$

and the acceleration vector $\boldsymbol{g}=g_x i+g_y j+g_z k$, we can rewrite the Navier-Stokes equations (1.5.12) as the following partial differential system

$$\frac{\partial(\rho u)}{\partial t}+\nabla.(\rho u\boldsymbol{V})=-\frac{\partial p}{\partial x}+\mu\left(\Delta u+\frac{\partial(\nabla.\boldsymbol{V})}{\partial x}\right)+\rho g_x \tag{1.5.17}$$

$$\frac{\partial(\rho v)}{\partial t}+\nabla.(\rho v\boldsymbol{V})=-\frac{\partial p}{\partial y}+\mu\left(\Delta v+\frac{\partial(\nabla.\boldsymbol{V})}{\partial y}\right)+\rho g_y \tag{1.5.18}$$

$$\frac{\partial(\rho w)}{\partial t}+\nabla.(\rho w\boldsymbol{V})=-\frac{\partial p}{\partial z}+\mu\left(\Delta w+\frac{\partial(\nabla.\boldsymbol{V})}{\partial z}\right)+\rho g_z \tag{1.5.19}$$

1.6. Equation of Energy

Assume that e be the thermal energy per unit mass in the volume. To earn the energy equation we perform the Conservation of energy law that it states

Rate of Increasing of Energy in Volume = Rate of Heat Flux into Volume
+ Rate of Work by Surface Force
+ Volumetric Heat Source, then we insert the values of (1.3.31), (1.3.32), and (1.3.33) into the energy conservation equality and we derive

$$\frac{D}{Dt}\iiint \rho e dv = -\oiint \boldsymbol{q}.\widehat{\boldsymbol{n}}dA + \oiint(\boldsymbol{\tau}.\boldsymbol{V}).\widehat{\boldsymbol{n}}dA + \iiint Sdv \qquad (1.6.1)$$

Now we apply divergence theorem for the equality (1.6.11) and we gain

$$\frac{D}{Dt}\iiint \rho e dv = -\iiint \nabla.\boldsymbol{q}dv + \iiint \nabla.(\boldsymbol{\tau}.\boldsymbol{V})dv + \iiint Sdv, \qquad (1.6.2)$$

and

$$\iiint \left(\rho\frac{De}{Dt} + \nabla.\boldsymbol{q} - (\boldsymbol{\tau}:\nabla\boldsymbol{V}) - S\right)dv = 0 \qquad (1.6.3)$$

The integral equality (1.6.3) establish for each control volume and specially for every infinitesimal control volume at every point in the flow, then

$$\rho\frac{De}{Dt} + \nabla.\boldsymbol{q} - (\boldsymbol{\tau}:\nabla\boldsymbol{V}) - S = 0 \qquad (1.6.4)$$

Suppose that the flow is steady and any quantities depend on time, then by inserting the values of (1.3.31) and (1.3.32) into the relation (1.6.4) we gain

$$\rho c\boldsymbol{V}.\nabla\theta = \nabla.\left(k_{eff}\nabla\theta\right) + (\boldsymbol{\tau}:\nabla\boldsymbol{V}) + S \qquad (1.6.5)$$

If the thermal conductivity k_{eff} is constant, we could restate the equation (1.6.5) as

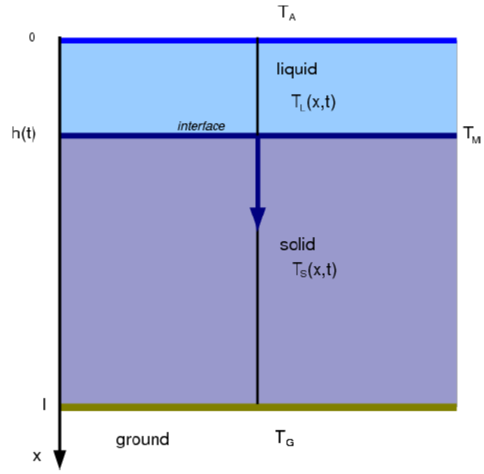
$$\rho c\boldsymbol{V}.\nabla\theta = k_{eff}\Delta\theta + (\boldsymbol{\tau}:\nabla\boldsymbol{V}) + S \qquad (1.6.6)$$

1.7. Stefan Condition

In the free boundary problems the boundary between solid and liquid phase changes in any time, and an important part in the modeling process is to prepare convenient model for moving boundary, and in this section we want to describe how we could earn the Stefan condition by separating the domain $\Omega = (0, l)$ to solid and liquid phase, then define

$$Liquid = \{(x,t): 0 < x < h(t), t > 0\}$$

$$Solid = \{(x,t): h(t) < x < l, t > 0\}$$



(1.7.1)

where $x = h(t)$ is the free boundary between solid and liquid parts (1.7.1). Hence we have the heat equation as

$$\theta_t = \alpha_L \theta_{xx} \; ; \; 0 < x < h(t),$$

$$\theta_t = \alpha_S \theta_{xx} \; ; \; h(t) < x < l,$$

where

$$\alpha_L = \frac{k_L}{\rho c_L}$$

$$\alpha_S = \frac{k_S}{\rho c_S}$$

Assume that θ_M is the melt temperature, then we can express the total enthalpy as

$$E(t) = A\left(\int_0^{h(t)} (\rho c_L(\theta(x,t) - \theta_M) + \rho L)dx + \int_{h(t)}^{l} \rho c_S(\theta(x,t) - \theta_M)dx\right), \tag{1.7.2}$$

and here L is the latent heat of the material. Now we differentiate the integral equation (1.7.2) by t and we get

$$\frac{1}{A}\frac{dE}{dt} = \rho c_L(\theta(h(t),t) - \theta_M)h'(t) + \rho L h'(t) + \int_0^{h(t)} \rho c_L \theta_t(x,t)dx - \rho c_S(\theta(h(t),t) - \theta_M)h'(t) + \int_{h(t)}^{l} \rho c_S \theta_t(x,t)dx \tag{1.7.3}$$

In the integral equality (1.7.3) we set $\theta(h(t),t) = \theta_M$, then we have

$$\frac{1}{A}\frac{dE}{dt} = \rho L h'(t) + \int_{0}^{h(t)} \rho c_L \theta_t(x,t)dx + \int_{h(t)}^{l} \rho c_S \theta_t(x,t)dx, \tag{1.7.4}$$

in this stage we replace $\theta_t(x,t)$ by $\theta_{xx}(x,t)$ from the heat equation in the relation (1.7.4) and we reach to

$$\frac{1}{A}\frac{dE}{dt} = \rho L h'(t) + \int_{0}^{h(t)} \rho c_L \alpha_L \theta_{xx}(x,t)dx + \int_{h(t)}^{l} \rho c_S \alpha_S \theta_{xx}(x,t)dx, \tag{1.7.5}$$

and finally we get

$$\frac{1}{A}\frac{dE}{dt} = \rho L h'(t) + k_L \theta_x(h(t),t) - k_L \theta_x(0,t) + k_S \theta_x(l,t) - k_S \theta_x(h(t),t) \tag{1.7.6}$$

Suppose the boundary conditions

$$q(0,t) = k_L \theta_x(0,t), \tag{1.7.7}$$

$$q(l,t) = k_S \theta_x(l,t), \tag{1.7.8}$$

also the global heat balance states that

$$\frac{dE}{dt} = A\big(q(0,t) - q(l,t)\big), \tag{1.7.9}$$

then by enforcing the boundary conditions (1.7.7) and (1.7.8), and the global heat balance (1.7.9), the equality (1.7.6) could be restated as

$$\rho L h'(t) = -k_L \theta_x(h(t),t) + k_S \theta_x(h(t),t) \tag{1.7.10}$$

The equation (1.7.10) is called Stefan condition and it expresses that the velocity $h'(t)$ of the free boundary $h(t)$ is proportional to the jump of the heat flux across the free boundary. If we choose only the melting process the Stefan condition (1.7.10) is shown as

$$\rho L h'(t) = -k_L \theta_x(h(t),t) \tag{1.7.11}$$

Assume the domain $\Omega \in \mathbb{R}^3$, and its boundary $\partial\Omega$, then the Stefan condition (1.7.11) in three- dimensional case could be stated as (see [41])

$$[\nabla\theta]_{-}^{+}.\boldsymbol{n}_x = -\lambda \boldsymbol{w}.\boldsymbol{n}_x = \lambda n_t \quad on\ \partial\Omega, \tag{1.7.12}$$

where $(\boldsymbol{n}_x, n_t)$ is the normal vector to the free boundary $\partial\Omega$, $\boldsymbol{w}$ is the velocity of the free boundary, and $\lambda > 0$ is the latent heat.

1.8. Heat Transfer Equations within Boundary Conditions

Now we will summarize the heat transfer equations, and prepare boundary conditions.

Continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\boldsymbol{V}) = 0$$

Navier-Stokes equations

$$\rho\left(\frac{\partial\boldsymbol{V}}{\partial t} + \boldsymbol{V}.\nabla\boldsymbol{V}\right) = -\nabla p + \nabla.\boldsymbol{\tau} + \rho\boldsymbol{g}$$

Energy equation

$$\rho c\boldsymbol{V}.\nabla\theta = k_{eff}\Delta\theta + (\boldsymbol{\tau}:\nabla\boldsymbol{V}) + S$$

Stefan condition

$$[\nabla\theta]_-^+.\boldsymbol{n}_x = -\lambda\boldsymbol{w}.\boldsymbol{n}_x = \lambda n_t \quad \text{on } \Gamma,$$

where

$$\Omega_1 = \{\theta > T_1\}, \qquad \Omega_2 = \{\theta < T_1\}, \qquad \Gamma = \{\theta = T_1\},$$

In the other words Ω_1 is the liquid phase, Ω_2 is the solid phase, and Γ is the free boundary between the liquid and solid phase.

Other boundary and initial conditions

$$\theta|_{\Gamma} = T_1$$

$$\theta|_{t=0} = \psi(x),\ x \in \Omega_1$$

$$\theta|_{t=0} = \varphi(x),\ x \in \Omega_2$$

$$\boldsymbol{V}|_{\Gamma\cup\Omega_2} = 0$$

1.9. Sobolev Spaces

In this section we will introduce the class of spaces that they play major role in the theory of modern partial differential equations, that is, Sobolev spaces. First we start by the weak derivative concept that was constructed on integration by parts relation.

Weak Derivative

Suppose that $\Omega \subset \mathbb{R}^n$ is a domain and $u \in L_1(\omega)$, for each bounded open set ω ($\overline{\omega} \subset \Omega$), then we call $D^\alpha u(x)$ the weak derivative of u if

$$\int_\Omega D^\alpha u(x).\, v(x) dx = (-1)^{|\alpha|} \int_\Omega u(x).\, D^\alpha v(x) dx \quad ; \; \forall v \in C_0^\infty(\Omega), \tag{1.9.1}$$

where $\boldsymbol{C}_0^\infty(\Omega)$ is the space of smooth functions (infinitely differentiable) on Ω with small support, and D^α is

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}},$$

which multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Sobolev Space

We define Sobolev space $W_p^k(\Omega)$ of order k as

$$W_p^k(\Omega) = \left\{u \in L_p(\Omega) : \; D^\alpha u \in L_p(\Omega), |\alpha| \le k\right\}, \tag{1.9.2}$$

where k is non-negative integer, and $p \in [1, \infty]$.

Sobolev Norm

Sobolev space $W_p^k(\Omega)$ is equipped with the norm that is called Sobolev norm and it is defined as

$$\|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha| \le k} \|D^\alpha u\|_{L_p(\Omega)}^p\right)^{\frac{1}{p}} ; 1 \le p < \infty, \tag{1.9.3}$$

and when $p = \infty$ we define

$$\|u\|_{W_\infty^k(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L_\infty(\Omega)} \tag{1.9.4}$$

Now we introduce an important special case when we choose $p = 2$, then the space $W_p^k(\Omega)$ would be a Hilbert space within inner product

$$(u, v)_{W_p^k(\Omega)} = \sum_{|\alpha| \le k} (D^\alpha u, D^\alpha v) \tag{1.9.5}$$

In this special case we use the notation $H^k(\Omega)$ instead of $W_p^k(\Omega)$. Let's immediately focus on $H^1(\Omega)$ and its norm, then

$$H^1(\Omega) = \left\{u \in L_2(\Omega) : \frac{\partial u}{\partial x_j} \in L_2(\Omega), j = 1,2,\dots,n\right\}, \qquad (1.9.6)$$

and we define its related norm as

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L_2(\Omega)} + \sum_{j=1}^{n} \left\|\frac{\partial u}{\partial x_j}\right\|^2_{L_2(\Omega)}\right)^{\frac{1}{2}} \qquad (1.9.7)$$

The other important space of functions is $H_0^1(\Omega)$ where it includes the functions in $H^1(\Omega)$ that they are vanished on the boundary $\partial\Omega$. We define $H_0^1(\Omega)$ as the closure of $H^1(\Omega)$ in the norm of $\|.\|_{H^1(\Omega)}$. Indeed $u \in H_0^1(\Omega)$ if and only if $u \in H^1(\Omega)$, and there exists the sequence $u_m \in C_0^\infty(\Omega)$ such that

$$\lim_{m\to\infty} u_m = u$$

Let's assume that Ω has smooth boundary $\partial\Omega$, it is easy to show that

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \; on \; \partial\Omega\} \qquad (1.9.8)$$

This class of functions are applied when the Dirichlet boundary condition is supposed for a partial differential equation. Also we note that $H_0^1(\Omega)$ is a Hilbert space within the same norm and inner product as $H^1(\Omega)$.

Chapter 2.

Stream Functions,

Mathematical Modeling

2.1. Stream Functions

We start the work by deriving the mathematical modelling of heat transfer in Garnissage furnace in two-dimensional case based on stream functions. This modelling will apply the mass, momentum, and energy conservation laws to achieve the continuity, Navier-Stokes, and heat conduction equations, and to illustrate the moving boundary between the solid and liquid phase we exert the Stefan condition. We will state the system according to stream functions, and then by weak formulation. Finally we invoke the finite element method to gain the numerical solution of the system.

Definition
Suppose that $\boldsymbol{V} = (u, v, 0)$ is the velocity vector field of two-dimensional flows such that $\boldsymbol{V} = \nabla \times \boldsymbol{\psi}$, where $\boldsymbol{\psi} = (0,0,\psi)$; then we call ψ the stream function. The stream function could be stated as

$$(u, v, 0) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix}$$

$$= \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right),$$

then we state the velocity vector components based on the stream function as

$$u = \frac{\partial \psi}{\partial y} \tag{2.1.1}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{2.1.2}$$

Before any mathematical details we need to transfer the continuity, Navier-Stokes, and energy conservation equation to the system according to stream function ψ. We will do this converting process in the next three sections.

2.2. Continuity Equation

Let's assume the flow is steady and has low speed, then from relation (1.4.7) we have the continuity equation as $\nabla . \boldsymbol{V} = 0$, then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.2.1}$$

now we replace the velocity components from (2.1.1), (2.1.2) into the equality (2.2.1), since we have supposed $\psi \in H_0^1(\Omega)$, then

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=\frac{\partial^2 \psi}{\partial x \partial y}-\frac{\partial^2 \psi}{\partial x \partial y}=0,$$

Theorem 2.1. The stream function ψ automatically satisfies in the continuity equation.

Due to theorem 2.1 we will omit the continuity equation in the transport system in this section.

2.3. Navier-Stokes Equations

Let's focus on the Navier-Stokes system in two dimension that in this case the velocity vector field is $\boldsymbol{V}=(u,v)$, and assume that the fluid is Newtonian, then stress is $\mu\frac{\partial \boldsymbol{V}}{\partial y}$, where μ is the dynamic viscosity, and also for the steady flow μ is constant, then we could rewrite the relations (1.5.17) and (1.5.18) as

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\Delta u+\rho g_x \tag{2.3.1}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\Delta v+\rho g_y \tag{2.3.2}$$

We divide the equations (2.3.1) and (2.3.2) by density and we earn

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\vartheta\Delta u+g_x, \tag{2.3.3}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\vartheta\Delta v+g_y, \tag{2.3.4}$$

,where $\vartheta=\frac{\mu}{\rho}$ is the kinematic viscosity. In this stage we differentiate the equation (2.3.3) respect to *y* and (2.3.4) respect to *x*, then we gain

$$\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+u\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y^2}=-\frac{1}{\rho}\frac{\partial^2 p}{\partial y\partial x}+\vartheta\frac{\partial}{\partial y}(\Delta u) \tag{2.3.5}$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+v\frac{\partial^2 v}{\partial x\partial y}=-\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}+\vartheta\frac{\partial}{\partial x}(\Delta v) \tag{2.3.6}$$

We subtract the equality (2.3.6) from (2.3.5), then we achieve

$$\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+u\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}-u\frac{\partial^2 v}{\partial x^2}-$$

$$\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - v\frac{\partial^2 v}{\partial x \partial y} = \vartheta\frac{\partial}{\partial y}(\Delta u) - \vartheta\frac{\partial}{\partial x}(\Delta v) \tag{2.3.7}$$

We continue the process by inserting the values of velocity components from (2.1.1) and (2.1.2) into the differential equation (2.3.7) and we derive the equality respect to stream function ψ as

$$\frac{\partial \psi}{\partial y}\frac{\partial^3 \psi}{\partial x \partial y^2} - \frac{\partial \psi}{\partial x}\frac{\partial^3 \psi}{\partial y^3} + \frac{\partial \psi}{\partial y}\frac{\partial^3 \psi}{\partial x^3} - \frac{\partial \psi}{\partial x}\frac{\partial^3 \psi}{\partial y \partial x^2} =$$

$$\vartheta\left(\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4}\right), \tag{2.3.8}$$

and we simplify (2.3.8) as

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = \vartheta\Delta^2\psi, \tag{2.3.9}$$

and finally we state the relation (2.3.9) as

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\Delta\psi) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\Delta\psi) = \vartheta\Delta^2\psi \tag{2.3.10}$$

Theorem 2.2. The momentum conservation equation for incompressible Newtonian fluid in two-dimension based on stream function is stated as

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\Delta\psi) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\Delta\psi) = \vartheta\Delta^2\psi$$

2.4. Energy Conservation Equation

Modeling of conservation equations respect to stream function ψ is finished by heat conduction equation, and remember that the flow has been supposed steady and incompressible, then we restate the relation (1.6.6) as

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{k}{\rho c}\Delta\theta + \frac{1}{\rho c}S \tag{2.4.1}$$

If we set the velocity components from (2.1.1) and (2.1.2) into the equation (2.4.1) we will have

$$\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \frac{k}{\rho c}\Delta\theta + \frac{1}{\rho c}S \tag{2.4.2}$$

Theorem 2.3. The energy conservation equation for incompressible Newtonian fluid in two-dimension based on stream function can be stated as

$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}=\frac{k}{\rho c}\Delta\theta+\frac{1}{\rho c}S$$

2.5. Weak Formulation

To follow the finite element method to find the numerical solution of the heat transfer system we need to construct the variational version of the system, that is, it is necessary to state the system in the integral equations form. Assume that $\eta\in H_0^1(\Omega)$, and multiply the equality (2.3.10) by η, and integrate the both sides of equality, then

$$\iint_\Omega\left(\frac{\partial}{\partial x}(\Delta\psi)\frac{\partial\psi}{\partial y}\eta\right)dxdy-\iint_\Omega\left(\frac{\partial}{\partial y}(\Delta\psi)\frac{\partial\psi}{\partial x}\eta\right)dxdy$$
$$=\iint_\Omega(\vartheta\Delta^2\psi\eta)dxdy \qquad (2.5.1)$$

We exert the integration by parts in the integral equality (2.5.1) and we earn

$$-\iint_\Omega\Delta\psi\left(\frac{\partial\eta}{\partial x}\frac{\partial\psi}{\partial y}+\frac{\partial^2\psi}{\partial x\partial y}\eta\right)dxdy+\iint_\Omega\Delta\psi\left(\frac{\partial\eta}{\partial y}\frac{\partial\psi}{\partial x}+\frac{\partial^2\psi}{\partial x\partial y}\eta\right)dxdy$$

$$=\vartheta\iint_\Omega(\Delta\psi\Delta\eta)dxdy, \qquad (2.5.2)$$

now we simplify the relation (2.5.2) as

$$\iint_\Omega\Delta\psi\left(\frac{\partial\eta}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial\eta}{\partial y}\frac{\partial\psi}{\partial x}+\vartheta\Delta\eta\right)dxdy=0 \qquad (2.5.3)$$

Theorem 2.4. Suppose that η is a smooth test function belongs to $H_0^1(\Omega)$, then the weak formulation of momentum conservation equation (2.3.10) is constructed as

$$\iint_\Omega\Delta\psi\left(\frac{\partial\eta}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial\eta}{\partial y}\frac{\partial\psi}{\partial x}+\vartheta\Delta\eta\right)dxdy=0$$

We complete the process by the same technic for energy equation (2.4.2) and we gain

$$\iint_\Omega\left(\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}\eta\right)dxdy-\iint_\Omega\left(\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}\eta\right)dxdy=\iint_\Omega\left(\frac{k}{\rho c}\Delta\theta\eta\right)dxdy$$

$$+\iint_{\Omega}\left(\frac{S}{\rho c}\eta\right)dxdy, \tag{2.5.4}$$

again we perform the integration by parts (Green's first identity) in the equality (2.5.4) and we derive

$$-\iint_{\Omega}\psi\left(\frac{\partial^2\theta}{\partial x\partial y}\eta+\frac{\partial\theta}{\partial x}\frac{\partial\eta}{\partial y}\right)dxdy+\iint_{\Omega}\psi\left(\frac{\partial^2\theta}{\partial x\partial y}\eta+\frac{\partial\theta}{\partial y}\frac{\partial\eta}{\partial x}\right)dxdy$$

$$=-\frac{k}{\rho c}\iint_{\Omega}(\nabla\theta.\nabla\eta)dxdy+\frac{k}{\rho c}\int_{\partial\Omega}\nabla\theta.\boldsymbol{n}_x\eta+\frac{S}{\rho c}\iint_{\Omega}\eta dxdy \tag{2.5.5}$$

We force the Stefan condition (1.7.12) in the relation (2.5.5) and we have

$$-\iint_{\Omega}\psi\left(\frac{\partial\theta}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\theta}{\partial y}\frac{\partial\eta}{\partial x}\right)dxdy=-\frac{k}{\rho c}\iint_{\Omega}\nabla\theta.\nabla\eta dxdy+$$

$$\frac{k}{\rho c}\int_{\partial\Omega}\lambda n_t\eta+\frac{S}{\rho c}\iint_{\Omega}\eta dxdy, \tag{2.5.6}$$

at last we call the divergence theorem for the relation (2.5.6) and we achieve

$$-\iint_{\Omega}\psi\left(\frac{\partial\theta}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\theta}{\partial y}\frac{\partial\eta}{\partial x}\right)dxdy=-\frac{k}{\rho c}\iint_{\Omega}\nabla\theta.\nabla\eta dxdy+$$

$$\frac{k\lambda}{\rho c}\iint_{\Omega}\frac{\partial\eta}{\partial t}dxdy+\frac{S}{\rho c}\iint_{\Omega}\eta dxdy \tag{2.5.7}$$

Theorem 2.5. Suppose that η is a smooth test function from $H_0^1(\Omega)$, then the weak formulation of energy conservation equation (2.4.2) is computed as

$$-\iint_{\Omega}\psi\left(\frac{\partial\theta}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\theta}{\partial y}\frac{\partial\eta}{\partial x}\right)dxdy=-\frac{k}{\rho c}\iint_{\Omega}\nabla\theta.\nabla\eta dxdy+$$

$$\frac{k\lambda}{\rho c}\iint_{\Omega}\frac{\partial\eta}{\partial t}dxdy+\frac{S}{\rho c}\iint_{\Omega}\eta dxdy$$

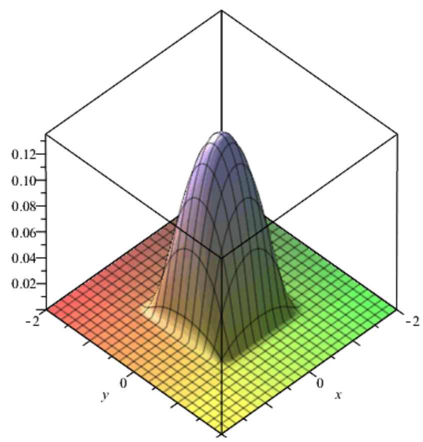
We interpret the transport phenomena system as the problem of finding $\psi,\theta\in H_0^1(\Omega)$ such that ψ,θ satisfy in the integral equations (2.5.3) and (2.5.7).

2.6. Discretization of Domain

To finish the process of the finite element method we will construct the finite-dimensional subspace $V_h \subset H_0^1(\Omega)$ which consists of test functions of the form

$$\phi(x,y) = \begin{cases} exp\left(\frac{1}{x^2-1}\right) exp\left(\frac{1}{y^2-1}\right) & |x|<1, |y|<1 \\ 0 & otherwise \end{cases}, \quad (2.6.1)$$

with the shape



(2.6.2)

Now we replace the approximate values ψ_h and θ_h instead of ψ and θ in the equalities (2.5.3) and (2.5.7), and we redefine the heat transfer problem as the problem of finding $\psi_h, \theta_h \in V_h$ such that ψ_h, θ_h satisfy in the integral equations

$$\iint_\Omega \Delta\psi_h \left(\frac{\partial \eta_h}{\partial x}\frac{\partial \psi_h}{\partial y} - \frac{\partial \eta_h}{\partial y}\frac{\partial \psi_h}{\partial x} + \vartheta \Delta\eta_h\right) dxdy = 0 \quad (2.6.3)$$

$$-\iint_\Omega \psi_h \left(\frac{\partial \theta_h}{\partial x}\frac{\partial \eta_h}{\partial y} - \frac{\partial \theta_h}{\partial y}\frac{\partial \eta_h}{\partial x}\right) dxdy = -\frac{k}{\rho c}\iint_\Omega (\nabla\theta_h . \nabla\eta_h) dxdy +$$

$$\frac{k\lambda}{\rho c}\iint_\Omega \frac{\partial \eta_h}{\partial t} + \frac{S}{\rho c}\iint_\Omega \eta_h dxdy, \quad (2.6.4)$$

for every $\eta_h \in V_h$.

Suppose that

$$\dim V_h = N(h),$$

and

$$V_h = span\{\phi_1, \phi_2, \phi_3, \dots, \phi_{N(h)}\},$$

where the basis functions $\phi_i(x,y), i = 1,2,\dots,N(h)$, have small support. Now we put on the approximate solutions ψ_h, θ_h in terms of the basis functions $\phi_i(x,y)$, we can write

$$\psi_h(x,y) = \sum_{i=1}^{N(h)} U_i \phi_i(x,y), \tag{2.6.5}$$

$$\theta_h(x,y) = \sum_{i=1}^{N(h)} V_i \phi_i(x,y), \tag{2.6.6}$$

where U_i, V_i , $i = 1,2,3,\dots,N(h)$, must to be determined. We replace ψ_h from (2.6.5) into the momentum integral equation (2.6.3) and we derive

$$\iint_{\Omega} \sum_{i=1}^{N(h)} U_i \Delta\phi_i \left(\frac{\partial \phi_k}{\partial x} \sum_{j=1}^{N(h)} U_j \frac{\partial \phi_j}{\partial y} - \frac{\partial \phi_k}{\partial y} \sum_{j=1}^{N(h)} U_j \frac{\partial \phi_j}{\partial x} + \vartheta \Delta\phi_k \right) dxdy = 0 \tag{2.6.7}$$

$$;\ k = 1,2,3,\dots,N(h)$$

The relation (2.6.7) could be expressed as

$$\sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_i U_j \iint_{\Omega} \Delta\phi_i \left(\frac{\partial \phi_j}{\partial y}\frac{\partial \phi_k}{\partial x} - \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_k}{\partial y} \right) dxdy +$$
$$\sum_{i=1}^{N(h)} U_i \iint_{\Omega} \vartheta \Delta\phi_i \Delta\phi_k dxdy = 0 \tag{2.6.8}$$

Also we simplify the equality (2.6.8) as the nonlinear system

$$\sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_i U_j\, a_{ijk} + \sum_{i=1}^{N(h)} U_i\, b_{ik} = 0, \tag{2.6.9}$$

where the coefficients are

$$a_{ijk} = \iint_{\Omega} \Delta\phi_i \left(\frac{\partial \phi_j}{\partial y}\frac{\partial \phi_k}{\partial x} - \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_k}{\partial y} \right) dxdy \tag{2.6.10}$$

$$b_{ik} = \vartheta \iint_{\Omega} \Delta\phi_i \Delta\phi_k dxdy \tag{2.6.11}$$

Theorem 2.6. Let's discrete the domain Ω by the finite-dimensional subspace $V_h \subset H_0^1(\Omega)$ which consists of smooth test functions $\phi_i(x,y), i = 1,2,\dots,N(h)$ with small support, where $\dim V_h = N(h)$, and $V_h = span\{\phi_1, \phi_2, \phi_3, \dots, \phi_{N(h)}\}$, then the equality (2.5.3) can be showed as the nonlinear system of equations

$$\sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_i U_j \, a_{ijk} + \sum_{i=1}^{N(h)} U_i \, b_{ik} = 0, \qquad k = 1,2,3,\dots,N(h),$$

where

$$a_{ijk} = \iint_\Omega \Delta\phi_i \left(\frac{\partial \phi_j}{\partial y} \frac{\partial \phi_k}{\partial x} - \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_k}{\partial y} \right) dxdy \,, b_{ik} = \vartheta \iint_\Omega \Delta\phi_i \Delta\phi_k dxdy.$$

In the final part we repeat the same approach for the energy conservation equation. First we suppose the test function $\xi(x,y,t)$ that it depends on time

$$\xi(x,y,t) = \begin{cases} exp\left(\frac{1}{t^2-1}\right) exp\left(\frac{1}{x^2-1}\right) exp\left(\frac{1}{y^2-1}\right) & |x| < 1, |y| < 1, |t| < 1 \\ 0 & otherwise \end{cases}, \tag{2.6.12}$$

then we insert θ_h from (2.6.6) into the relation (2.64) and we achieve

$$-\iint_\Omega \psi_h \left(\left(\sum_{i=1}^{N(h)} V_i \frac{\partial \xi_i}{\partial x} \right) \frac{\partial \xi_j}{\partial y} - \left(\sum_{i=1}^{N(h)} V_i \frac{\partial \xi_i}{\partial y} \right) \frac{\partial \xi_j}{\partial x} \right) dxdy = $$
$$-\frac{k}{\rho c} \iint_\Omega \left(\sum_{i=1}^{N(h)} V_i \nabla \xi_i \,. \nabla \xi_j \right) dxdy + \frac{k\lambda}{\rho c} \iint_\Omega \frac{\partial \xi_j}{\partial t} + \frac{S}{\rho c} \iint_\Omega \xi_j dxdy \tag{2.6.13}$$

The integral equality (2.6.13) is stated as

$$\sum_{i=1}^{N(h)} V_i \left(\iint_\Omega \left(-\psi_h \frac{\partial \xi_i}{\partial x} \frac{\partial \xi_j}{\partial y} + \psi_h \frac{\partial \xi_i}{\partial y} \frac{\partial \xi_j}{\partial x} + \frac{k}{\rho c} \nabla \xi_i . \nabla \xi_j \right) dxdy \right) =$$
$$\frac{k\lambda}{\rho c} \iint_\Omega \frac{\partial \xi_j}{\partial t} + \frac{S}{\rho c} \iint_\Omega \xi_j dxdy, \tag{2.6.14}$$

and at the end we earn the linear system

$$\sum_{i=1}^{N(h)} c_{ij} V_i = d_j, \tag{2.6.15}$$

$$j = 1,2,3, \dots, N(h),$$

where the coefficients are

$$c_{ij} = \iint_{\Omega} \left(-\psi_h \frac{\partial \xi_i}{\partial x} \frac{\partial \xi_j}{\partial y} + \psi_h \frac{\partial \xi_i}{\partial y} \frac{\partial \xi_j}{\partial x} + \frac{k}{\rho c} \nabla \xi_i . \nabla \xi_j \right) dxdy \tag{2.6.16}$$

$$d_j = \frac{k\lambda}{\rho c} \iint_{\Omega} \frac{\partial \xi_j}{\partial t} + \frac{S}{\rho c} \iint_{\Omega} \xi_j dxdy \tag{2.6.17}$$

Theorem 2.7. Let's discrete the domain Ω by the finite-dimensional subspace $V_h \subset H_0^1(\Omega)$ which consists of smooth test functions $\xi_i(x, y), i = 1,2, \dots, N(h)$ with small support, where $\dim V_h = N(h)$, and $V_h = span\{\xi_1, \xi_2, \xi_3, \dots, \xi_{N(h)}\}$, then the equality (2.5.7) can be showed as the linear system of equations

$$\sum_{i=1}^{N(h)} c_{ij} V_i = d_j \,, \qquad j = 1,2,3, \dots, N(h),$$

where

$$c_{ij} = \iint_{\Omega} \left(-\psi_h \frac{\partial \xi_i}{\partial x} \frac{\partial \xi_j}{\partial y} + \psi_h \frac{\partial \xi_i}{\partial y} \frac{\partial \xi_j}{\partial x} + \frac{k}{\rho c} \nabla \xi_i . \nabla \xi_j \right) dxdy,$$

$$d_j = \frac{k\lambda}{\rho c} \iint_{\Omega} \frac{\partial \xi_j}{\partial t} + \frac{S}{\rho c} \iint_{\Omega} \xi_j dxdy.$$

Thus by considering the systems (2.6.9) and (2.6.15) we redefine the transport phenomena as the problem of finding $(U_1, U_2, \dots, U_{N(h)}) \in R^{N(h)}$ and $(V_1, V_2, \dots, V_{N(h)}) \in R^{N(h)}$, such that they satisfy in the systems (2.6.9) and (2.6.15).

As we see the nonlinear system (2.6.9) we has $N(h)$ equations within $N(h)$ unknowns $U_1, U_2, \dots, U_{N(h)}$, then we will execute the Newton's method to search the numerical solution of the system. Also the linear system (2.6.15) has $N(h)$ equations within $N(h)$ unknowns $V_1, V_2, \dots, V_{N(h)}$, then we can perform every classical numerical methods to solve this linear system.

2.7. the Newton's Method

In the final stage of the process of numerical approach we show that how the Newton method could be applied to investigate the numerical solution of nonlinear system (2.6.9), to start the method define the functions

$$F_k\left(U_1, U_2, \dots, U_{N(h)}\right) = \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_i U_j \, a_{ijk} + \sum_{i=1}^{N(h)} U_i \, b_{ik} \tag{2.7.1}$$

$$;\ k = 1,2,3, \dots, N(h)$$

Newton's method works according to following iterative process

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^{n} - (\boldsymbol{DF})^{-1}(\boldsymbol{U}^{n})\boldsymbol{F}(\boldsymbol{U}^{n}), \tag{2.7.2}$$

where $\boldsymbol{U_0}$ is the initial vector as

$$\boldsymbol{U_0} = \left(U_{1_0}, U_{2_0}, \dots, U_{N(h)_0}\right)^T \tag{2.7.3}$$

The Jacobean matrix $\boldsymbol{DF}$ in the relation (2.7.2) is defined

$$\boldsymbol{DF} = \begin{pmatrix} \frac{\partial F_1}{\partial U_1} & \frac{\partial F_1}{\partial U_2} & \cdots & \frac{\partial F_1}{\partial U_{N(h)}} \\ \frac{\partial F_2}{\partial U_1} & \frac{\partial F_2}{\partial U_2} & \cdots & \frac{\partial F_2}{\partial U_{N(h)}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_{N(h)}}{\partial U_1} & \frac{\partial F_{N(h)}}{\partial U_2} & \cdots & \frac{\partial F_{N(h)}}{\partial U_{N(h)}} \end{pmatrix}, \tag{2.7.4}$$

and finally the value vector $\boldsymbol{F}$ is

$$\boldsymbol{F} = \left(F_1(U), F_2(U), \dots, F_{N(h)}(U)\right)^T. \tag{2.7.5}$$

Chapter 3.

Cylindrical Coordinates, Mathematical Modeling, and Numerical Solution

3.1. Vector Analysis

In the current chapter we will prepare some basic relations in vector analysis to apply them in the modeling process in cylindrical coordinate system. In the previous chapter we had noticed that the Garnissage tank has cylinder shape and our objective is to use convenient symmetric properties of furnace in the mathematical modeling process. The great advantage of this modeling is it reduces one dimension, therefore we will achieve the modeling in three dimension space instead of four dimension.

We start the modeling in cylindrical coordinate immediately after defining some necessary operators in vector analysis, then we derive the mathematical modeling and also its weak formulation, and finally we discrete the domain to replace the approximate variables.

Gradient
Assume that e_r, e_φ and e_z are unit normal vectors in the cylindrical coordinate, that is,

$$e_r = (cos\varphi, sin\varphi, 0),$$

$$e_\varphi = (-sin\varphi, cos\varphi, 0),$$

$$e_z = (0,0,1).$$

We compute the partial derivatives

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} cos\varphi - \frac{1}{r}\frac{\partial f}{\partial \varphi} sin\varphi,$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r} sin\varphi + \frac{1}{r}\frac{\partial f}{\partial \varphi} cos\varphi,$$

and insert them into the gradient operator $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$, then we have

$$\nabla f = \left(\frac{\partial f}{\partial r} cos\varphi - \frac{1}{r}\frac{\partial f}{\partial \varphi} sin\varphi\right)\left(e_r cos\varphi - e_\varphi sin\varphi\right)$$

$$+\left(\frac{\partial f}{\partial r} sin\varphi + \frac{1}{r}\frac{\partial f}{\partial \varphi} cos\varphi\right)\left(e_r sin\varphi + e_\varphi cos\varphi\right) + \frac{\partial f}{\partial z} e_z,$$

therefore we get the gradient operator in the cylindrical coordinate system as

$$\nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r}\frac{\partial f}{\partial \varphi} e_\varphi + \frac{\partial f}{\partial z} e_z \qquad (3.1.1)$$

Divergence

To determine the divergence operator in the cylindrical coordinate we write

$$\nabla.\boldsymbol{V} = \left(\frac{\partial}{\partial r}e_r + \frac{1}{r}\frac{\partial}{\partial \varphi}e_\varphi + \frac{\partial}{\partial z}e_z\right).(u,v,w),$$

then we will get

$$\nabla.\boldsymbol{V} = \frac{\partial u}{\partial r}cos\varphi - \frac{1}{r}\frac{\partial u}{\partial \varphi}sin\varphi + \frac{\partial v}{\partial r}sin\varphi + \frac{1}{r}\frac{\partial v}{\partial \varphi}cos\varphi + \frac{\partial w}{\partial z}, \qquad (3.1.2)$$

in the other hand we can express the velocity vector field as

$$\boldsymbol{V} = u_r e_r + u_\varphi e_\varphi + u_z e_z, \qquad (3.1.3)$$

where the velocity components u_r, u_φ, and u_z are

$$u_r = u\ cos\varphi + v\ sin\varphi, \qquad (3.1.4)$$

$$u_\varphi = -u\ sin\varphi + v\ cos\varphi, \qquad (3.1.5)$$

$$u_z = w. \qquad (3.1.6)$$

We differentiate the relations (3.1.4) and (3.1.5) by r and φ to gain

$$\frac{\partial u_r}{\partial r} = \frac{\partial u}{\partial r}cos\varphi + \frac{\partial v}{\partial r}sin\varphi, \qquad (3.1.7)$$

$$\frac{\partial u_\varphi}{\partial r} = -\frac{\partial u}{\partial r}sin\varphi + \frac{\partial v}{\partial r}cos\varphi, \qquad (3.1.8)$$

$$\frac{\partial u_r}{\partial \varphi} = u_\varphi + \frac{\partial u}{\partial \varphi}cos\varphi + \frac{\partial v}{\partial \varphi}sin\varphi, \qquad (3.1.9)$$

$$\frac{\partial u_\varphi}{\partial \varphi} = -u_r - \frac{\partial u}{\partial \varphi}sin\varphi + \frac{\partial v}{\partial \varphi}cos\varphi. \qquad (3.1.10)$$

From the partial derivative equalities (3.1.7) and (3.1.8) we earn $\frac{\partial u}{\partial r}$ and $\frac{\partial v}{\partial r}$ as

$$\frac{\partial u}{\partial r} = \frac{\partial u_r}{\partial r}cos\varphi - \frac{\partial u_\varphi}{\partial r}sin\varphi, \qquad (3.1.11)$$

$$\frac{\partial v}{\partial r} = \frac{\partial u_\varphi}{\partial r}cos\varphi + \frac{\partial u_r}{\partial r}sin\varphi, \qquad (3.1.12)$$

and also from (3.1.9) and (3.1.10) we gain $\frac{\partial u}{\partial \varphi}$ and $\frac{\partial v}{\partial \varphi}$ as

$$\frac{\partial u}{\partial \varphi} = \left(\frac{\partial u_r}{\partial \varphi} - u_\varphi\right) cos\varphi - \left(\frac{\partial u_\varphi}{\partial \varphi} + u_r\right) sin\varphi, \qquad (3.1.13)$$

$$\frac{\partial v}{\partial \varphi} = \left(\frac{\partial u_\varphi}{\partial \varphi} + u_r\right) cos\varphi + \left(\frac{\partial u_r}{\partial \varphi} - u_\varphi\right) sin\varphi. \qquad (3.1.14)$$

If we replace the values of recent partial derivatives into the equality (3.1.2) we will derive divergence operator as

$$\nabla.\boldsymbol{V} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} \qquad (3.1.15)$$

3.2. Mathematical Modeling

In this section we will construct the mathematical modeling of heat transfer in the cylindrical coordinate system respect to the velocity vector field components u_r, u_φ, and u_z, let's start by the continuity equation and just apply the divergence operator (3.1.15) for the continuity equation to get

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{1}{r}\frac{\partial(\rho u_\varphi)}{\partial \varphi} + \frac{\partial(\rho u_z)}{\partial z} = 0 \qquad (3.2.1)$$

For the steady flow with low speed we rewrite the equation (3.2.1) as

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} = 0 \qquad (3.2.2)$$

In the Lagrangian reference the velocity is only a function of time, when in the Eulerian reference the velocity is a function of position that is the function of time. Let us assume the velocity vector field as

$$\boldsymbol{V}(r,\varphi,z,t) = u_r(r,\varphi,z,t)e_r + u_\varphi(r,\varphi,z,t)e_\varphi + u_z(r,\varphi,z,t)e_z, \qquad (3.2.3)$$

then the material derivative of $\boldsymbol{V}$ is written as

$$\frac{D\boldsymbol{V}}{Dt} = \frac{\partial \boldsymbol{V}}{\partial t} + \frac{\partial \boldsymbol{V}}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial \boldsymbol{V}}{\partial \varphi}\frac{\partial \varphi}{\partial t} + \frac{\partial \boldsymbol{V}}{\partial z}\frac{\partial z}{\partial t} \qquad (3.2.4)$$

To compute the material derivative we need to derive all of partial derivatives in the differential equality (3.2.4), then we compute the partial derivatives as

$$\frac{\partial \boldsymbol{V}}{\partial t}=\frac{\partial u_r}{\partial t}e_r+\frac{\partial u_\varphi}{\partial t}e_\varphi+\frac{\partial u_z}{\partial t}e_z,$$

$$\frac{\partial \boldsymbol{V}}{\partial r}=\frac{\partial u_r}{\partial r}e_r+\frac{\partial u_\varphi}{\partial r}e_\varphi+\frac{\partial u_z}{\partial r}e_z,$$

$$\frac{\partial \boldsymbol{V}}{\partial \varphi}=\frac{\partial u_r}{\partial \varphi}e_r+u_r e_\varphi+\frac{\partial u_\varphi}{\partial \varphi}e_\varphi-u_\varphi e_r+\frac{\partial u_z}{\partial \varphi}e_z,$$

$$\frac{\partial \boldsymbol{V}}{\partial z}=\frac{\partial u_r}{\partial z}e_r+\frac{\partial u_\varphi}{\partial z}e_\varphi+\frac{\partial u_z}{\partial z}e_z,$$

then we insert the recent partial derivatives into the material derivative (3.2.4) and we will earn

$$\begin{aligned}\frac{D\boldsymbol{V}}{Dt}&=\frac{\partial u_r}{\partial t}e_r+\frac{\partial u_\varphi}{\partial t}e_\varphi+\frac{\partial u_z}{\partial t}e_z\\&+u_r\left(\frac{\partial u_r}{\partial r}e_r+\frac{\partial u_\varphi}{\partial r}e_\varphi+\frac{\partial u_z}{\partial r}e_z\right),\\&+\frac{u_\varphi}{r}\left(\frac{\partial u_r}{\partial \varphi}e_r+u_r e_\varphi+\frac{\partial u_\varphi}{\partial \varphi}e_\varphi-u_\varphi e_r+\frac{\partial u_z}{\partial \varphi}e_z\right)\\&+u_z\left(\frac{\partial u_r}{\partial z}e_r+\frac{\partial u_\varphi}{\partial z}e_\varphi+\frac{\partial u_z}{\partial z}e_z\right),\end{aligned} \tag{3.2.5}$$

and after some simplification and reordering we gain as

$$\begin{aligned}\frac{D\boldsymbol{V}}{Dt}&=\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\varphi}{r}\frac{\partial u_r}{\partial \varphi}+u_z\frac{\partial u_r}{\partial z}-\frac{{u_\varphi}^2}{r}\right)e_r\\&+\left(\frac{\partial u_\varphi}{\partial t}+u_r\frac{\partial u_\varphi}{\partial r}+\frac{u_\varphi}{r}\frac{\partial u_\varphi}{\partial \varphi}+u_z\frac{\partial u_\varphi}{\partial z}+\frac{u_r u_\varphi}{r}\right)e_\varphi\\&+\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\varphi}{r}\frac{\partial u_z}{\partial \varphi}+u_z\frac{\partial u_z}{\partial z}\right)e_z\end{aligned} \tag{3.2.6}$$

If we apply the relation (3.2.6) for the Navier-Stokes equations (1.5.17), (1.5.18), and (1.5.19) we would state the system in the cylindrical coordinate as

$$r: \ \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_r}{\partial \varphi} + u_z\frac{\partial u_r}{\partial z} - \frac{{u_\varphi}^2}{r}\right) =$$

$$-\frac{\partial p}{\partial r} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \varphi^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial \varphi}\right) + \rho g_r, \quad (3.2.7)$$

$$\varphi: \ \rho\left(\frac{\partial u_\varphi}{\partial t} + u_r\frac{\partial u_\varphi}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_\varphi}{\partial \varphi} + u_z\frac{\partial u_\varphi}{\partial z} + \frac{u_r u_\varphi}{r}\right) =$$

$$-\frac{1}{r}\frac{\partial p}{\partial \varphi} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\varphi}{\partial \varphi^2} + \frac{\partial^2 u_\varphi}{\partial z^2} - \frac{u_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \varphi}\right) + \rho g_\varphi, \quad (3.2.8)$$

$$z: \ \rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_z}{\partial \varphi} + u_z\frac{\partial u_z}{\partial z}\right) =$$

$$-\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \varphi^2} + \frac{\partial^2 u_z}{\partial z^2}\right) + \rho g_z. \quad (3.2.9)$$

According to thermodynamic first law if we follow the same approach as what we had done to the momentum equations we derive the energy conservation equation in the cylindrical coordinate as

$$\frac{\partial \theta}{\partial t} + u_r\frac{\partial \theta}{\partial r} + \frac{u_\varphi}{r}\frac{\partial \theta}{\partial \varphi} + u_z\frac{\partial \theta}{\partial z} =$$

$$\frac{k}{\rho c}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \varphi^2} + \frac{\partial^2 \theta}{\partial z^2}\right) + 2\vartheta|\boldsymbol{\tau}|^2 + \frac{1}{\rho c}F \quad (3.2.10)$$

To take the advantage of symmetry, the cylindrical representation of the incompressible Navier-Stokes equations are chosen, and they are one of the most common systems that they are used to describe the behavior of the fluids in the furnaces. We will use the axisymmetric flow with the assumption of no tangential velocity, that is

$$u_\varphi = 0,$$

and also the other quantities that they are dependent on φ, therefore we omit all φ-dependent quantities in the transport phenomena system and we achieve the more simplified system consist of continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{\partial(\rho u_z)}{\partial z} = 0 \quad (3.2.11)$$

Navier-Stokes equation correspond to r

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}\right)= \\ -\frac{\partial p}{\partial r}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}\right)+\rho g_r \tag{3.2.12}$$

Navier-Stokes equation correspond to z

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}\right)= \\ -\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{\partial^2 u_z}{\partial z^2}\right)+\rho g_z \tag{3.2.13}$$

Heat conduction equation

$$\frac{\partial\theta}{\partial t}+u_r\frac{\partial\theta}{\partial r}+u_z\frac{\partial\theta}{\partial z}=\frac{k}{\rho c}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\frac{\partial^2\theta}{\partial z^2}\right)+2\vartheta|\boldsymbol{\tau}|^2+\frac{1}{\rho c}F \tag{3.2.14}$$

3.3. Weak Formulation

Finite element method has chosen to search the numerical solution of transport phenomena system, then in the current stage it is necessary to derive the variational version of system, thus to follow the objective we will apply the convenient smooth test function $\eta(r,\varphi,z,t)$ with small support, but the details of test functions construction would be prepared in the next chapter. Suppose that the density ρ is constant because the flow had been assumed steady with low speed, hence we multiply the equation (3.2.11) by η and after integration we will derive

$$\int_{\Omega}\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\eta+\int_{\Omega}\frac{\partial u_z}{\partial z}\eta=0, \tag{3.3.1}$$

by enforcing integration by parts in the relation (3.3.1) we have

$$\int_{\Omega}u_r\left(\frac{\eta}{r}-\frac{\partial\eta}{\partial r}\right)-\int_{\Omega}u_z\frac{\partial\eta}{\partial z}=0 \tag{3.3.2}$$

Theorem 3.1. Suppose that η is a smooth test function belongs to $H_0^1(\Omega)$, then the weak formulation of continuity equation (3.2.11) is constructed as

$$\int_{\Omega} u_r \left(\frac{\eta}{r} - \frac{\partial \eta}{\partial r}\right) - \int_{\Omega} u_z \frac{\partial \eta}{\partial z} = 0.$$

In the second step we focus on the Navier-Stokes equation (3.2.12) and again it is multiplied by η and also is integrated, then we earn

$$\int_{\Omega} \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z}\right) \eta =$$

$$-\frac{1}{\rho} \int_{\Omega} \frac{\partial p}{\partial r} \eta + \vartheta \int_{\Omega} \left(\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u_r}{\partial r}\right) + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2}\right) \eta + \int_{\Omega} g_r \eta, \tag{3.3.3}$$

we handle on the integration by parts in the integral equality (3.3.3) and we gain

$$-\int_{\Omega} u_r \frac{\partial \eta}{\partial t} - \frac{1}{2} \int_{\Omega} {u_r}^2 \frac{\partial \eta}{\partial r} - \int_{\Omega} u_r \left(u_z \frac{\partial \eta}{\partial z} + \eta \frac{\partial u_z}{\partial z}\right) = \frac{1}{\rho} \int_{\Omega} p \frac{\partial \eta}{\partial r} +$$

$$\vartheta \int_{\Omega} u_r \left(\frac{\partial^2 \eta}{\partial r^2} - \frac{1}{r} \frac{\partial \eta}{\partial r} + \frac{\eta}{r^2}\right) + \vartheta \int_{\Omega} u_r \frac{\partial^2 \eta}{\partial z^2} + \int_{\Omega} \left(-\vartheta \frac{u_r}{r^2} + g_r\right) \eta \tag{3.3.4}$$

We simplify the relation (3.3.4) to derive the weak mathematical modeling of Navier-Stokes equation correspond to r as

$$-\int_{\Omega} u_r \frac{\partial \eta}{\partial t} - \frac{1}{2} \int_{\Omega} {u_r}^2 \frac{\partial \eta}{\partial r} - \int_{\Omega} u_r \left(u_z \frac{\partial \eta}{\partial z} + \eta \frac{\partial u_z}{\partial z}\right) =$$

$$\frac{1}{\rho} \int_{\Omega} p \frac{\partial \eta}{\partial r} + \vartheta \int_{\Omega} u_r \left(\frac{\partial^2 \eta}{\partial r^2} - \frac{1}{r} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial z^2}\right) + \int_{\Omega} g_r \eta \tag{3.3.5}$$

Let's repeat the same procedure for the z-component by test function multiplying and integrating

$$-\int_{\Omega} u_z \frac{\partial \eta}{\partial t} - \int_{\Omega} u_z \left(u_r \frac{\partial \eta}{\partial r} + \eta \frac{\partial u_r}{\partial r}\right) - \frac{1}{2} \int_{\Omega} {u_z}^2 \frac{\partial \eta}{\partial z} =$$

$$\frac{1}{\rho} \int_{\Omega} p \frac{\partial \eta}{\partial z} + \vartheta \int_{\Omega} u_z \frac{\partial^2 \eta}{\partial r \partial z} + \vartheta \int_{\Omega} u_z \frac{\partial^2 \eta}{\partial z^2} + \int_{\Omega} g_z \eta \tag{3.3.6}$$

To get more simplicity in the relations (3.3.5) and (3.3.6), suppose that $\vartheta = 1$, then these integral equalities are shown as

$$\int_\Omega u_r\left(\frac{\partial \eta}{\partial t}+\left(\frac{1}{2}u_r+\frac{1}{r}\right)\frac{\partial \eta}{\partial r}+u_z\frac{\partial \eta}{\partial z}+\eta\frac{\partial u_z}{\partial z}-\frac{\partial^2 \eta}{\partial r^2}-\frac{\partial^2 \eta}{\partial z^2}\right)= -\frac{1}{\rho}\int_\Omega p\frac{\partial \eta}{\partial r}-\int_\Omega g_r\eta, \tag{3.3.7}$$

and

$$\int_\Omega u_z\left(\frac{\partial \eta}{\partial t}+u_r\frac{\partial \eta}{\partial r}+\eta\frac{\partial u_r}{\partial r}+\frac{1}{2}u_z\frac{\partial \eta}{\partial z}+\frac{\partial^2 \eta}{\partial r\partial z}+\frac{\partial^2 \eta}{\partial z^2}\right)= -\frac{1}{\rho}\int_\Omega p\frac{\partial \eta}{\partial z}-\int_\Omega g_z\eta \tag{3.3.8}$$

Theorem 3.2. Suppose that η is a smooth test function belongs to $H_0^1(\Omega)$, then the weak formulation of momentum conservation equations (3.2.12) and (3.2.13) respectively are constructed as

$$-\int_\Omega u_r\frac{\partial \eta}{\partial t}-\frac{1}{2}\int_\Omega {u_r}^2\frac{\partial \eta}{\partial r}-\int_\Omega u_r\left(u_z\frac{\partial \eta}{\partial z}+\eta\frac{\partial u_z}{\partial z}\right)= \frac{1}{\rho}\int_\Omega p\frac{\partial \eta}{\partial r}+\vartheta\int_\Omega u_r\left(\frac{\partial^2 \eta}{\partial r^2}-\frac{1}{r}\frac{\partial \eta}{\partial r}+\frac{\partial^2 \eta}{\partial z^2}\right)+\int_\Omega g_r\eta,$$

$$\int_\Omega u_z\left(\frac{\partial \eta}{\partial t}+u_r\frac{\partial \eta}{\partial r}+\eta\frac{\partial u_r}{\partial r}+\frac{1}{2}u_z\frac{\partial \eta}{\partial z}+\frac{\partial^2 \eta}{\partial r\partial z}+\frac{\partial^2 \eta}{\partial z^2}\right)=-\frac{1}{\rho}\int_\Omega p\frac{\partial \eta}{\partial z}-\int_\Omega g_z\eta.$$

Finally we attain the energy conservation equation, as the continuity and Navier-Stokes equation we multiply the equality (3.2.14) by η and integrate to obtain

$$\int_G \frac{\partial \theta}{\partial t}\eta+\int_G u_r\frac{\partial \theta}{\partial r}\eta+\int_G u_z\frac{\partial \theta}{\partial z}\eta = \frac{k}{\rho c}\int_G\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right)+\frac{\partial^2 \theta}{\partial z^2}\right)\eta+\int_G\left(2\vartheta|\boldsymbol{\tau}|^2+\frac{1}{\rho c}F\right)\eta, \tag{3.3.9}$$

now we use the Green's first identity in the integral relation (3.3.9) and we derive

$$-\int_G \theta \frac{\partial \eta}{\partial t} - \int_G \theta \left(u_r \frac{\partial \eta}{\partial r} + u_z \frac{\partial \eta}{\partial z}\right) =$$

$$-\frac{k}{\rho c}\int_G \nabla\theta.\nabla\eta + \frac{k}{\rho c}\int_\Gamma [\nabla\theta]_-^+.\boldsymbol{n}_x\eta + \int_G \left(2\vartheta|\boldsymbol{\tau}|^2 + \frac{1}{\rho c}F\right)\eta \tag{3.3.10}$$

Define

$$H(\theta) = \begin{cases} L & ;\ \theta > T_1 \\ [0, L] & ;\ \theta = T_1 \\ 0 & ;\ \theta < T_1 \end{cases}$$

Stefan condition in three-dimension in the integral form is

$$\int_\Gamma ([\nabla\theta]_-^+.\boldsymbol{n}_x)\eta = -L\int_\Gamma (\boldsymbol{w}.\boldsymbol{n}_x)\eta = L\int_\Gamma n_t\eta \tag{3.3.11}$$

Also by invoking the function $H(\theta)$ we can write

$$\int_{\mathrm{G}} H(\theta)\frac{\partial \eta}{\partial t} = \int_{\Omega_1} H(\theta)\frac{\partial \eta}{\partial t} + \int_{\Omega_2} H(\theta)\frac{\partial \eta}{\partial t} + \int_\Gamma H(\theta)\frac{\partial \eta}{\partial t} = L\int_{\Omega_1}\frac{\partial \eta}{\partial t}$$

Divergence theorem in one-dimension states that

$$\int_{\Omega_1}\frac{\partial \eta}{\partial t} = \int_\Gamma n_t\eta$$

Then we will obtain the equality

$$\int_\Gamma ([\nabla\theta]_-^+.\boldsymbol{n}_x)\eta = \int_{\mathrm{G}} H(\theta)\frac{\partial \eta}{\partial t}$$

In this step we apply the relation (3.3.11) into the integral equation (3.3.10) to achieve

$$-\int_G \theta \frac{\partial \eta}{\partial t} - \int_G \theta \left(u_r \frac{\partial \eta}{\partial r} + u_z \frac{\partial \eta}{\partial z}\right) =$$

$$-\frac{k}{\rho c}\int_G \left(\frac{\partial \theta}{\partial r}\frac{\partial \eta}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial \eta}{\partial z}\right) + \frac{k}{\rho c}\int_G H(\theta)\frac{\partial \eta}{\partial t} + \int_G \left(2\vartheta|\boldsymbol{\tau}|^2 + \frac{1}{\rho c}F\right)\eta \tag{3.3.12}$$

If we use integration by parts in the equality (3.3.12) we get

$$\int_G \left(\theta + \frac{k}{\rho c} H(\theta)\right) \frac{\partial \eta}{\partial t} + \int_G \theta \left(u_r \frac{\partial \eta}{\partial r} + u_z \frac{\partial \eta}{\partial z} + \frac{k}{\rho c}\left(\frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 \eta}{\partial z^2}\right)\right) = -\int_G \left(\frac{1}{\rho c} F + 2\vartheta |\boldsymbol{\tau}|^2\right) \eta \tag{3.3.13}$$

Theorem 3.3. Suppose that η is a smooth test function belongs to $H_0^1(\Omega)$, then the weak formulation of heat conduction equation (3.2.14) is constructed as

$$\int_G \left(\theta + \frac{k}{\rho c} H(\theta)\right) \frac{\partial \eta}{\partial t} + \int_G \theta \left(u_r \frac{\partial \eta}{\partial r} + u_z \frac{\partial \eta}{\partial z} + \frac{k}{\rho c}\left(\frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 \eta}{\partial z^2}\right)\right) = -\int_G \left(\frac{1}{\rho c} F + 2\vartheta |\boldsymbol{\tau}|^2\right) \eta,$$

where

$$H(\theta) = \begin{cases} L & ;\ \theta > T_1 \\ [0, L] & ;\ \theta = T_1, \\ 0 & ;\ \theta < T_1 \end{cases}$$

and L is latent heat.

3.4. Discretization of Domain

Assume that $V_h \subset H_0^1(\Omega)$, which consists of test functions that they are piecewise continuous with the fix degree , and also suppose that

$$\dim V_h = N(h),$$

and

$$V_h = span\{\phi_1, \phi_2, \phi_3, \dots, \phi_{N(h)}\},$$

where the basis functions $\phi_i(r, \varphi, z, t), i = 1,2, \dots, N(h)$, have small support. If we choose the finite dimensional subspace V_h, and also the correspond variables in V_h, then we redefine the heat transfer problem as the problem of finding u_{r_h}, u_{z_h}, p_h, and θ_h such that there satisfy in the continuity equation (3.2.11), Navier-Stokes equations (3.2.12), (3,2,13), and heat conduction equation (3.2.14).

Now we express the approximate solutions u_{r_h}, u_{z_h}, p_h, and θ_h in terms of the basis functions $\phi_i(r, \varphi, z, t)$, we could write

$$u_{r_h}(r, \varphi, z, t) = \sum_{i=1}^{N(h)} U_{r_i} \phi_i(r, \varphi, z, t), \tag{3.4.1}$$

$$u_{z_h}(r,\varphi,z,t) = \sum_{i=1}^{N(h)} U_{z_i}\phi_i(r,\varphi,z,t), \tag{3.4.2}$$

$$p_h(r,\varphi,z,t) = \sum_{i=1}^{N(h)} P_i\phi_i(r,\varphi,z,t), \tag{3.4.3}$$

$$\theta_h(r,\varphi,z,t) = \sum_{i=1}^{N(h)} \theta_i\phi_i(r,\varphi,z,t), \tag{3.4.4}$$

where U_{r_i}, U_{z_i}, P_i , $i = 1,2,3,\dots,N(h)$, are to be determined. We insert the values of u_{r_h} and $u_{z_h} \in V_h$ instead of u_r and u_z in the continuity equation (3.3.2) and we obtain

$$\int_\Omega u_{r_h}\left(\frac{\eta_h}{r} - \frac{\partial \eta_h}{\partial r}\right) - \int_\Omega u_{z_h}\frac{\partial \eta_h}{\partial z} = 0, \tag{3.4.5}$$

for every smooth test function $\eta_h \in V_h$. We replace the approximate values of u_{r_h} and u_{z_h} from the relations (3.4.1) and (3.4.2) into the integral equation (3.4.5) we earn

$$\int_\Omega \sum_{i=1}^{N(h)} U_{r_i}\phi_i\left(\frac{\phi_k}{r} - \frac{\partial \phi_k}{\partial r}\right) - \int_\Omega \sum_{i=1}^{N(h)} U_{z_i}\phi_i\frac{\partial \phi_k}{\partial z} = 0, \tag{3.4.6}$$

then we rewrite the relation (3.4.6) as

$$\sum_{i=1}^{N(h)} U_{r_i}\int_\Omega \phi_i\left(\frac{\phi_k}{r} - \frac{\partial \phi_k}{\partial r}\right) - \sum_{i=1}^{N(h)} U_{z_i}\int_\Omega \phi_i\frac{\partial \phi_k}{\partial z} = 0 \tag{3.4.7}$$

At last we simplify the equation (3.4.7) and we gain the linear system

$$\sum_{i=1}^{N(h)} \left(U_{r_i}a_{ik} - U_{z_i}b_{ik}\right) = 0, \tag{3.4.8}$$

where the coefficients are

$$a_{ik} = \int_\Omega \phi_i\left(\frac{\phi_k}{r} - \frac{\partial \phi_k}{\partial r}\right), \tag{3.4.9}$$

$$b_{ik} = \int_{\Omega} \phi_i \frac{\partial \phi_k}{\partial z}. \tag{3.4.10}$$

Theorem 3.4. Let's discrete the domain Ω by the finite-dimensional subspace $V_h \subset H_0^1(\Omega)$ which consists of smooth test functions $\phi_i(r, \varphi, z, t), i = 1,2, \dots, N(h)$ with small support, where dim $V_h = N(h)$, and $V_h = span\{\phi_1, \phi_2, \phi_3, \dots, \phi_{N(h)}\}$, then the equality (3.3.2) can be showed as the linear system of equations

$$\sum_{i=1}^{N(h)} \left(U_{r_i} a_{ik} - U_{z_i} b_{ik}\right) = 0, \qquad k = 1,2, \dots, N(h),$$

where

$$a_{ik} = \int_{\Omega} \phi_i \left(\frac{\phi_k}{r} - \frac{\partial \phi_k}{\partial r}\right), b_{ik} = \int_{\Omega} \phi_i \frac{\partial \phi_k}{\partial z}.$$

After the continuity equation we refer to the variational formulation of Navier-Stokes equations and we replace u_{r_h}, u_{z_h}, and p_h instead of u_r, u_z, and p in the equation (3.3.5) correspond to r, then we will achieve

$$\int_{\Omega} u_{r_h} \left(\frac{\partial \eta_h}{\partial t} + \left(\frac{1}{2} u_{r_h} + \frac{1}{r}\right) \frac{\partial \eta_h}{\partial r} + u_{z_h} \frac{\partial \eta_h}{\partial z} + \eta_h \frac{\partial u_{z_h}}{\partial z} - \frac{\partial^2 \eta_h}{\partial r^2} - \frac{\partial^2 \eta_h}{\partial z^2}\right) = -\frac{1}{\rho} \int_{\Omega} p_h \frac{\partial \eta_h}{\partial r} - \int_{\Omega} g_r \eta_h \tag{3.4.11}$$

In this step of weak modeling we set the approximate values of u_{r_h}, u_{z_h}, and p_h from relations (3.4.1), (3.4.2), and (3.4.3) into the integral equality (3.4.11) to attain

$$\int_{\Omega} \sum_{i=1}^{N(h)} U_{r_i} \phi_i \left(\frac{\partial \phi_k}{\partial t} + \left(\frac{1}{2} \sum_{j=1}^{N(h)} U_{r_j} \phi_j + \frac{1}{r}\right) \frac{\partial \phi_k}{\partial r} + \sum_{j=1}^{N(h)} U_{z_j} \left(\phi_j \frac{\partial \phi_k}{\partial z} + \phi_k \frac{\partial \phi_j}{\partial z}\right) - \frac{\partial^2 \phi_k}{\partial r^2} - \frac{\partial^2 \phi_k}{\partial z^2}\right) = -\frac{1}{\rho} \int_{\Omega} \sum_{i=1}^{N(h)} P_i \phi_i \frac{\partial \phi_k}{\partial r} - \int_{\Omega} g_r \phi_k \tag{3.4.12}$$

We rewrite the equation (3.4.12) as

$$\sum_{i=1}^{N(h)} U_{r_i} \int_{\Omega} \phi_i \left(\frac{\partial \phi_k}{\partial t} - \frac{\partial^2 \phi_k}{\partial r^2} - \frac{\partial^2 \phi_k}{\partial z^2} + \frac{1}{r} \frac{\partial \phi_k}{\partial r} \right) + \frac{1}{2} \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{r_j} \int_{\Omega} \phi_i \phi_j \frac{\partial \phi_k}{\partial r} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{z_j} \int_{\Omega} \phi_i \left(\phi_j \frac{\partial \phi_k}{\partial z} + \phi_k \frac{\partial \phi_j}{\partial z} \right) + \sum_{i=1}^{N(h)} P_i \int_{\Omega} \frac{1}{\rho} \phi_i \frac{\partial \phi_k}{\partial r} + \int_{\Omega} g_r \phi_k = 0, \tag{3.4.13}$$

and at the end we derive the nonlinear system as

$$\sum_{i=1}^{N(h)} U_{r_i} A_{ik} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{r_j} B_{ijk} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{z_j} C_{ijk} + \sum_{i=1}^{N(h)} P_i D_{ik} + E_k = 0, \tag{3.4.14}$$

where the coefficients are

$$A_{ik} = \int_{\Omega} \phi_i \left(\frac{\partial \phi_k}{\partial t} - \frac{\partial^2 \phi_k}{\partial r^2} - \frac{\partial^2 \phi_k}{\partial z^2} + \frac{1}{r} \frac{\partial \phi_k}{\partial r} \right), \tag{3.4.15}$$

$$B_{ijk} = \frac{1}{2} \int_{\Omega} \phi_i \phi_j \frac{\partial \phi_k}{\partial r}, \tag{3.4.16}$$

$$C_{ijk} = \int_{\Omega} \phi_i \left(\phi_j \frac{\partial \phi_k}{\partial z} + \phi_k \frac{\partial \phi_j}{\partial z} \right), \tag{3.4.17}$$

$$D_{ik} = \frac{1}{\rho} \int_{\Omega} \phi_i \frac{\partial \phi_k}{\partial r}, \tag{3.4.18}$$

$$E_k = \int_{\Omega} g_r \phi_k. \tag{3.4.19}$$

We continue the work by concentration on the z-component equation (3.3.8) of the Navier-Stokes system and we replace the approximate values instead of exact one to obtain

$$\int_{\Omega} u_{z_h}\left(\frac{\partial\eta_h}{\partial t}+u_{r_h}\frac{\partial\eta_h}{\partial r}+\eta_h\frac{\partial u_{r_h}}{\partial r}+\frac{1}{2}u_{z_h}\frac{\partial\eta_h}{\partial z}+\frac{\partial^2\eta_h}{\partial r\partial z}+\frac{\partial^2\eta_h}{\partial z^2}\right)= -\frac{1}{\rho}\int_{\Omega} p_h\frac{\partial\eta_h}{\partial z}-\int_{\Omega} g_z\eta_h, \tag{3.4.20}$$

then by applying the u_{r_h}, u_{z_h}, and p_h we will reach to

$$\begin{aligned}\int_{\Omega}\sum_{i=1}^{N(h)} U_{z_i}\phi_i\Bigg(\frac{\partial\phi_k}{\partial t}+\sum_{j=1}^{N(h)} U_{r_j}\phi_j\frac{\partial\phi_k}{\partial r}+\phi_k\frac{\partial}{\partial r}\left(\sum_{j=1}^{N(h)} U_{r_j}\phi_j\right)\\ +\frac{1}{2}\sum_{j=1}^{N(h)} U_{z_j}\phi_j\frac{\partial\phi_k}{\partial z}+\frac{\partial^2\phi_k}{\partial r\partial z}+\frac{\partial^2\phi_k}{\partial z^2}\Bigg)\\ =-\frac{1}{\rho}\int_{\Omega}\sum_{i=1}^{N(h)} P_i\phi_i\frac{\partial\phi_k}{\partial z}-\int_{\Omega} g_z\phi_k\end{aligned} \tag{3.4.21}$$

The relation (3.4.21) could be simplified that

$$\begin{aligned}\sum_{i=1}^{N(h)} U_{z_i}\int_{\Omega}\phi_i\left(\frac{\partial\phi_k}{\partial t}+\frac{\partial^2\phi_k}{\partial r\partial z}+\frac{\partial^2\phi_k}{\partial z^2}\right)\\ +\frac{1}{2}\sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)} U_{z_i}U_{z_j}\int_{\Omega}\phi_i\phi_j\frac{\partial\phi_k}{\partial z}\\ +\sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)} U_{z_i}U_{r_j}\int_{\Omega}\phi_i\left(\phi_j\frac{\partial\phi_k}{\partial r}+\phi_k\frac{\partial\phi_j}{\partial r}\right)\\ +\sum_{i=1}^{N(h)} P_i\int_{\Omega}\frac{1}{\rho}\phi_i\frac{\partial\phi_k}{\partial z}+\int_{\Omega} g_z\phi_k=0\end{aligned} \tag{3.4.22}$$

At last we state the equality (3.4.22) as the nonlinear following system

$$\begin{aligned}\sum_{i=1}^{N(h)} U_{z_i}A'_{ik}+\frac{1}{2}\sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)} U_{z_i}U_{z_j}B'_{ijk}+\sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)} U_{z_i}U_{r_j}C'_{ijk}\\ +\sum_{i=1}^{N(h)} P_iD'_{ik}+E'_k=0,\end{aligned} \tag{3.4.23}$$

where the coefficients are as

$$A'_{ik} = \int_{\Omega} \phi_i \left(\frac{\partial \phi_k}{\partial t} + \frac{\partial^2 \phi_k}{\partial r \partial z} + \frac{\partial^2 \phi_k}{\partial z^2} \right), \tag{3.4.24}$$

$$B'_{ijk} = \int_{\Omega} \phi_i \phi_j \frac{\partial \phi_k}{\partial z}, \tag{3.4.25}$$

$$C'_{ijk} = \int_{\Omega} \phi_i \left(\phi_j \frac{\partial \phi_k}{\partial r} + \phi_k \frac{\partial \phi_j}{\partial r} \right), \tag{3.4.26}$$

$$D'_{ik} = \int_{\Omega} \frac{1}{\rho} \phi_i \frac{\partial \phi_k}{\partial z}, \tag{3.4.27}$$

$$E'_{k} = \int_{\Omega} g_z \phi_k. \tag{3.4.28}$$

Theorem 3.5. Let's discrete the domain Ω by the finite-dimensional subspace $V_h \subset H_0^1(\Omega)$ which consists of smooth test functions $\phi_i(r, \varphi, z, t), i = 1,2, \dots, N(h)$ with small support, where $\dim V_h = N(h)$, and $V_h = span\{\phi_1, \phi_2, \phi_3, \dots, \phi_{N(h)}\}$, then the equalities (3.3.7) and (3.3.8) can be computed as the nonlinear system of equations

$$\sum_{i=1}^{N(h)} U_{r_i} A_{ik} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{r_j} B_{ijk} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{z_j} C_{ijk} + \sum_{i=1}^{N(h)} P_i D_{ik} + E_k = 0,$$

$$\sum_{i=1}^{N(h)} U_{z_i} A'_{ik} + \frac{1}{2} \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{z_i} U_{z_j} B'_{ijk} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{z_i} U_{r_j} C'_{ijk} + \sum_{i=1}^{N(h)} P_i D'_{ik} + E'_k = 0,$$

$$k = 1,2, \dots, N(h),$$

where the coefficients of the system are stated in the relations (3.4.15),…,(34.19) and (3.4.24),…,(3.4.28).

We finish this section by focusing on the weak modeling of the energy conservation equation and we replace $u_{r_h}, u_{z_h}, \theta_h \in V_h$ instead of u_r, u_z, θ in the conservation equation (3.3.13) and we will gain

$$\int_{G} \left(\theta_h + \frac{k}{\rho c} H(\theta_h) \right) \frac{\partial \eta_h}{\partial t}$$

$$+\int_G \theta_h\left(u_{r_h}\frac{\partial\eta_h}{\partial r}+u_{z_h}\frac{\partial\eta_h}{\partial z}+\frac{k}{\rho c}\left(\frac{\partial^2\eta_h}{\partial r^2}+\frac{\partial^2\eta_h}{\partial z^2}\right)\right)$$
$$=-\int_G\left(\frac{1}{\rho c}F+2\vartheta|\boldsymbol{\tau}_h|^2\right)\eta_h \tag{3.4.29}$$

Now we insert the approximate values u_{r_h}, u_{z_h}, and θ_h into the integral equality (3.4.29) when we omit $\frac{\partial^2\eta_h}{\partial r^2}+\frac{\partial^2\eta_h}{\partial z^2}$, since it is zero and we attain to

$$\int_G\left(\sum_{i=1}^{N(h)}\theta_i\phi_i+\frac{k}{\rho c}H(\theta_h)\right)\frac{\partial\phi_k}{\partial t}$$
$$+\int_G\sum_{i=1}^{N(h)}\theta_i\phi_i\left(\frac{\partial\phi_k}{\partial r}\sum_{j=1}^{N(h)}U_{r_j}\phi_j+\frac{\partial\phi_k}{\partial z}\sum_{j=1}^{N(h)}U_{z_j}\phi_j\right) \tag{3.4.30}$$
$$=-\int_G\left(\frac{1}{\rho c}F+2\vartheta|\boldsymbol{\tau}_h|^2\right)\phi_k$$

Equality (3.4.30) in the liquid phase is written as

$$\int_{\Omega_1}\left(\sum_{i=1}^{N(h)}\theta_i\phi_i+\frac{k}{\rho c}L\right)\frac{\partial\phi_k}{\partial t}$$
$$+\int_{\Omega_1}\sum_{i=1}^{N(h)}\theta_i\phi_i\left(\frac{\partial\phi_k}{\partial r}\sum_{j=1}^{N(h)}U_{r_j}\phi_j+\frac{\partial\phi_k}{\partial z}\sum_{j=1}^{N(h)}U_{z_j}\phi_j\right) \tag{3.4.31}$$
$$=-\int_{\Omega_1}\left(\frac{1}{\rho c}F+2\vartheta|\boldsymbol{\tau}_h|^2\right)\phi_k,$$

and so

$$\sum_{i=1}^{N(h)}\theta_i\int_{\Omega_1}\phi_i\frac{\partial\phi_k}{\partial t}+$$
$$\sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)}\theta_i\int_{\Omega_1}\phi_i\phi_j\left(U_{r_j}\frac{\partial\phi_k}{\partial r}+U_{z_j}\frac{\partial\phi_k}{\partial z}\right) \tag{3.4.32}$$
$$=-2\int_{\Omega_1}\vartheta|\boldsymbol{\tau}_h|^2\phi_k-\frac{1}{\rho c}\int_{\Omega_1}\left(F\phi_k-kL\frac{\partial\phi_k}{\partial t}\right)$$

Finally we achieve the linear system with $N(h)$ equations within $N(h)$ unknowns θ_i as

$$\sum_{i=1}^{N(h)} \theta_i F_{ik} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} \theta_i G_{ijk} = H_k + I_k, \tag{3.4.32}$$

where the coefficients are as

$$F_{ik} = \int_{\Omega_1} \phi_i \frac{\partial \phi_k}{\partial t}, \tag{3.4.33}$$

$$G_{ijk} = \int_{\Omega_1} \phi_i \phi_j \left(U_{r\,j} \frac{\partial \phi_k}{\partial r} + U_{z\,j} \frac{\partial \phi_k}{\partial z} \right), \tag{3.4.34}$$

$$H_k = -2 \int_{\Omega_1} \vartheta |\boldsymbol{\tau}_h|^2 \phi_k, \tag{3.4.35}$$

$$I_k = -\int_{\Omega_1} \frac{1}{\rho c} \left(F\phi_k - kL \frac{\partial \phi_k}{\partial t} \right). \tag{3.4.35}$$

It is necessary to mention that for the stress tensor $\boldsymbol{\tau}$ we have

$$|\boldsymbol{\tau}|^2 = \left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial r}\right)^2 + \left(\frac{\partial u_r}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2, \tag{3.4.36}$$

then after the discretization of domain we renew the stress as

$$|\boldsymbol{\tau}_h|^2 = \left(\frac{\partial u_{r_h}}{\partial r}\right)^2 + \left(\frac{\partial u_{z_h}}{\partial r}\right)^2 + \left(\frac{\partial u_{r_h}}{\partial z}\right)^2 + \left(\frac{\partial u_{z_h}}{\partial z}\right)^2, \tag{3.4.37}$$

and by applying the relation (3.4.37) we compute H_k as

$$\begin{aligned} H_k &= -2 \int_{\Omega_1} \vartheta |\boldsymbol{\tau}_h|^2 \phi_k \\ &= -2 \int_{\Omega_1} \vartheta \left(\left(\frac{\partial u_{r_h}}{\partial r}\right)^2 + \left(\frac{\partial u_{z_h}}{\partial r}\right)^2 + \left(\frac{\partial u_{r_h}}{\partial z}\right)^2 + \left(\frac{\partial u_{z_h}}{\partial z}\right)^2 \right) \phi_k, \end{aligned} \tag{3.4.38}$$

and finally

$$H_k = -2 \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} \int_{\Omega_1} \vartheta \left(U_{r_i} U_{r_j} + U_{z_i} U_{z_j} \right) \left(\frac{\partial \phi_i}{\partial r} \frac{\partial \phi_j}{\partial r} + \frac{\partial \phi_i}{\partial z} \frac{\partial \phi_j}{\partial z} \right) \phi_k \tag{3.4.39}$$

At last we refer to the energy conservation system in solid phase and we earn

$$\sum_{i=1}^{N(h)} \theta_i \int_{\Omega_2} \phi_i \frac{\partial \phi_k}{\partial t} = -\frac{1}{\rho c} F \int_{\Omega_2} \phi_k, \tag{3.4.40}$$

and so we get the system of

$$\sum_{i=1}^{N(h)} \theta_i F'_{ik} = I'_k, \tag{3.4.41}$$

where

$$F'_{ik} = \int_{\Omega_2} \phi_i \frac{\partial \phi_k}{\partial t}, \tag{3.4.42}$$

$$I'_k = -\frac{1}{\rho c} F \int_{\Omega_2} \phi_k. \tag{3.4.43}$$

Theorem 3.6. Let's discrete the domain Ω by the finite-dimensional subspace $V_h \subset H_0^1(\Omega)$ which consists of smooth test functions $\phi_i(r, \varphi, z, t), i = 1,2, \dots, N(h)$ with small support, where $\dim V_h = N(h)$, and $V_h = span\{\phi_1, \phi_2, \phi_3, \dots, \phi_{N(h)}\}$, then the equality (3.3.13) can be computed as the linear system of equations

$$\sum_{i=1}^{N(h)} \theta_i F_{ik} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} \theta_i G_{ijk} = H_k + I_k,$$

where the coefficients of the system are stated in the relations (3,4,33),…, (3.4.39). Also for the solid phase we will have the system of linear equations as

$$\sum_{i=1}^{N(h)} \theta_i F'_{ik} = I'_k,$$

where

$$F'_{ik} = \int_{\Omega_2} \phi_i \frac{\partial \phi_k}{\partial t}, I'_k = -\frac{1}{\rho c} F \int_{\Omega_2} \phi_k.$$

One essential part is the construction of smooth test functions that they are piecewise continuous polynomials with the fixed degree. It is necessary because we need them to determine the coefficients of the transport system before we attain to their numerical solution.

3.5. Mathematical Modeling of Heat Transfer

In section (3.4) we prepared the whole details of the discretization of the domain and now we recall the heat transfer problem as the problem of finding $\left(U_{r_1}, U_{r_2}, \dots, U_{r_{N(h)}}\right) \in R^{N(h)}$, $\left(U_{z_1}, U_{z_2}, \dots, U_{z_{N(h)}}\right) \in R^{N(h)}$, and $\left(\theta_1, \theta_2, \dots, \theta_{N(h)}\right) \in R^{N(h)}$, that they are satisfy in the system of equation (3.5.1),…, (3.5.4).

$$\sum_{i=1}^{N(h)} \left(U_{r_i} a_{ik} - U_{z_i} b_{ik}\right) = 0, \tag{3.5.1}$$

$$\begin{gathered}\sum_{i=1}^{N(h)} U_{r_i} A_{ik} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{r_j} B_{ijk} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{z_j} C_{ijk} \\ + \sum_{i=1}^{N(h)} P_i D_{ik} + E_k = 0,\end{gathered} \tag{3.5.2}$$

$$\begin{gathered}\sum_{i=1}^{N(h)} U_{z_i} A'_{ik} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{z_i} U_{z_j} B'_{ijk} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{z_i} U_{r_j} C'_{ijk} \\ + \sum_{i=1}^{N(h)} P_i D'_{ik} + E'_k = 0,\end{gathered} \tag{3.5.3}$$

$$\sum_{i=1}^{N(h)} \theta_i F_{ik} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} \theta_i \left(U_{r_j} + U_{z_j}\right) G_{jk} + H_k = 0, \tag{3.5.4}$$

$$k = 1,2,3, \dots, N(h),$$

where the coefficients of the system are listed in the table (3.5.1).

Table 3.5.1.

$a_{ik} = \int_{\Omega} \phi_i \left(\frac{\phi_k}{r} - \frac{\partial \phi_k}{\partial r}\right)$	$b_{ik} = \int_{\Omega} \phi_i \frac{\partial \phi_k}{\partial z}$
$A_{ik} = \int_{\Omega} \phi_i \left(\frac{\partial \phi_k}{\partial t} - \frac{\partial^2 \phi_k}{\partial r^2} - \frac{\partial^2 \phi_k}{\partial z^2} + \frac{1}{r}\frac{\partial \phi_k}{\partial r}\right)$	$A'_{ik} = \int_{\Omega} \phi_i \left(\frac{\partial \phi_k}{\partial t} + \frac{\partial^2 \phi_k}{\partial r \partial z} + \frac{\partial^2 \phi_k}{\partial z^2}\right)$
$B_{ijk} = \frac{1}{2}\int_{\Omega} \phi_i \phi_j \frac{\partial \phi_k}{\partial r}$	$B'_{ijk} = \frac{1}{2}\int_{\Omega} \phi_i \phi_j \frac{\partial \phi_k}{\partial z}$
$C_{ijk} = \int_{\Omega} \phi_i \left(\phi_j \frac{\partial \phi_k}{\partial z} + \phi_k \frac{\partial \phi_j}{\partial z}\right)$	$C'_{ijk} = \int_{\Omega} \phi_i \left(\phi_j \frac{\partial \phi_k}{\partial r} + \phi_k \frac{\partial \phi_j}{\partial r}\right)$
$D_{ik} = \frac{1}{\rho}\int_{\Omega} \phi_i \frac{\partial \phi_k}{\partial r}$	$D'_{ik} = \frac{1}{\rho}\int_{\Omega} \phi_i \frac{\partial \phi_k}{\partial z}$
$E_k = \int_{\Omega} g_r \phi_k$	$E'_k = \int_{\Omega} g_z \phi_k$
$F_{ik} = \int_{\Omega} \phi_i \left(\frac{\partial \phi_k}{\partial t} + \frac{k}{\rho c}\left(\frac{\partial^2 \phi_k}{\partial r^2} + \frac{\partial^2 \phi_k}{\partial z^2}\right)\right)$	$G_{jk} = \int_{\Omega} \left(\phi_k \frac{\partial \phi_j}{\partial r} + \phi_j \frac{\partial \phi_k}{\partial r}\right)$
$H_k = \frac{1}{\rho c}\int_{\Omega} \left(k\lambda \frac{\partial \phi_k}{\partial t} + S\phi_k\right)$	

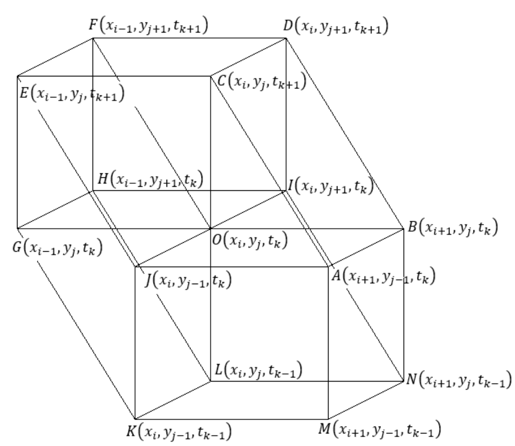
The Coefficients of System

It is trivial we need to determine the values of the coefficients to derive the numerical solution of system, then we must define the sufficient domain and discrete it to compute the relevant integrals, and at last we would found all coefficients.

3.6. Computing the unknown coefficients

We continue the process by computing the coefficients in the transport system, for this objective we need to divide the domain Ω to the mesh cubes, the figure (3.6.1) shows the ijk −mesh cube.

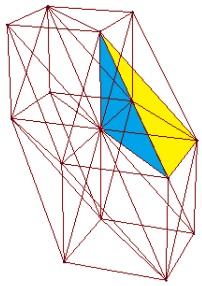
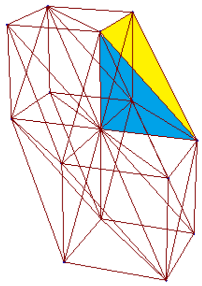
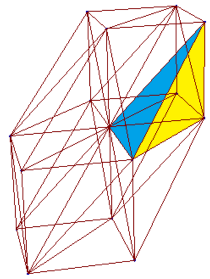
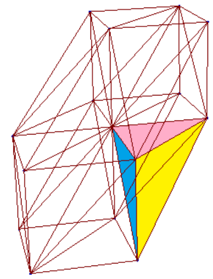
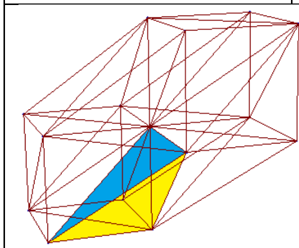
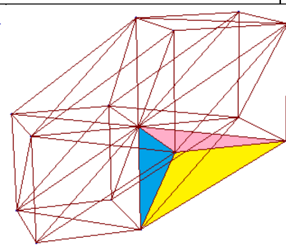
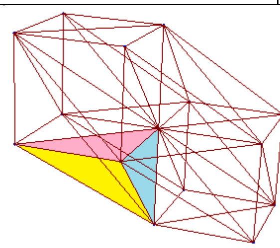
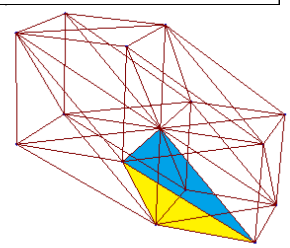
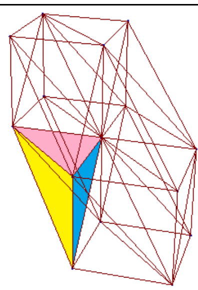
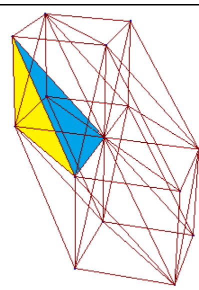
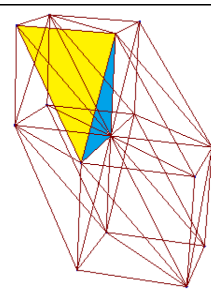
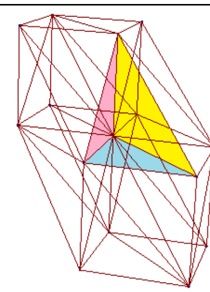
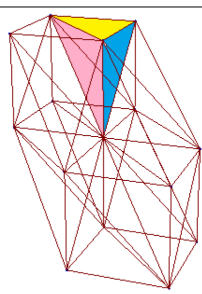
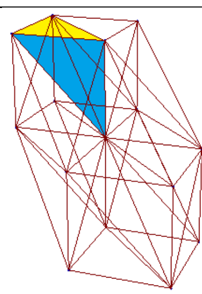
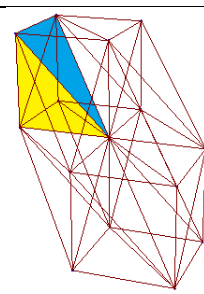
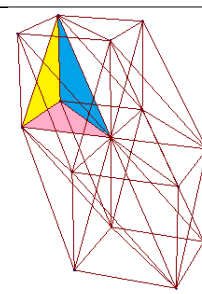
Figure 3.6.1.

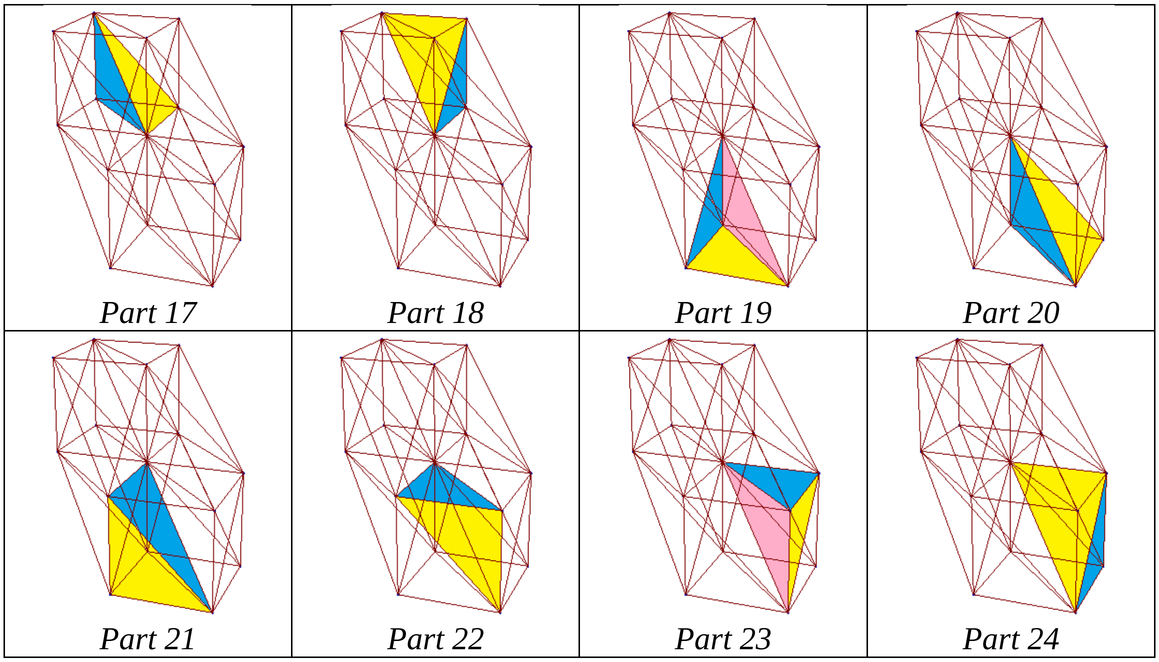


ijk −mesh cube

and we will construct the test functions on ijk −mesh cube. To define the test functions we divide the ijk −mesh cube into the 24 tetrahedrons that they are presented in the table (3.6.1).

Table 3.6.1.

 *Part 01*	 *Part 02*	 *Part 03*	 *Part 04*
 *Part 05*	 *Part 06*	 *Part 07*	 *Part 08*
 *Part 09*	 *Part 10*	 *Part 11*	 *Part 12*
 *Part 13*	 *Part 14*	 *Part 15*	 *Part 16*



24 Tetrahedrons in ijk −mesh cube

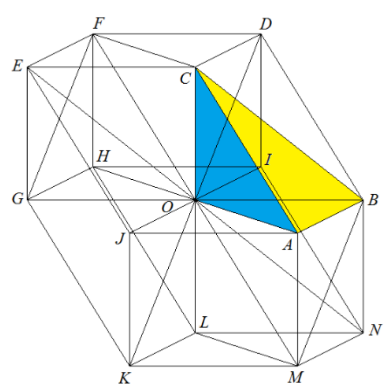
We are looking for the piecewise continuous linear function $\phi = \phi(x, y, t)$ where it equals to 1 in the origin node of ijk −mesh cube, and 0 in the other nodes. Suppose that

$$\phi(x, y, t) = a + bx + cy + dt,$$

for $(x, y, z) \in OABC$ in the figure (3.6.2) and

$$\phi\left(O(x_i, y_j, t_k)\right) = 1, \qquad \phi\left(A(x_{i+1}, y_{j-1}, t_k)\right) = 0,$$
$$\phi\left(B(x_{i+1}, y_j, t_k)\right) = 0, \qquad \phi\left(C(x_i, y_j, t_{k+1})\right) = 0.$$

Figure 3.6.2.



Tetrahedron 01 in ijk −mesh cube

then we will have

$$\begin{cases} a + x_i b + y_j c + t_k d = 1 \\ a + x_{i+1} b + y_{j-1} c + t_k d = 0 \\ a + x_{i+1} b + y_j c + t_k d = 0 \\ a + x_i b + y_j c + t_{k+1} d = 0 \end{cases} \tag{3.6.1}$$

We get the solution of the linear system (3.6.1) as

$$a = \frac{1}{h}(x_i + t_k) + 1, b = d = -\frac{1}{h}, c = 0,$$

then we derive the test function

$$\phi(x, y, t) = \frac{1}{h}(-x + x_i - t + t_k) + 1 \tag{3.6.2}$$

We perform the same approach for all of the other tetrahedrons and we demonstrate them in the table (3.6.2).

Table 3.6.2.

Parts 1, 2 $\phi(x,y,t) = \frac{1}{h}(-x + x_i - t + t_k) + 1$	*Parts 3, 4* $\phi(x,y,t) = \frac{1}{h}(-x + x_i - y + y_j) + 1$
Parts 5, 6 $\phi(x,y,t) = \frac{1}{h}(-y + y_j + t - t_k) + 1$	*Parts 7, 8* $\phi(x,y,t) = \frac{1}{h}(x - x_i + t - t_k) + 1$
Parts 9, 10 $\phi(x,y,t) = \frac{1}{h}(x - x_i + y - y_j) + 1$	*Parts 11, 12* $\phi(x,y,t) = \frac{1}{h}(y - y_j - t + t_k) + 1$
Parts 13, 14 $\phi(x,y,t) = \frac{1}{h}(-t + t_k) + 1$	*Parts 15, 16* $\phi(x,y,t) = \frac{1}{h}(x - x_i) + 1$
Parts 17, 18 $\phi(x,y,t) = \frac{1}{h}(-y + y_j) + 1$	*Parts 19, 20* $\phi(x,y,t) = \frac{1}{h}(t - t_k) + 1$
Parts 21, 22 $\phi(x,y,t) = \frac{1}{h}(y - y_j) + 1$	*Parts 23, 24* $\phi(x,y,t) = \frac{1}{h}(-x + x_i) + 1$

Test Functions defined on 24 tetrahedrons

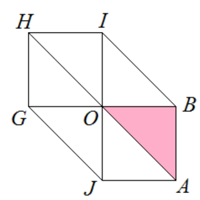
3.7. The Continuity Equation

We are in the position to compute the coefficients of the continuity equation a_{ik} and b_{ik} from (2.9), then we continue the process by determining the a_{ik} for $k = i, k = i - (N^2 + N - 1)$, and $k = i + (N^2 + N - 1)$, and it is trivial that $a_{ik} = 0$ in the other cases. Also it is necessary to note that we will use the variables x and y instead of r and z in the integral equations. Then we divide the integral as

$$a_{ii} = \int_{\Omega} \phi_i \left(\frac{\phi_i}{r} - \frac{\partial \phi_i}{\partial r}\right) = \iiint_{\Omega} \phi_i \left(\frac{\phi_i}{x} - \frac{\partial \phi_i}{\partial x}\right) dxdydt = \iiint_{part\ 01} \phi_i \left(\frac{\phi_i}{x} - \frac{\partial \phi_i}{\partial x}\right) dxdydt + \iiint_{part\ 02} \phi_i \left(\frac{\phi_i}{x} - \frac{\partial \phi_i}{\partial x}\right) dxdydt + \cdots + \iiint_{part\ 24} \phi_i \left(\frac{\phi_i}{x} - \frac{\partial \phi_i}{\partial x}\right) dxdydt \tag{3.7.1}$$

For the computation of the integral in the part 01 we get the image of the tetrahedron $OABC$ over the plane XOY

Figure 3.7.1.



$$\iiint_{part\ 01} \phi_i \left(\frac{\phi_i}{x} - \frac{\partial \phi_i}{\partial x}\right) dxdydt =$$
$$\int_{x_i}^{x_{i+1}} \int_{-x+x_i+y_j}^{y_j} \int_{t_k}^{-x+x_i+t_{k+1}} \left(\frac{1}{h}(-x + x_i - t + t_k) + 1\right) \left(\frac{\frac{1}{h}(-x + x_i - t + t_k) + 1}{x} + \frac{1}{h}\right) dtdydx =$$
$$\frac{1}{72h}(24x_i^3 + 60x_i^2 h + 44x_i h^2 + 9h^3) - \frac{1}{3h^2} x_i (x_i + h)^3 \ln\left(1 + \frac{h}{x_i}\right) \tag{3.7.2}$$

We perform the same process for the other integrals on the 24 tetrahedrons and we will have the result in the table (3.7.1).

Table 3.7.1

01, 04	$\frac{1}{72h}\left(24x_i^3+60x_i^2h+44x_ih^2+9h^3\right)-\frac{1}{3h^2}x_i(x_i+h)^3\, ln\left(1+\frac{h}{x_i}\right)$
02, 03	$-\frac{1}{144h}\left(12x_i^3+42x_i^2h+52x_ih^2+19h^3\right)+\frac{1}{12h^2}(x_i+h)^4\, ln\left(1+\frac{h}{x_i}\right)$
05, 12, 19, 21	$-\frac{1}{144h}\left(12x_i^3+42x_i^2h+52x_ih^2+25h^3\right)+\frac{1}{12h^2}(x_i+h)^4\, ln\left(1+\frac{h}{x_i}\right)$
06, 11, 13, 18	$\frac{1}{144h}\left(-12x_i^3+42x_i^2h-52x_ih^2+25h^3\right)-\frac{1}{12h^2}(x_i-h)^4\, ln\left(1-\frac{h}{x_i}\right)$
07, 10	$\frac{1}{72h}\left(24x_i^3-60x_i^2h+44x_ih^2-9h^3\right)+\frac{1}{3h^2}x_i(x_i-h)^3\, ln\left(1-\frac{h}{x_i}\right)$
08, 09	$\frac{1}{144h}\left(-12x_i^3+42x_i^2h-52x_ih^2+19h^3\right)-\frac{1}{12h^2}(x_i-h)^4\, ln\left(1-\frac{h}{x_i}\right)$
14, 17	$\frac{1}{36h}\left(12x_i^3-30x_i^2h+22x_ih^2-3h^3\right)+\frac{1}{3h^2}x_i(x_i-h)^3\, ln\left(1-\frac{h}{x_i}\right)$
15, 16	$\frac{1}{12h}(x_i-h)\left(-6x_i^2+3x_ih+h^2\right)-\frac{1}{2h^2}x_i^2(x_i-h)^2\, ln\left(1-\frac{h}{x_i}\right)$
20, 22	$\frac{1}{36h}\left(12x_i^3+30x_i^2h+22x_ih^2+3h^3\right)-\frac{1}{3h^2}x_i(x_i+h)^3\, ln\left(1+\frac{h}{x_i}\right)$
23, 24	$-\frac{1}{12h}(x_i+h)\left(6x_i^2+3x_ih-h^2\right)+\frac{1}{2h^2}x_i^2(x_i+h)^2\, ln\left(1+\frac{h}{x_i}\right)$

Values of integrals for a_{ik}

Finally we finish the computation process by summation of the values of integrals in the table (3.7.1) and we get

$$a_{ii}=\frac{(x_i+h)(6x_i^2-8x_i(x_i+h)^2+3(x_i+h)^3)}{6h^2}ln\left(1+\frac{h}{x_i}\right)-\frac{x_i^4-4x_ih^3+3h^4}{6h^2}ln\left(1-\frac{h}{x_i}\right)-\frac{x_i(3x_i^2+h^2)}{9h}, \qquad (3.7.3)$$

and by the same approach we derive

$$a_{i,i-(N^2+N-1)}=\frac{1}{72h}(12x_i^3-6x_i^2h+40x_ih^2-3h^3)-\frac{1}{6h^2}(x_i^4+3x_i^2h^2+2x_ih^3-2h^4)\,ln\left(1+\frac{h}{x_i}\right), \qquad (3.7.4)$$

$$a_{i,i+(N^2+N-1)}=\frac{1}{72h}(12x_i^3+6x_i^2h+40x_ih^2+3h^3)+\frac{1}{6h^2}(x_i^4+3x_i^2h^2-2x_ih^3-2h^4)\,ln\left(1-\frac{h}{x_i}\right). \qquad (3.7.5)$$

Then we will get the following theorem. Also for the coefficient b_{ik} we get

$$\begin{aligned} b_{ii} &= 0, \\ b_{i,i-(N^2+N-1)} &= \frac{h^2}{12}, \\ b_{i,i+(N^2+N-1)} &= -\frac{h^2}{12}. \end{aligned} \tag{3.7.6}$$

Theorem 3.7. Suppose the linear system of equations

$$\sum_{i=1}^{N(h)} \left(U_{r_i} a_{ik} - U_{z_i} b_{ik} \right) = 0, \qquad k = 1,2,\dots,N(h),$$

where

$$a_{ik} = \int_{\Omega} \phi_i \left(\frac{\phi_k}{r} - \frac{\partial \phi_k}{\partial r} \right), b_{ik} = \int_{\Omega} \phi_i \frac{\partial \phi_k}{\partial z},$$

and ϕ_i , $i = 1,2,\dots,N(h)$ are the test functions in table (3.6.2), then the coefficients a_{ik} and b_{ik} are prepared in the relations (3.7.3), …,(3.7.6).

Suppose that the domain $\Omega = [0.3, 0.9] \times [0.3, 0.9] \times [0.3, 0.9]$, and $N = 4$, then $h = 0.15$, and

$$x_1 = 0.3, x_2 = 0.45, x_3 = 0.6, x_4 = 0.75$$

After inserting the values in the system (3.5.1) we express the continuity linear system as

$$(P, Q) \begin{pmatrix} U_r \\ U_z \end{pmatrix} = 0, \tag{3.7.7}$$

where

$$P =$$

$$\begin{pmatrix}
0.41 & 0 & \dots & 0 & 0.0009 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0.62 & 0 & \dots & 0 & 0.0001 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0.67 & 0 & \dots & 0 & -0.0003 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & \vdots & 0 & 0.41 & 0 & \dots & 0 & -0.0006 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0.003 & 0 & \vdots & 0 & 0.41 & 0 & \dots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0.002 & 0 & \vdots & 0 & 0.62 & 0 & \dots & \ddots & 0.0009 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.002 & 0 & \vdots & 0 & 0.67 & 0 & \ddots & 0 & 0.0001 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.002 & 0 & \vdots & 0 & 0.41 & \ddots & \vdots & 0 & -0.0003 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.003 & 0 & \vdots & 0 & \ddots & 0 & \vdots & 0 & -0.0006 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.002 & 0 & \vdots & \ddots & 0.41 & 0 & \vdots & 0 & 0.0009 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.002 & 0 & \ddots & 0 & 0.41 & 0 & \vdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.002 & \ddots & 0 & 0 & 0.62 & 0 & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0.003 & 0 & 0 & 0 & 0.41
\end{pmatrix}, \tag{3.7.8}$$

and

$$Q =$$

$$\begin{pmatrix}
0 & 0 & \dots & 0 & 0.002 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0.002 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0 & 0 & \dots & 0 & 0.002 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & \vdots & 0 & 0 & 0 & \dots & 0 & 0.002 & \ddots & 0 & 0 & 0 & 0 & 0 \\
-0.002 & 0 & \vdots & 0 & 0 & 0 & \dots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & -0.002 & 0 & \vdots & 0 & 0 & 0 & \dots & \ddots & 0.002 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.002 & 0 & \vdots & 0 & 0 & 0 & \ddots & 0 & 0.002 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.002 & 0 & \vdots & 0 & 0 & \ddots & \vdots & 0 & 0.002 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.002 & 0 & \vdots & 0 & \ddots & 0 & \vdots & 0 & 0.002 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.002 & 0 & \vdots & \ddots & 0 & 0 & \vdots & 0 & 0.002 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.002 & 0 & \ddots & 0 & 0 & 0 & \vdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.002 & \ddots & 0 & 0 & 0 & 0 & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -0.002 & 0 & 0 & 0 & 0
\end{pmatrix} \qquad (3.7.9)$$

Theorem 3.8. Suppose the linear system of equations

$$\sum_{i=1}^{N(h)} \left(U_{r_i} a_{ik} - U_{z_i} b_{ik} \right) = 0, \qquad k = 1,2,\dots,N(h),$$

where

$$a_{ik} = \int_{\Omega} \phi_i \left(\frac{\phi_k}{r} - \frac{\partial \phi_k}{\partial r} \right), b_{ik} = \int_{\Omega} \phi_i \frac{\partial \phi_k}{\partial z},$$

and ϕ_i , $i = 1,2,\dots,N(h)$ are the test functions in table (3.6.2), then the system can be stated as $PU_r + QU_z = 0$, where the matrices P and Q are introduced in (3.7.8) and (3.7.9).

Now we obtain a linear system of equations that it has 128 variables $U_{r_1}, U_{r_2}, \dots, U_{r_{64}}$, and $U_{z_1}, U_{z_2}, \dots, U_{z_{64}}$ within 64 equations. We stop here and then focus on the Navier-Stokes system to earn the remainder equations.

P;

3.8. The Navier-Stokes Equations

In this stage we will compute the coefficients of the Navier-Stokes system (3.5.2) and (3.5.3). We start by determining the coefficient A_{ik}, and as we have done before we suppose the ijk −mesh cube that it was divided to 24 tetrahedrons in the table (3.6.1), then after integration process in the whole parts we gain

$$A_{ii} = -\frac{x_i^3 + h^3}{3h^2} ln\left(1 + \frac{h}{x_i}\right) + \frac{x_i^3 - h^3}{3h^2} ln\left(1 - \frac{h}{x_i}\right) + \frac{6x_i^2 + 2h^2}{9h} \qquad (3.8.1)$$

$$A_{i,i-(N^2+N-1)} = -\frac{1}{18h}(6x_i^2 + 15x_i h + 11h^2) + \frac{(x_i + h)^3}{3h^2} ln\left(1 + \frac{h}{x_i}\right) - \frac{h^2}{12} \qquad (3.8.2)$$

$$A_{i,i+(N^2+N-1)} = -\frac{1}{18h}(6x_i^2 - 15x_i h + 11h^2) + \frac{(x_i - h)^3}{3h^2} ln\left(1 - \frac{h}{x_i}\right) + \frac{h^2}{12} \qquad (3.8.3)$$

Now we insert the values of A_{ik} from (3.8.1), (3.8.2), and (3.8.3) into the $\sum_{i=1}^{N(h)} U_{r_i} A_{ik}$ and we earn the coefficient matrix as

$$A =$$

$$\begin{pmatrix}
0.008 & 0 & \dots & 0 & 0.004 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0.003 & 0 & \dots & 0 & 0.002 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0.002 & 0 & \dots & 0 & 0.001 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & \vdots & 0 & 0.001 & 0 & \dots & 0 & 0.0005 & \ddots & 0 & 0 & 0 & 0 & 0 \\
-0.07 & 0 & \vdots & 0 & 0.008 & 0 & \dots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & -0.33 & 0 & \vdots & 0 & 0.003 & 0 & \dots & \ddots & 0.004 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.78 & 0 & \vdots & 0 & 0.002 & 0 & \ddots & 0 & 0.002 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.43 & 0 & \vdots & 0 & 0.001 & \ddots & \vdots & 0 & 0.001 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.07 & 0 & \vdots & 0 & \ddots & 0 & \vdots & 0 & 0.0005 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.33 & 0 & \vdots & \ddots & 0.001 & 0 & \vdots & 0 & 0.004 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.78 & 0 & \ddots & 0 & 0.008 & 0 & \vdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.43 & \ddots & 0 & 0 & 0.003 & 0 & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0.002 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -0.07 & 0 & 0 & 0 & 0.001
\end{pmatrix} \quad (3.8.4)$$

We continue the process of finding the coefficients of the Navier-Stokes system by focusing on the coefficient B_{ijk} from $\sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{r_j} B_{ijk}$, and we achieve

$$B_{iii} = 0,$$

$$B_{i,i-(N^2+N-1),i} = -\frac{h^2}{20}, \qquad B_{i,i+(N^2+N-1),i} = \frac{h^2}{20},$$

$$B_{i,i-(N^2+N-1),i-(N^2+N-1)} = \frac{h^2}{20}, \qquad B_{i,i+(N^2+N-1),i+(N^2+N-1)} = -\frac{h^2}{20},$$

$$B_{i,i,i-(N^2+N-1)} = \frac{h^2}{60}, \qquad B_{i,i,i+(N^2+N-1)} = -\frac{h^2}{60}.$$

then by applying the values B_{ijk} we will have

$$\sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{r_j} B_{ijk} =$$

$$\begin{cases} \frac{h^2}{10} U_{r_k} U_{r_{k+(N^2+N-1)}} + \frac{h^2}{60} U^2_{r_{k+(N^2+N-1)}} & ;\ k < 20 \\ -\frac{h^2}{10} U_{r_k} U_{r_{k-(N^2+N-1)}} + \frac{h^2}{10} U_{r_k} U_{r_{k+(N^2+N-1)}} - \frac{h^2}{60} U^2_{r_{k-(N^2+N-1)}} + \frac{h^2}{60} U^2_{r_{k+(N^2+N-1)}} & ;\ 20 \le k \le 45 \\ -\frac{h^2}{10} U_{r_k} U_{r_{k-(N^2+N-1)}} - \frac{h^2}{60} U^2_{r_{k-(N^2+N-1)}} & ;\ k > 45 \end{cases}$$

We continue the same process to compute the values of C_{ijk}, and we get

$$C_{iii} = 0,$$

$$C_{i,i-(N^2+N-1),i} = \frac{h^2}{15}, \qquad C_{i,i+(N^2+N-1),i} = -\frac{h^2}{15},$$

$$C_{i,i,i-(N^2+N-1)} = \frac{h^2}{15}, \qquad C_{i,i,i+(N^2+N-1)} = -\frac{h^2}{15},$$

$$C_{i,i-(N^2+N-1),i-(N^2+N-1)} = -\frac{h^2}{5}, \qquad C_{i,i+(N^2+N-1),i+(N^2+N-1)} = \frac{h^2}{5}.$$

then we set the values C_{ijk} and we derive

$$\sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{z_j} C_{ijk} =$$

$$\begin{cases} -\frac{h^2}{15} U_{r_k} U_{z_{k+(N^2+N-1)}} - \frac{h^2}{5} U_{z_k} U_{r_{k+(N^2+N-1)}} + \frac{h^2}{15} U_{r_{k+(N^2+N-1)}} U_{z_{k+(N^2+N-1)}} & ;\ k < 20 \\ -\frac{h^2}{15} U_{r_{k-(N^2+N-1)}} U_{z_{k-(N^2+N-1)}} + \frac{h^2}{5} U_{r_{k-(N^2+N-1)}} U_{z_k} + \frac{h^2}{15} U_{r_k} U_{z_{k-(N^2+N-1)}} - \frac{h^2}{15} U_{r_k} U_{z_{k+(N^2+N-1)}} \\ \qquad - \frac{h^2}{5} U_{r_{k+(N^2+N-1)}} U_{z_k} + \frac{h^2}{15} U_{r_{k+(N^2+N-1)}} U_{z_{k+(N^2+N-1)}} & ;\ 20 \le k \le 45 \\ \frac{h^2}{15} U_{r_k} U_{z_{k-(N^2+N-1)}} - \frac{h^2}{15} U_{r_{k-(N^2+N-1)}} U_{z_{k-(N^2+N-1)}} + \frac{h^2}{5} U_{r_{k-(N^2+N-1)}} U_{z_k} & ;\ k > 45 \end{cases}$$

We terminate this part by computing the coefficients D_{ik}, and we derive

$$D_{ii} = 0, \qquad D_{i,i-(N^2+N-1)} = \frac{h^2}{12}, \qquad D_{i,i+(N^2+N-1)} = -\frac{h^2}{12}.$$

and we will get the coefficient matrix $(D_{ik})_{64\times 64}$ from $\sum_{i=1}^{N(h)} P_i D_{ik}$ as

$$D =$$

$$\frac{1}{\rho}\begin{pmatrix} 0 & 0 & \cdots & 0 & 0.002 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0.002 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & \cdots & 0 & 0.002 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & \vdots & 0 & 0 & 0 & \cdots & 0 & 0.002 & \ddots & 0 & 0 & 0 & 0 & 0 \\ -0.002 & 0 & \vdots & 0 & 0 & 0 & \cdots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.002 & 0 & \vdots & 0 & 0 & 0 & \cdots & \ddots & 0.002 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.002 & 0 & \vdots & 0 & 0 & 0 & \ddots & 0 & 0.002 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.002 & 0 & \vdots & 0 & 0 & \ddots & \vdots & 0 & 0.002 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.002 & 0 & \vdots & 0 & \ddots & 0 & \vdots & 0 & 0.002 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.002 & 0 & \vdots & \ddots & 0 & 0 & \vdots & 0 & 0.002 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.002 & 0 & \ddots & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.002 & \ddots & 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -0.002 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.8.5)$$

We note that the last coefficient E_i in the system (3.5.2) is

$$E_i = g_r h^3$$

Now we gain the nonlinear system 64 variables $U_{r_1}, U_{r_2}, \ldots, U_{r_{64}}$ within 64 equations, then we repeat precisely the same process for the second part of the Navier-Stokes equations that is the system (3.5.3), and again we get the other nonlinear system that also it has 64 variables $U_{z_1}, U_{z_2}, \dots, U_{z_{64}}$ within 64 equations. Hear we present the details of second Navier-Stokes system.

Coefficients A'_{ik} and the matrix $A' = (A'_{ik})_{64\times 64}$

$$A'_{ii} = 0, \qquad A'_{i,i-(N^2+N-1)} = -\frac{h^2}{12}, \qquad A'_{i,i+(N^2+N-1)} = \frac{h^2}{12}.$$

$$A' =$$

$$\begin{pmatrix}
0 & 0 & \dots & 0 & -0.002 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & -0.002 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0 & 0 & \dots & 0 & -0.002 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & \vdots & 0 & 0 & 0 & \dots & 0 & -0.002 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0.002 & 0 & \vdots & 0 & 0 & 0 & \dots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0.002 & 0 & \vdots & 0 & 0 & 0 & \dots & \ddots & -0.002 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.002 & 0 & \vdots & 0 & 0 & 0 & \ddots & 0 & -0.002 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.002 & 0 & \vdots & 0 & 0 & \ddots & \vdots & 0 & -0.002 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.002 & 0 & \vdots & 0 & \ddots & 0 & \vdots & 0 & -0.002 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.002 & 0 & \vdots & \ddots & 0 & 0 & \vdots & 0 & -0.002 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.002 & 0 & \ddots & 0 & 0 & 0 & \vdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.002 & \ddots & 0 & 0 & 0 & 0 & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0.002 & 0 & 0 & 0 & 0
\end{pmatrix} \qquad (3.8.6)$$

Coefficients B'_{ijk} and the value $\sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{z_i} U_{z_j} B'_{ijk}$

$$B'_{iii} = 0$$

$$B'_{i,i-(N^2+N-1),i} = -\frac{h^2}{20} \qquad B'_{i,i+(N^2+N-1),i} = \frac{h^2}{20}$$

$$B'_{i,i-(N^2+N-1),i-(N^2+N-1)} = \frac{h^2}{20} \qquad B'_{i,i+(N^2+N-1),i+(N^2+N-1)} = -\frac{h^2}{20}$$

$$B'_{i,i,i-(N^2+N-1)} = \frac{h^2}{60} \qquad B'_{i,i,i+(N^2+N-1)} = -\frac{h^2}{60}$$

$$\sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{z_i} U_{z_j} B'_{ijk} =$$

$$\begin{cases}
-\frac{h^2}{10} U_{z_k} U_{z_{k+(N^2+N-1)}} - \frac{h^2}{60} U^2_{z_{k+(N^2+N-1)}} & ;\ k < 20 \\
\frac{h^2}{10} U_{z_k} U_{z_{k-(N^2+N-1)}} - \frac{h^2}{10} U_{z_k} U_{z_{k+(N^2+N-1)}} + \frac{h^2}{60} U^2_{z_{k-(N^2+N-1)}} - \frac{h^2}{60} U^2_{z_{k+(N^2+N-1)}} & ;\ 20 \le k \le 45 \\
\frac{h^2}{10} U_{z_k} U_{z_{k-(N^2+N-1)}} + \frac{h^2}{60} U^2_{z_{k-(N^2+N-1)}} & ;\ k > 45
\end{cases}$$

Coefficients C'_{ijk} and the value $\sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{z_i} U_{r_j} C'_{ijk}$

$C'_{iii} = 0,$

$C'_{i,i-(N^2+N-1),i} = -\frac{h^2}{15},$ $\qquad C'_{i,i+(N^2+N-1),i} = \frac{h^2}{15},$

$C'_{i,i,i-(N^2+N-1)} = -\frac{h^2}{15},$ $\qquad C'_{i,i,i+(N^2+N-1)} = \frac{h^2}{15},$

$C'_{i,i-(N^2+N-1),i-(N^2+N-1)} = \frac{h^2}{5},$ $\qquad C'_{i,i+(N^2+N-1),i+(N^2+N-1)} = -\frac{h^2}{5}.$

$$\sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{z_i} U_{r_j} C'_{ijk} =$$

$$\begin{cases} \frac{h^2}{15} U_{z_k} U_{r_{k+(N^2+N-1)}} + \frac{h^2}{5} U_{r_k} U_{z_{k+(N^2+N-1)}} - \frac{h^2}{15} U_{z_{k+(N^2+N-1)}} U_{r_{k+(N^2+N-1)}} & ;\ k < 20 \\ \frac{h^2}{15} U_{z_{k-(N^2+N-1)}} U_{r_{k-(N^2+N-1)}} - \frac{h^2}{5} U_{z_{k-(N^2+N-1)}} U_{r_k} - \frac{h^2}{15} U_{z_k} U_{r_{k-(N^2+N-1)}} + \frac{h^2}{15} U_{z_k} U_{r_{k+(N^2+N-1)}} \\ \qquad + \frac{h^2}{5} U_{z_{k+(N^2+N-1)}} U_{r_k} - \frac{h^2}{15} U_{z_{k+(N^2+N-1)}} U_{r_{k+(N^2+N-1)}} & ;\ 20 \le k \le 45 \\ -\frac{h^2}{15} U_{z_k} U_{r_{k-(N^2+N-1)}} + \frac{h^2}{15} U_{z_{k-(N^2+N-1)}} U_{r_{k-(N^2+N-1)}} - \frac{h^2}{5} U_{z_{k-(N^2+N-1)}} U_{r_k} & ;\ k > 45 \end{cases}$$

Coefficients D'_{ik} and the matrix $(D'_{ik})_{64\times 64}$

$D'_{ii} = 0,$ $\qquad D'_{i,i-(N^2+N-1)} = -\frac{h^2}{12},$ $\qquad D'_{i,i+(N^2+N-1)} = \frac{h^2}{12}.$

$$D' =$$

$$\frac{1}{\rho} \begin{pmatrix} 0 & 0 & \dots & 0 & -0.002 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -0.002 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & \dots & 0 & -0.002 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & \vdots & 0 & 0 & 0 & \dots & 0 & -0.002 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0.002 & 0 & \vdots & 0 & 0 & 0 & \dots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.002 & 0 & \vdots & 0 & 0 & 0 & \dots & \ddots & -0.002 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.002 & 0 & \vdots & 0 & 0 & 0 & \ddots & 0 & -0.002 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.002 & 0 & \vdots & 0 & 0 & \ddots & \vdots & 0 & -0.002 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.002 & 0 & \vdots & 0 & \ddots & 0 & \vdots & 0 & -0.002 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.002 & 0 & \vdots & \ddots & 0 & 0 & \vdots & 0 & -0.002 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.002 & 0 & \ddots & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.002 & \ddots & 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0.002 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (3.8.7)$$

and

$$E'_i = g_z h^3$$

We will try to omit the pressure from both systems, then we add correspond equations in the systems (3.5.2) and (3.5.3) to each other to cancel the pressure, and finally we derive the new system with 64 equations within 128 variables.

Theorem 3.9. Suppose the nonlinear system of equations

$$\sum_{i=1}^{N(h)} U_{r_i} A_{ik} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{r_j} B_{ijk} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{z_j} C_{ijk} + \sum_{i=1}^{N(h)} P_i D_{ik} + E_k = 0,$$

$$\sum_{i=1}^{N(h)} U_{z_i} A'_{ik} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{z_i} U_{z_j} B'_{ijk} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{z_i} U_{r_j} C'_{ijk} + \sum_{i=1}^{N(h)} P_i D'_{ik} + E'_k = 0,$$

where the coefficients were listed in the table (3.5.1), and $\phi_i , i = 1,2, \dots , N(h)$ are the test functions in table (3.6.2), then the nonlinear system are reduced to

$for\ k < 20$

$$0.008\, U_{r_1} + 0.004\, U_{r_{20}} - 0.002\, U_{z_{20}} + 0.002\, U_{r_1} U_{r_{20}} - 0.002\, U_{z_1} U_{z_{20}} + 0.003\, U_{r_1} U_{z_{20}} - 0.003\, U_{z_1} U_{r_{20}} + 0.0004\, U_{r_{20}}^2 - 0.0004\, U_{z_{20}}^2 + 0.003\, (g_r + g_z) = 0,$$

$$0.003\, U_{r_2} + 0.002\, U_{r_{21}} - 0.002\, U_{z_{21}} + 0.002\, U_{r_2} U_{r_{21}} - 0.002\, U_{z_2} U_{z_{21}} + 0.003\, U_{r_2} U_{z_{21}} - 0.003\, U_{z_2} U_{r_{21}} + 0.0004\, U_{r_{21}}^2 - 0.0004\, U_{z_{21}}^2 + 0.003\, (g_r + g_z) = 0,$$

$$\vdots$$

$$0.002\, U_{r_{19}} + 0.001\, U_{r_{38}} - 0.002\, U_{z_{38}} + 0.002\, U_{r_{19}} U_{r_{38}} - 0.002\, U_{z_{19}} U_{z_{38}} + 0.003\, U_{r_{19}} U_{z_{38}} - 0.003\, U_{z_{19}} U_{r_{38}} + 0.0004\, U_{r_{38}}^2 - 0.0004\, U_{z_{38}}^2 + 0.003\, (g_r + g_z) = 0,$$

$for\ 20 \le k \le 45$

$$-0.07\, U_{r_1} + 0.001\, U_{r_{20}} + 0.001\, U_{r_{39}} + 0.002\, U_{z_1} - 0.002\, U_{z_{39}} - 0.002\, U_{r_1} U_{r_{20}} + 0.002\, U_{r_{20}} U_{r_{39}} + 0.003\, U_{r_1} U_{z_{20}} - 0.003\, U_{r_{20}} U_{z_1} + 0.003\, U_{r_{20}} U_{z_{39}} - 0.003\, U_{r_{39}} U_{z_{20}} + 0.002\, U_{z_1} U_{z_{20}} - 0.002\, U_{z_{20}} U_{z_{39}} - 0.0004\, U_{r_1}^2 + 0.0004\, U_{r_{39}}^2 + 0.0004\, U_{z_1}^2 - 0.0004\, U_{z_{39}}^2 + 0.003\, (g_r + g_z) = 0,$$

$$-0.33\, U_{r_2} + 0.008\, U_{r_{21}} + 0.0005\, U_{r_{40}} + 0.002\, U_{z_2} - 0.002\, U_{z_{40}} - 0.002\, U_{r_2} U_{r_{21}} + 0.002\, U_{r_{21}} U_{r_{40}} + 0.003\, U_{r_2} U_{z_{21}} - 0.003\, U_{r_{21}} U_{z_2} + 0.003\, U_{r_{21}} U_{z_{40}} - 0.003\, U_{r_{40}} U_{z_{21}} + 0.002\, U_{z_2} U_{z_{21}} - 0.002\, U_{z_{21}} U_{z_{40}} - 0.0004\, U_{r_2}^2 + 0.0004\, U_{r_{40}}^2 + 0.0004\, U_{z_2}^2 - 0.0004\, U_{z_{40}}^2 + 0.003\, (g_r + g_z) = 0,$$

$$\vdots$$

$$-0.33\, U_{r_{26}} + 0.008\, U_{r_{45}} + 0.0005\, U_{r_{64}} + 0.002\, U_{z_{26}} - 0.002\, U_{z_{64}} - 0.002\, U_{r_{26}} U_{r_{45}} + 0.002\, U_{r_{45}} U_{r_{64}} + 0.003\, U_{r_{26}} U_{z_{45}} - 0.003\, U_{r_{45}} U_{z_{26}} + 0.003\, U_{r_{45}} U_{z_{64}} - 0.003\, U_{r_{64}} U_{z_{45}} + 0.002\, U_{z_{26}} U_{z_{45}} - 0.002\, U_{z_{45}} U_{z_{64}} - 0.0004\, U_{r_{26}}^2 + 0.0004\, U_{r_{64}}^2 + 0.0004\, U_{z_{26}}^2 - 0.0004\, U_{z_{64}}^2 + 0.003\, (g_r + g_z) = 0,$$

$for\ 46 \leq k \leq 64$

$$-0.78\,U_{r_{27}} + 0.003\,U_{r_{46}} + 0.002\,U_{z_{27}} - 0.002\,U_{r_{46}}U_{r_{27}} + 0.002\,U_{z_{46}}U_{z_{27}} + 0.003\,U_{r_{27}}U_{z_{46}}$$
$$-0.003\,U_{r_{46}}U_{z_{27}} - 0.0004\,U_{r_{27}}^2 + 0.0004\,U_{z_{27}}^2 + 0.003\,(g_r + g_z) = 0,$$

$$-1.43\,U_{r_{28}} + 0.002\,U_{r_{47}} + 0.002\,U_{z_{28}} - 0.002\,U_{r_{47}}U_{r_{28}} + 0.002\,U_{z_{47}}U_{z_{28}} + 0.003\,U_{r_{28}}U_{z_{47}}$$
$$-0.003\,U_{r_{47}}U_{z_{28}} - 0.0004\,U_{r_{28}}^2 + 0.0004\,U_{z_{28}}^2 + 0.003\,(g_r + g_z) = 0,$$

$$\vdots$$

$$-0.07\,U_{r_{45}} + 0.001\,U_{r_{64}} + 0.002\,U_{z_{45}} - 0.002\,U_{r_{64}}U_{r_{45}} + 0.002\,U_{z_{64}}U_{z_{45}} + 0.003\,U_{r_{45}}U_{z_{64}}$$
$$-0.003\,U_{r_{64}}U_{z_{45}} - 0.0004\,U_{r_{45}}^2 + 0.0004\,U_{z_{45}}^2 + 0.003\,(g_r + g_z) = 0,$$

3.9. Numerical Solution

As we have shown in the section (3.7) the continuity equation could be stated as a linear system in 64 equations in 128 variables. The matrix P has diagonal dominance, then it is invertible and so from (3.7.7) we will get the U_r uniquely as

$$U_r = -P^{-1}QU_z \tag{3.9.1}$$

Also remember from section (3.8) that the Navier-Stokes system has the style

$$AU_r + \psi_1(U_r, U_z) + Dp + E = 0, \tag{3.9.2}$$

and

$$A'U_z + \psi_2(U_r, U_z) + D'p + E' = 0, \tag{3.9.3}$$

where the matrices A, D, A', and D' are stated in the relations (3.8.4), (3.8.5), (3.8.6), and (3.8.7). Also we know that

$$D = -D',$$

and

$$E = h^3 g_r \mathbf{1},$$

$$E' = h^3 g_z \mathbf{1}.$$

After the summation of (3.9.2) and (3.9.3) we will reach to

$$AU_r + A'U_z + \psi(U_r, U_z) + E + E' = 0 \tag{3.9.4}$$

Now we replace the value of U_r from (3.9.1) into the (3.9.4) to earn

$$(A' - AP^{-1}Q)U_z + \psi(U_z) = -h^3(g_r + g_z)\mathbf{1} \tag{3.9.5}$$

The system (3.9.5) has 64 equations within 64 variables. Our purpose is to search for its numerical solution by invoking the Newton's method, thus we need the initial solution to start the process of iterations, so we assume that the nonlinear part of the system (3.9.5) be zero, that is

$$\psi(U_z) = 0,$$

and then

$$(A' - AP^{-1}Q)U_z = -h^3(g_r + g_z)\mathbf{1} \tag{3.9.6}$$

From the system (3.9.6) immediately we compute

$U_{z_{21}} = -16.5$ $U_{z_{22}} = -16.5$ $U_{z_{23}} = -16.5$ $U_{z_{25}} = -16.5$

$U_{z_{26}} = -16.5$ $U_{z_{27}} = -16.5$ $U_{z_{29}} = -16.5$ $U_{z_{30}} = -16.5$

$U_{z_{31}} = -16.5$ $U_{z_{33}} = -16.5$ $U_{z_{34}} = -16.5$ $U_{z_{35}} = -16.5$

$U_{z_{37}} = -16.5$ $U_{z_{38}} = -16.5$

Then we perform the Newton's iterations

$$U_z^{k+1} = U_z^k - (DF)^{-1}(U_z^k).F(U_z^k),$$

where $F: \mathbb{R}^{64} \to \mathbb{R}^{64}$ is defined as

$$F(U_z) = \begin{pmatrix} F_1(U_z) \\ F_2(U_z) \\ \vdots \\ F_{64}(U_z) \end{pmatrix},$$

and DF is the Jacobian matrix of

$$DF = \begin{pmatrix} \frac{\partial F_1}{\partial U_{z_1}} & \frac{\partial F_1}{\partial U_{z_2}} & \cdots & \frac{\partial F_1}{\partial U_{z_{64}}} \\ \frac{\partial F_2}{\partial U_{z_1}} & \frac{\partial F_2}{\partial U_{z_2}} & \cdots & \frac{\partial F_2}{\partial U_{z_{64}}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_{64}}{\partial U_{z_1}} & \frac{\partial F_{64}}{\partial U_{z_2}} & \cdots & \frac{\partial F_{64}}{\partial U_{z_{64}}} \end{pmatrix}$$

We repeat the Newton's algorithm 30 times and we produce the solution as

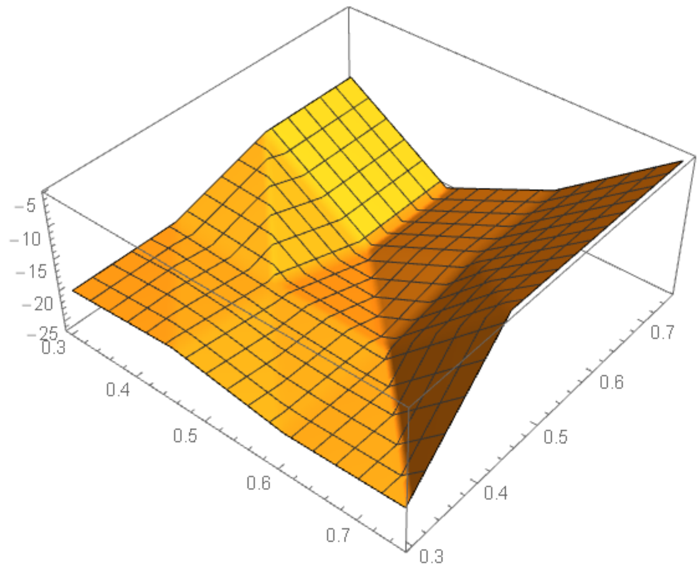
$U_{z_1} = -19.964396203271495$ $U_{z_2} = -19.032631603831394$
$U_{z_3} = -21.585584196054814$ $U_{z_4} = -21.152491963049943$
$U_{z_5} = -19.96439620327149$ $U_{z_6} = -21.027391916196642$
$U_{z_7} = -21.585584196045044$ $U_{z_8} = -5.153356893381632$
$U_{z_9} = -15.080509375582018$ $U_{z_{10}} = -26.18565106460826$
$U_{z_{11}} = -18.131959256356694$ $U_{z_{12}} = -5.153356893381636$
$U_{z_{13}} = -15.080351223281776$ $U_{z_{14}} = -26.185651064608262$
$U_{z_{15}} = -18.131959256356694$ $U_{z_{16}} = -5.153356893381636$
$U_{z_{17}} = -15.080509375582018$ $U_{z_{18}} = -26.185651064608262$
$U_{z_{19}} = -18.131387860413064$ $U_{z_{20}} = 0.9044792016387821$
$U_{z_{21}} = 0.9512084266258982$ $U_{z_{22}} = 0.8210418438489437$
$U_{z_{23}} = 0.8364611015836216$ $U_{z_{24}} = 0.9044792016387821$
$U_{z_{25}} = 0.8527276053620682$ $U_{z_{26}} = 0.8210418438493451$
$U_{z_{27}} = 14.525657477973708$ $U_{z_{28}} = 1.2569219242317637$
$U_{z_{29}} = 0.9665538943001553$ $U_{z_{30}} = 1.0909165624593518$
$U_{z_{31}} = 14.525657477973725$ $U_{z_{32}} = 1.256935110922197$
$U_{z_{33}} = 0.9665538943001551$ $U_{z_{34}} = 1.0909165624593518$
$U_{z_{35}} = 14.525657477973725$ $U_{z_{36}} = 1.2569219242317635$
$U_{z_{37}} = 0.9665538943001551$ $U_{z_{38}} = 1.0909428660709701$
$U_{z_{39}} = -16.968385693584583$ $U_{z_{40}} = -15.817504516369528$
$U_{z_{41}} = -19.118623948423114$ $U_{z_{42}} = -18.95208878856998$
$U_{z_{43}} = -16.96838569358458$ $U_{z_{44}} = -18.34556685135166$
$U_{z_{45}} = -19.118623948411262$ $U_{z_{46}} = -2.8815383240026664$
$U_{z_{47}} = -8.018433353991593$ $U_{z_{48}} = 12.77607584911194$
$U_{z_{49}} = 3.629362362017885$ $U_{z_{50}} = -2.881538324002671$
$U_{z_{51}} = -8.01837578716835$ $U_{z_{52}} = 12.776075849111944$
$U_{z_{53}} = 3.629362362017885$ $U_{z_{54}} = -2.8815383240026704$
$U_{z_{55}} = -8.018433353991593$ $U_{z_{56}} = 12.776075849111944$
$U_{z_{57}} = 3.6285109525148713$ $U_{z_{58}} = 1.365358543778119$
$U_{z_{59}} = 1.0708011223774292$ $U_{z_{60}} = 1.8262529864312615$
$U_{z_{61}} = 1.7875947952490874$ $U_{z_{62}} = 1.3653585437781197$
$U_{z_{63}} = 1.8781813493164021$ $U_{z_{64}} = 1.8262529864328112$

Now we refer to the relation (3.3.2) to simulate the function

$$u_{z_h}(r,\varphi,z,t) = \sum_{i=1}^{N(h)} U_{z_i}\phi_i(r,\varphi,z,t),$$

and the result is exhibited in the three-dimensional figure (3.9.1)

Figure 3.9.1



In this part we apply the values of U_z to compute the U_r from (3.9.1) and we will derive

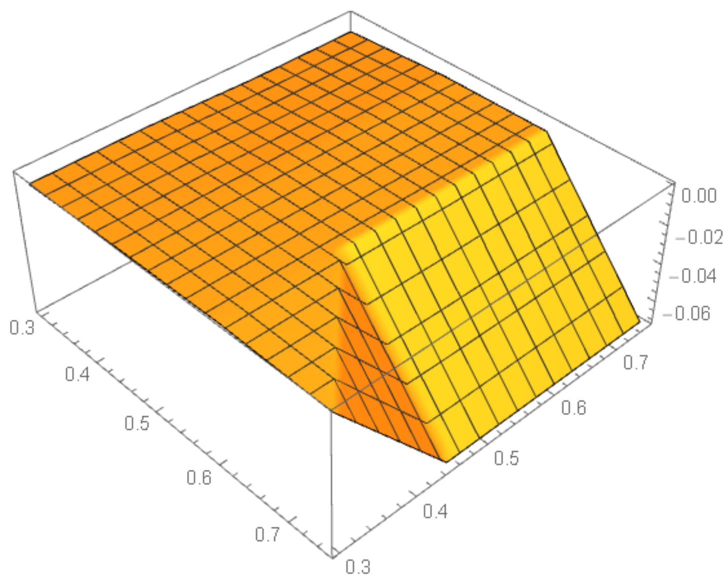
$U_{r_1} = -0.00438008$	$U_{r_2} = -0.00306589$	$U_{r_3} = -0.00245443$
$U_{r_4} = -0.00408989$	$U_{r_5} = -0.00438008$	$U_{r_6} = -0.00274863$
$U_{r_7} = -0.00245443$	$U_{r_8} = -0.0708664$	$U_{r_9} = -0.00605582$
$U_{r_{10}} = -0.00308726$	$U_{r_{11}} = -0.00328789$	$U_{r_{12}} = -0.0708664$
$U_{r_{13}} = -0.00605588$	$U_{r_{14}} = -0.00308726$	$U_{r_{15}} = -0.00328789$
$U_{r_{16}} = -0.0708664$	$U_{r_{17}} = -0.00605582$	$U_{r_{18}} = -0.00308726$
$U_{r_{19}} = -0.00328797$	$U_{r_{20}} = -0.0145846$	$U_{21} = -0.0156674$
$U_{r_{22}} = -0.00794927$	$U_{r_{23}} = -0.00655752$	$U_{r_{24}} = -0.0145846$
$U_{r_{25}} = -0.0130577$	$U_{r_{26}} = -0.00794927$	$U_{r_{27}} = -0.00654902$
$U_{r_{28}} = -0.0343992$	$U_{r_{29}} = -0.190056$	$U_{r_{30}} = -0.0701881$
$U_{r_{31}} = -0.00654902$	$U_{r_{32}} = -0.0343987$	$U_{r_{33}} = -0.190056$
$U_{r_{34}} = -0.0701881$	$U_{r_{35}} = -0.00654902$	$U_{r_{36}} = -0.0343992$
$U_{r_{37}} = -0.190056$	$U_{r_{38}} = -0.0701836$	$U_{r_{39}} = -0.00135673$
$U_{r_{40}} = -0.000537834$	$U_{r_{41}} = -0.00466005$	$U_{r_{42}} = -0.00303211$
$U_{r_{43}} = -0.00135673$	$U_{r_{44}} = -0.00498679$	$U_{r_{45}} = -0.00466005$
$U_{r_{46}} = 0.0468781$	$U_{r_{47}} = 0.00385469$	$U_{r_{48}} = 0.00610555$
$U_{r_{49}} = 0.00566393$	$U_{r_{50}} = 0.0468781$	$U_{r_{51}} = 0.00385473$
$U_{r_{52}} = 0.00610555$	$U_{r_{53}} = 0.00566393$	$U_{r_{54}} = 0.0468781$
$U_{r_{55}} = 0.00385469$	$U_{r_{56}} = 0.00610555$	$U_{r_{57}} = 0.00624877$
$U_{r_{58}} = -0.0547324$	$U_{r_{59}} = -0.0472148$	$U_{r_{60}} = -0.0932275$
$U_{r_{61}} = -0.0924344$	$U_{r_{62}} = -0.0547324$	$U_{r_{63}} = -0.054748$
$U_{r_{64}} = -0.0932275$		

then we insert the values of U_r into the relation (3.3.1) to simulate the function

$$u_{r_h}(r,\varphi,z,t) = \sum_{i=1}^{N(h)} U_{r_i}\phi_i(r,\varphi,z,t),$$

and we will show the result in the three-dimensional figure (3.9.2)

Figure 3.9.2.



In the final stage we will return to the energy conservation system (3.4.32), and as we mentioned before to get the solution of this system we need the values of U_r and U_z, then in this position we can apply the computed values of U_r and U_z to earn the linear system of equations in 64 equations in 64 variables. We notify that during the computation process we need the coefficients

$$H_{iii} = 10h,$$

$$H_{i,i-(N^2+N-1),i} = 0, \qquad H_{i,i+(N^2+N-1),i} = 0,$$

$$H_{i,i-(N^2+N-1),i-(N^2+N-1)} = 0, \qquad H_{i,i+(N^2+N-1),i+(N^2+N-1)} = 0,$$

$$H_{i,i,i-(N^2+N-1)} = \frac{25}{3}h, \qquad H_{i,i,i+(N^2+N-1)} = \frac{25}{3}h,$$

and

$$I_k = -\frac{F}{\rho c}h^3$$

Also we use the following algorithm to prepare the coefficients of G_{ijk}

```
Array[G1, {64, 64, 64}];
For[i = 1, i ≤ 64, i++, For[j = 1, j ≤ 64, j++, For[k = 1, k ≤ 64, k++, G1[i, j, k] = 0]]]
For[i = 1, i ≤ 64, i++, G1[i, i, i - 19] = .00075 * (UR[i] - UZ[i])];
For[i = 1, i ≤ 64, i++, G1[i, i - 19, i] = .00225 * (-UR[i - 19] + UZ[i - 19])];
For[i = 1, i ≤ 64, i++, G1[i, i - 19, i - 19] = .00225 * (UR[i - 19] - UZ[i - 19])];
For[i = 1, i ≤ 64, i++, G1[i, i, i + 19] = .00075 * (-UR[i] + UZ[i])];
For[i = 1, i ≤ 64, i++, G1[i, i + 19, i] = .00225 * (UR[i + 19] - UZ[i + 19])]
For[i = 1, i ≤ 64, i++, G1[i, i + 19, i + 19] = .00225 * (-UR[i + 19] + UZ[i + 19])];

Array[F, {64, 64}];
For[i = 1, i ≤ 64, i++, For[j = 1, j ≤ 64, j++, F[i, j] = 0]];
For[i = 1, i ≤ 45, i++, F[i + 19, i] = .002];
For[i = 20, i ≤ 64, i++, F[i - 19, i] = -.002];


Array[T, 64];
For[k = 1, k ≤ 64, k++,
     sum1 = 0;
     sum2 = 0;
     For[i = 1, i ≤ 64, i++,
          sum2 = sum2 + T[i] * F[i, k];
          For[j = 1, j ≤ 64, j++, sum1 = sum1 + T[i] * G1[i, j, k]]];
 Print[sum1 + sum2];
]
```

The solution of the system (3.3.32) is as

$\theta_1 = 267464.50784935005$
$\theta_2 = 248522.63133185773$
$\theta_3 = 307632.2765236602$
$\theta_4 = 307560.1642197024$
$\theta_5 = 279837.7344858227$
$\theta_6 = 321624.62060482206$
$\theta_7 = 427070.98850310955$
$\theta_8 = 196160.35809189378$
$\theta_9 = 3066.7775345885548$
$\theta_{10} = 186800.78275498643$
$\theta_{11} = 121034.49058102771$
$\theta_{12} = 196594.31919427816$
$\theta_{13} = 2152.4412377153467$
$\theta_{14} = 188106.54167680247$
$\theta_{15} = 121684.13564829151$
$\theta_{16} = 197507.5771545514$
$\theta_{17} = -74.86551356291443$
$\theta_{18} = 191421.1027365251$
$\theta_{19} = 123429.09980863145$
$\theta_{20} = -13968.224603060831$
$\theta_{21} = -12457.95824857949$
$\theta_{22} = -16553.23955494196$
$\theta_{23} = -15559.48049623543$
$\theta_{24} = -13416.836196379161$
$\theta_{25} = -14444.301955614843$
$\theta_{26} = -12094.511632460784$
$\theta_{27} = 2288889.553754123$
$\theta_{28} = -19412.911905873087$
$\theta_{29} = -26223.652534756788$
$\theta_{30} = -16086.278156650058$
$\theta_{31} = 2294428.3796616173$
$\theta_{32} = -19492.949247765693$
$\theta_{33} = -26169.651772698493$
$\theta_{34} = -16049.706435758939$
$\theta_{35} = 2306132.915719399$
$\theta_{36} = -19683.559356278733$
$\theta_{37} = -26029.21974382987$
$\theta_{38} = -15944.722098429129$
$\theta_{39} = 171548.63305087356$
$\theta_{40} = 160669.13298697892$
$\theta_{41} = 196734.19273537697$
$\theta_{42} = 196477.20367911315$
$\theta_{43} = 186387.49327337768$
$\theta_{44} = 220899.00388040626$
$\theta_{45} = 333148.22129448433$
$\theta_{46} = -178843.43872993847$
$\theta_{47} = -198087.48171731568$
$\theta_{48} = 85489.45076041944$

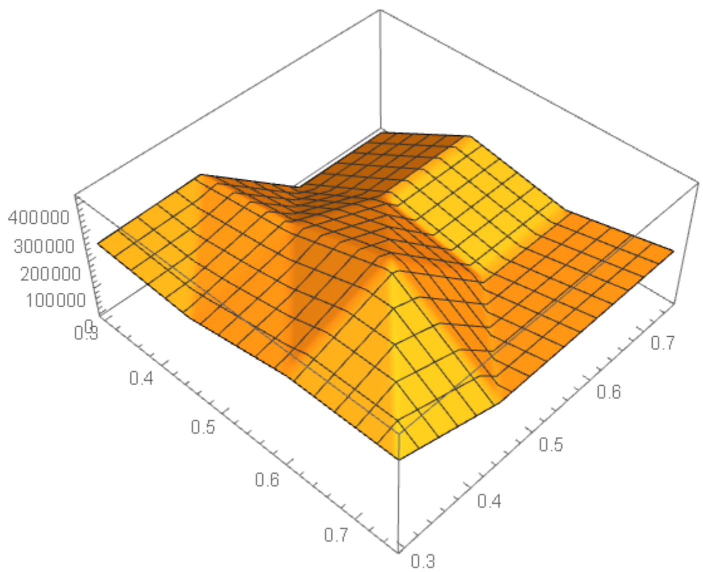
$\theta_{49} = 23049.55145269956$ $\theta_{40} = -179359.75305180202$
$\theta_{51} = -200293.62139168158$ $\theta_{52} = 82636.40038827318$
$\theta_{53} = 20157.041229944673$ $\theta_{54} = -180465.94960577425$
$\theta_{55} = -205556.80096722223$ $\theta_{56} = 75306.14089300479$
$\theta_{57} = 12048.141272602259$ $\theta_{58} = -31523.731118851112$
$\theta_{59} = -31697.68839039283$ $\theta_{60} = -31297.291049577845$
$\theta_{61} = -30510.651315069914$ $\theta_{62} = -30577.26948649386$
$\theta_{63} = -26586.217706801373$ $\theta_{64} = -19116.1681524927$

then we enter the values of θ into the relation (3.3.4) to simulate the function

$$\theta_h(r, \varphi, z, t) = \sum_{i=1}^{N(h)} \theta_i \phi_i(r, \varphi, z, t),$$

and we will prepare the result in the three-dimensional figure (4.6.3)

Figure 4.6.3.



Numerical solution of the transport phenomena and its mathematical simulation, where they have been gotten in the current work, are suitable tools to describe the environment of furnaces and fluid flow inside them. Also they help to designers to determine the optimal position of the electrical boosters. In particular designers analyze the mathematical simulation to decide about the optimized style of the furnaces, then we hope this work is useful for them.

3.10. Summary and Conclusion

In this work we performed the process to get the mathematical modeling of heat transfer in the Garnissage furnace in three dimension in the cylindrical coordinate system. The cylindrical coordinate system has chosen for the modeling process because of its symmetric advantages, then we applied the physical conservation laws, that is the mass, momentum, and energy conservation laws, to achieve the continuity, Navier-Stokes, and heat equation. To modeling the free boundary between the solid and liquid phase we used the three dimensional version of Stefan condition.

When we derived the mathematical modeling of the transport phenomena immediately we started to rearrange the modeling to the weak formulation by handling the sufficient test functions. In this part we divided the domain by cubes and every cubic parts had 24 tetrahedrons. Then we defined the test functions on the tetrahedrons and we got the coefficients after solving the integrals on the mentioned tetrahedrons, and we completed the finite element technic by constructing the system of equations with 128 variables within 64 linear equations and 64 nonlinear equations. Newton's method has been used to achieve the numerical solution of the system and we simulated the numerical solution of the heat transfer system where that was applicable for furnace designers.

Chapter 4.

Elliptic Equations in Unit Disk

4.1. Properly Elliptic Equation

In the final chapter we refer to special kind of boundary value problems, thus we consider the Dirichlet problem for the fourth order elliptic equation with constant coefficients in the unit disc. The characteristic equation has one double root and two simple roots. We consider two cases, first if double root in the upper half-plane and the simple roots in lower-half plane, that is the equation is properly elliptic, and second case if all the roots belong to the upper-half plane, that is improperly elliptic case. The solution must be found in the class of functions Hölder continuous with first order derivatives up to the boundary, then we obtain the new formula for the determination of the defect numbers. The solvability conditions and the solutions of homogeneous and inhomogeneous problems are gained in explicit form. The numerical results shows that the defect numbers may be only zero and one. For improperly elliptic case the set of boundary functions provided the normal solvability of the Dirichlet problem is determined.

4.2. Mathematical Formulation

Let $D = \{(x, y): x^2 + y^2 < 1\}$ be the unit disc in the complex plane, and we consider the fourth order elliptic differential equation in the domain D

$$\sum_{k=0}^{4} A_k \frac{\partial^4 u}{\partial x^k \partial y^{4-k}} (x, y) = 0, \quad (x, y) \in D, \tag{4.2.1}$$

where A_k are the complex constants ($A_0 \neq 0$). We assume that the roots λ_j ($j = 1,2,3,4$) of the characteristic equation

$$\sum_{k=0}^{4} A_k \lambda^{4-k} = 0, \tag{4.2.2}$$

satisfy the condition

$$\begin{gathered} \lambda_1 = \lambda_2 \neq i, \quad \Im\lambda_1 > 0, \\ \lambda_3 \neq \lambda_4, \quad \lambda_j \neq -i, \quad \Im\lambda_j < 0, \quad j = 3,4, \end{gathered} \tag{4.2.3}$$

that is the equation (4.2.1) is properly elliptic. We want to find the solution of the equation (4.2.1) in the class of $C^4(D) \cap C^{(1,\alpha)}(\overline{D})$, which satisfies the Dirichlet conditions

$$u\Big|_{\Gamma} = f(x, y), \quad \frac{\partial u}{\partial N}\Big|_{\Gamma} = g(x, y), \quad (x, y) \in \Gamma, \tag{4.2.4}$$

where Γ is the boundary of D, also $f \in C^{(1,\alpha)}(\Gamma)$ and $g \in C^{(\alpha)}(\Gamma)$ are the given functions, and $\frac{\partial}{\partial N} = -\frac{\partial}{\partial r}$ is a differentiation in the inner normal direction to the boundary Γ, and

$$z = x + iy = re^{i\varphi}$$

It is well known that for the properly elliptic equation (4.2.1) the Dirichlet problem is Fredholmian . We want to determine the defect numbers of the problem

$$\sum_{k=0}^{4} A_k \frac{\partial^4 u}{\partial x^k \partial y^{4-k}} (x,y) = 0, \ \ (x,y) \in D,$$

$$u\bigg|_{\Gamma} = f(x,y), \ \ \frac{\partial u}{\partial N}\bigg|_{\Gamma} = g(x,y), \ \ (x,y) \in \Gamma,$$

that is the number of the linearly independent solutions of the homogeneous problem

$$\sum_{k=0}^{4} A_k \frac{\partial^4 u}{\partial x^k \partial y^{4-k}} (x,y) = 0, \ \ (x,y) \in D,$$

$$u\bigg|_{\Gamma} = 0, \ \ \frac{\partial u}{\partial N}\bigg|_{\Gamma} = 0, \ \ (x,y) \in \Gamma,$$

and the number of the linearly independent solvability conditions of the inhomogeneous problem. In this work we consider the both homogeneous and inhomogeneous problems, when the roots of the equation

$$\sum_{k=0}^{4} A_k \lambda^{4-k} = 0,$$

satisfy the conditions

$$\lambda_1 = \lambda_2 \neq i, \ \ \Im\lambda_1 > 0,$$

$$\lambda_3 \neq \lambda_4, \ \ \lambda_j \neq -i, \ \ \Im\lambda_j < 0, \ \ j = 3,4.$$

We derive the new formula to determining the defect numbers of the problem and we find the solutions in the explicit form of the homogeneous problem and the solvability conditions of inhomogeneous problem. For exact formulation of the obtained results let's represent the equation (4.2.1) and boundary conditions (4.2.3) in the complex form, by applying the operators of complex differentiation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

By forcing the conditions (4.2.3) the equation (4.2.1) can be restated as

$$\left(\frac{\partial}{\partial \bar{z}} - \mu\frac{\partial}{\partial z}\right)^2 \left(\frac{\partial}{\partial z} - \nu_1\frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z} - \nu_2\frac{\partial}{\partial \bar{z}}\right)u(x,y) = 0, \tag{4.2.5}$$

where

$$\mu = \frac{i-\lambda_1}{i+\lambda_1}, \qquad \nu_j = \frac{i+\lambda_{2+j}}{i-\lambda_{2+j}}, \qquad j = 1,2$$

By using the conditions (4.2.3), we have

$$\begin{gathered} |\mu| < 1,\ \ \nu_1 \neq \nu_2, \\ |\nu_j| < 1,\ \ j = 1,2, \qquad \mu\nu_1\nu_2 \neq 0 \end{gathered} \tag{4.2.6}$$

Boundary conditions (4.2.4) are reduced to equivalent form

$$\begin{gathered} \left.\frac{\partial u}{\partial \bar{z}}\right|_{\Gamma} = F(x,y), \\ \left.\frac{\partial u}{\partial z}\right|_{\Gamma} = G(x,y),\ \ (x,y) \in \Gamma, \\ u(1,0) = f(1,0), \end{gathered} \tag{4.2.7}$$

the functions F and G from the class of $C^{(\alpha)}(\Gamma)$ are determined by the formulas

$$\begin{gathered} F(x,y) = \frac{z}{2}\left(g(x,y) + i\frac{\partial f}{\partial \varphi}(x,y)\right), \\ G(x,y) = \frac{\bar{z}}{2}\left(g(x,y) - i\frac{\partial f}{\partial \varphi}(x,y)\right), \\ z = re^{i\varphi} \in \Gamma \end{gathered} \tag{4.2.8}$$

Theorem 4.1. Assume that $\sigma = \mu\nu_1$ and $\tau = \mu\nu_2$, then the problem

$$\sum_{k=0}^{4} A_k \frac{\partial^4 u}{\partial x^k \partial y^{4-k}}\,(x,y) = 0,\;\; (x,y) \in D,$$

$$u\Big|_\Gamma = f(x,y),\;\; \frac{\partial u}{\partial N}\Big|_\Gamma = g(x,y),\;\; (x,y) \in \Gamma,$$

is uniquely solvable if and only if the conditions

$$P_k(\sigma,\tau) = \sum_{m=0}^{k-1}\sum_{p=0}^{m}(m-p)(\sigma\tau)^p \sum_{j=0}^{m-p-1} \sigma^j \tau^{m-p-j} \neq 0, \qquad k = 3,4,\ldots \tag{4.2.9}$$

hold. If the conditions (4.2.9) fail, that is $P_{k_0}(\sigma,\tau) = 0$ for some value $k_0 > 2$, then the homogeneous problem (4.2.1) and (4.2.4) has one linearly independent solution which is polynomial of order $k_0 + 1$. The corresponding inhomogeneous problem has a solution if the boundary functions F and G satisfy one linearly independent orthogonality condition. Therefore, the defect numbers of the problem are equal to the quantity of the numbers k_0 for which $P_{k_0}(\sigma,\tau) = 0$.

Proof of the theorem 4.1. The general solution of the equation (4.2.5) may be represented in the form

$$u(x,y) = \Phi_0(z+\mu\bar{z}) + \frac{\partial}{\partial\varphi}\Phi_1(z+\mu\bar{z}) + \Psi_0(\bar{z}+\nu_1 z) + \Psi_1(\bar{z}+\nu_2 z) \tag{4.2.10}$$

where Φ_j and Ψ_j, $j = 0{,}1$ are the functions, that must be determined which they are analytic in the domains

$$G = \{z+\mu\bar{z} : z \in D\},$$

$$D_j = \left\{\bar{z}+\nu_{j+1} z : z \in D\right\}$$

respectively. We substitute the function (4.2.10) in the boundary equations (4.2.7). By applying the operator identity

$$\frac{\partial^{k+m}}{\partial z^k \partial \bar{z}^m}\frac{\partial^l}{\partial\varphi^l} = \left(\frac{\partial}{\partial\varphi} + (k-m)iI\right)^l \frac{\partial^{k+m}}{\partial z^k \partial \bar{z}^m},$$

we get

$$\mu\Phi_0'(z+\mu\bar{z})+\mu\left(\frac{\partial}{\partial\varphi}-iI\right)\Phi_1'(z+\mu\bar{z})+\Psi_0'(\bar{z}+\nu_1 z)$$
$$+\Psi_1'(\bar{z}+\nu_2 z)=F(z), z\in\Gamma,$$

$$\Phi_0'(z+\mu\bar{z})+\left(\frac{\partial}{\partial\varphi}+iI\right)\Phi_1'(z+\mu\bar{z})+\nu_1\Psi_0'(\bar{z}+\nu_1 z)$$
$$+\nu_2\Psi_1'(\bar{z}+\nu_2 z)=G(z), z\in\Gamma \quad (4.2.11)$$

Now we represent the function $\Omega(z+\mu\bar{z})$ where Ω is analytic in the domain $G=\{z+\mu\bar{z} : z\in D\}$ on the circumference Γ using analytic in D functions. It was proved that the function $\Omega(z+\mu\bar{z})$ for $|z|=1$, could be written as

$$\Omega(z+\mu\bar{z})=\omega(z)+\omega(\mu\bar{z}), \quad (4.2.12)$$

where ω is analytic function in the unit disk. If we have the function ω, then the function Ω is specify as

$$\Omega(\zeta)=\omega\left(\frac{\zeta+\sqrt{\zeta^2-4\mu}}{2}\right)+\omega\left(\frac{\zeta-\sqrt{\zeta^2-4\mu}}{2}\right), \quad (4.2.13)$$

for $\zeta\in G$. In these formulas we choose the branch of $\sqrt{\zeta^2-4\mu}$, which it is analytic outside the segment $[-2\sqrt{\mu},2\sqrt{\mu}]$ and satisfies the condition

$$\lim_{\zeta\to\infty}\zeta^{-1}\sqrt{\zeta^2-4\mu}=1$$

We use (4.2.12) and rewrite the functions $\Phi_j',\ \Psi_j'$ on the circumference Γ

$$\Phi_j'(z+\mu\bar{z})=\vartheta_j(z)+\vartheta_j(\mu\bar{z})=\sum_{k=0}^{\infty}A_{kj}\,z^k+\sum_{k=0}^{\infty}A_{kj}\mu^k\,z^{-k},$$
$$\Psi_j'\big(\bar{z}+\nu_{j+1}z\big)=\rho_j(\bar{z})+\rho_j\big(\nu_{j+1}z\big)=\sum_{k=0}^{\infty}B_{kj}\,z^{-k}+\sum_{k=0}^{\infty}B_{kj}\nu_{j+1}^k z^k, \quad (4.2.14)$$
$$j=0,1,\ \ z\in\Gamma.$$

We want to determine unknown functions ϑ_j and ρ_j, that these functions are analytic in the unit disc D, therefore they determined by the corresponding Taylor coefficients A_{kj} and B_{kj}. For the determination of these coefficients let's substitute the expansions (4.2.14) and Fourier expansions of the functions F and G

$$F(z)=\sum_{k=0}^{\infty}F_k\,z^k+\sum_{k=1}^{\infty}F_{-k}\,z^{-k}=F_+(z)+F_-(z),$$

$$G(z)=\sum_{k=0}^{\infty}G_k\,z^k+\sum_{k=1}^{\infty}G_{-k}\,z^{-k}=G_+(z)+G_-(z), \tag{4.2.15}$$

in the boundary equations (4.2.11). We get

$$\sum_{k=0}^{\infty}A_{k0}\mu\,z^k+\sum_{k=0}^{\infty}A_{k0}\mu^{k+1}\,z^{-k}+\sum_{k=0}^{\infty}A_{k1}(ik-i)\mu\,z^k-$$

$$\sum_{k=0}^{\infty}A_{k1}(ik+i)\mu^{k+1}\,z^{-k}+\sum_{k=0}^{\infty}B_{k0}\,z^{-k}+\sum_{k=0}^{\infty}B_{k0}\nu_1^k\,z^k+$$

$$\sum_{k=0}^{\infty}B_{k1}\,z^{-k}+\sum_{k=0}^{\infty}B_{k1}\nu_2^k\,z^k=\sum_{k=0}^{\infty}F_k\,z^k+\sum_{k=1}^{\infty}F_{-k}\,\bar{z}^k,$$

$$|z|=1,$$

$$\tag{4.2.16}$$

$$\sum_{k=0}^{\infty}A_{k0}\,z^k+\sum_{k=0}^{\infty}A_{k0}\mu^k\,z^{-k}+\sum_{k=0}^{\infty}A_{k1}(ik+i)\,z^k$$

$$+\sum_{k=0}^{\infty}A_{k1}(i-ik)\mu^k\,z^{-k}+\sum_{k=0}^{\infty}B_{k0}\nu_1\,z^{-k}+\sum_{k=0}^{\infty}B_{k0}\nu_1^{k+1}\,z^k+$$

$$\sum_{k=0}^{\infty}B_{k1}\nu_2\,z^{-k}+\sum_{k=0}^{\infty}B_{k1}\nu_2^{k+1}\,z^k=\sum_{k=0}^{\infty}G_k\,z^k+\sum_{k=1}^{\infty}G_{-k}\,\bar{z}^k,$$

$$|z|=1.$$

Equating the coefficients by the same degrees of z and $\bar{z}$, we obtain the system for the determination of the unknown coefficients A_{kj} and B_{kj}. For $k=0$ we have

$$2\mu A_{00}-2\mu iA_{01}+2B_{00}+2B_{01}=F_0,$$

$$2A_{00}+2iA_{01}+2\nu_1B_{00}+2\nu_2B_{01}=G_0 \tag{4.2.17}$$

If $k \geq 1$ we have fourth order system of linear equations for the determination A_{kj} and B_{kj}

$$\begin{cases} A_{k0} + i(k+1)A_{k1} + \nu_1^{k+1}B_{k0} + \nu_2^{k+1}B_{k2} = G_k \\ \mu A_{k0} + i(k-1)\mu A_{k1} + \nu_1^k B_{k0} + \nu_2^k B_{k2} = F_k \\ \mu^k A_{k0} - i(k-1)\mu^k A_{k1} + \nu_1 B_{k0} + \nu_2 B_{k2} = G_{-k} \\ \mu^{k+1} A_{k0} - i(k+1)\mu^{k+1} A_{k1} + B_{k0} + B_{k2} = F_{-k} \end{cases} \tag{4.2.18}$$

We consider the determinant of the main matrix of the system (4.2.18)

$$S_k = det\widetilde{S_k} = det\begin{pmatrix} 1 & i(k+1) & \nu_1^{k+1} & \nu_2^{k+1} \\ \mu & i(k-1)\mu & \nu_1^k & \nu_2^k \\ \mu^k & -i(k-1)\mu^k & \nu_1 & \nu_2 \\ \mu^{k+1} & -i(k+1)\mu^{k+1} & 1 & 1 \end{pmatrix} \tag{4.2.19}$$

After transformation, using denotations of the theorem 4.1, we reach to

$$S_k(\sigma,\tau) = i\begin{vmatrix} 1 & k+1 & \sigma^{k+1} & \tau^{k+1} \\ 1 & k-1 & \sigma^k & \tau^k \\ 1 & -k+1 & \sigma & \tau \\ 1 & -k-1 & 1 & 1 \end{vmatrix}$$

$$= i(1-\sigma)^2(1-\tau)^2 \sum_{m=0}^{k-1}\sum_{p=0}^{m}(m-p)(\sigma^p\tau^m - \sigma^m\tau^p) \tag{4.2.20}$$

$$= i(1-\sigma)^2(1-\tau)^2\Theta_k(\sigma,\tau)$$

And finally, the function Θ_k could be restated as

$$\Theta_k(\sigma,\tau) = (\tau-\sigma)P_k(\sigma,\tau),$$

where P_k is a function that was defined in the theorem 4.1. According to the conditions of the theorem 4.1, $\sigma \neq 1$, $\tau \neq 1$, and $\sigma \neq \tau$, then for $k > 2$ we have $S_k \neq 0$ if and only if the conditions (4.2.9) hold. Since S_2 is a generalized Vandermonde determinant with different terms then $S_2 \neq 0$.

Let us assume that the conditions (4.2.9) hold, then the coefficients A_{kj} and B_{kj} for $k \geq 2$ are uniquely determined. Determinant S_1 of the system (4.2.18) for $k = 1$ is equal to zero, because second and third rows are the same, but by taking into account the relations (4.2.8) we have

$$F_1 = G_{-1} = \frac{1}{4\pi}\int_0^{2\pi} g(\cos\varphi, \sin\varphi)d\varphi,$$

that is the system (4.2.18) has a solution (not unique) for $k = 1$, and at last we determine (not uniquely too) the coefficients A_{0j} and B_{0j} from the system (4.2.17), Thus the coefficients A_{kj} and B_{kj} may be found for arbitrary k, therefore we earn the functions ϑ_j and ρ_j , when we compute the Φ_j and Ψ_j by the relation (4.2.13).

Since

$$\lim_{k\to\infty} S_k = -2i,$$

hence the coefficients A_{kj} and B_{kj} have the same rate of decreasing as the coefficients F_k and G_k, therefore the resulting function

$$u(x,y) = \Phi_0(z + \mu\bar{z}) + \frac{\partial}{\partial\varphi}\Phi_1(z + \mu\bar{z}) + \Psi_0(\bar{z} + \nu_1 z) + \Psi_1(\bar{z} + \nu_2 z),$$

that is the solution of the problem (4.2.1), (4.2.4), belongs to the prescribe class of $C^{(1,\alpha)}(\Gamma)$. Now let's consider the homogeneous problem

$$\sum_{k=0}^{4} A_k \frac{\partial^4 u}{\partial x^k \partial y^{4-k}}\ (x,y) = 0,\ \ (x,y) \in D,$$

$$u\Big|_\Gamma = 0,\ \ \frac{\partial u}{\partial N}\Big|_\Gamma = 0,\ \ (x,y) \in \Gamma,$$

if the conditions (4.2.9) hold, the corresponding Taylor coefficients $A_{kj} = B_{kj} = 0$ for $k > 1$, then nontrivial solution of the homogeneous problem (4.2.1), (4.2.4) may be at last second order polynomial, but the theorem 4.1 from [8] (p. 84) implies that every nontrivial polynomial which satisfies homogeneous conditions (4.2.4) must be divisible by $(1 - z\bar{z})^2$, that is it must has at least fourth order, hence if the conditions (4.2.9) hold, then the corresponding homogeneous problem has only zero solution.

If the conditions (4.2.9) fail for some $k_0 > 2$, that is $S_{k_0} = 0$, then the corresponding homogeneous problem has one linearly independent solution which is determined by nonzero solution $A_{k_0 j}$ and $B_{k_0 j}$ of the system (4.2.18).

4.3. Some Numerical Results

In this stage we will investigate more about the conditions (4.2.9). First we start by (4.2.5) and we gain

$$\left(P_3(\sigma,\tau)\frac{\partial^4}{\partial z^2\partial \bar{z}^2}+E\frac{\partial^4}{\partial z\partial \bar{z}^3}+H\frac{\partial^4}{\partial z^3\partial \bar{z}}+\mu^2\frac{\partial^4}{\partial z^4}+\nu_1\nu_2\frac{\partial^4}{\partial \bar{z}^4}\right)u=0, \quad (4.3.1)$$

where $P_3(\sigma,\tau)$ defined in (4.2.9). For $k=3$

$$P_3(\sigma,\tau)=1+2(\sigma+\tau)+\sigma\tau=1+2\mu(\nu_1+\nu_2)+\mu^2\nu_1\nu_2,$$

and the constants E and H are as

$$E=-(\nu_1+\nu_2+2\mu\nu_1\nu_2)$$

$$H=-\left(2\mu+\mu^2(\nu_1+\nu_2)\right)$$

Let's illustrate the results of the theorem (4.1) in the cases of $k=3,4$. We will refer to the homogeneous problem and by invoking the homogeneous conditions (4.2.4) we know that the seeking solution must be divisible by $(1-z\bar{z})^2$. Suppose that the function $u_0(z,\bar{z})=\alpha(1-z\bar{z})^2$, $\alpha\neq 0$ is a solution of the homogeneous problem (4.2.1), (4.2.4), this function satisfies the homogeneous boundary conditions (4.2.4). By substituting the function in the equation (4.3.1), we derive

$$P_3(\sigma,\tau)4\alpha=0$$

So the function u_0 is non-zero solution of the homogeneous problem (4.2.1), (4.2.4) if and only if $P_3(\sigma,\tau)=0$. For example, if the constants μ, ν_1, and ν_2 satisfy the conditions

$$\mu\nu_1=\sigma=-\frac{2}{3},$$

$$\mu\nu_2=\tau=-\frac{1}{4},$$

then the function u_0 is a non-zero solution of the homogeneous problem (4.2.1), (4.2.4). Now let us state the non-zero solution of the homogeneous problem (4.2.1), (4.2.4) in the form of

$$u_1(z,\bar{z})=(1-z\bar{z})^2(\beta z+\gamma\bar{z}),$$

$$|\beta|+|\gamma|\neq 0$$

After inserting u_1 function in the equation (4.3.1) we gain

$$P_3(\sigma,\tau)(12\beta z+12\gamma\bar{z})+12E\gamma z+12H\beta\bar{z}=0$$

This equality holds if and only if β and γ satisfy the system

$$P_3(\sigma,\tau)\beta + E\gamma = 0,$$
$$H\beta + P_3(\sigma,\tau)\gamma = 0 \tag{4.3.2}$$

This homogeneous system of linear equations has non-zero solution if and only if the determinant of the main matrix of this system is equal to zero. By using denotations of the theorem 4.1, $\sigma = \mu\nu_1, \tau = \mu\nu_2$ this determinant could be written as

$$P_3^2(\sigma,\tau) - EH = (1 + 2(\sigma+\tau) + \sigma\tau)^2 - (\sigma+\tau+2)(\sigma+\tau+2\sigma\tau) = P_4(\sigma,\tau),$$

thus we achieve that u_1 is the non-zero solution of the homogeneous problem (4.2.1), (4.2.4) if and only if $P_4(\sigma,\tau) = 0$. Let us show that P_4 may be equal zero for σ and τ such that

$$|\sigma| < 1, \quad |\tau| < 1$$

First we represent

$$P_4(\sigma,\tau) = 1 + 2(\sigma+\tau) + 3\sigma^2 + 4\sigma\tau + 3\tau^2 + 2\sigma\tau(\sigma+\tau) + \sigma^2\tau^2,$$

in the style of

$$P_4(\sigma,\tau) = 1 + 2\sigma + 3\sigma^2 + 2\tau(1 + 2\sigma + \sigma^2) + \tau^2(3 + 2\sigma + \sigma^2)$$
$$= E_0 + E_1\tau + E_2\tau^2, \tag{4.3.3}$$

then fix arbitrary σ, $|\sigma| < 1$ and consider second order polynomial

$$P(\tau) = E_0 + E_1\tau + E_2\tau^2$$

We will use Schur transform to show that two roots of this polynomial lie in the unit disc. We have

$$P^*(\tau) = \tau^2\overline{p\left(\frac{1}{\bar{\tau}}\right)} = \bar{E}_2 + \bar{E}_1\tau + \bar{E}_0\tau^2,$$

therefore Schur transform Tp of the polynomial (4.3.3) is as

$$Tp(\tau) = \bar{E}_0 p(\tau) - E_2 P^*(\tau) = (\bar{E}_0 E_1 - E_2\bar{E}_1)\tau + |E_0|^2 - |E_2|^2$$

Let us calculate the constants

$$\gamma_1 = Tp(0) = |E_0|^2 - |E_2|^2,$$

$$\gamma_2 = T^2 p(0) = (|E_0|^2 - |E_2|^2)^2 - |\bar{E}_0 E_1 - E_2\bar{E}_1|^2$$

We apply the identity $\sigma = \rho e^{i\varphi}$ to reach

$$\gamma_1 = -8(1-\rho^2)^2(1+\rho^2+\rho\cos\varphi),$$

$$\gamma_2 = 16(1-\rho^2)^2(3+3\rho^2+4\rho(1+\rho^2)\cos\varphi - 4\rho^2\sin^2\varphi)$$

Taking into account inequality $0 < \rho < 1$, we obtain $\gamma_1 < 0$ and $\gamma_2 > 0$, therefore we will have that all two roots of the polynomial (4.3.3) lie in the unit disc. For example, by assumption $\sigma = 0.5$ we achieve $E_0 = \frac{11}{4}$, $E_1 = \frac{9}{2}$, and $E_2 = \frac{17}{4}$. Also the roots of the polynomial (4.3.3) are equal to

$$\tau_{1,2} = \frac{-9 \pm i\sqrt{108}}{17},$$

and then $|\tau_{1,2}| < 1$, thus in this case if

$$\mu\nu_1 = 0.5,$$

$$\mu\nu_2 = \frac{-9 + i\sqrt{108}}{17},$$

then the function u_1 is a non-zero solution of the homogeneous problem (4.2.1), (4.2.4), where β and γ is nontrivial solution of the system (4.3.2).

We can continue the process in the same approach, for example

$$u_2(z,\bar{z}) = (1-z\bar{z})^2(\beta z^2 + \gamma\bar{z}^2 + \delta z\bar{z}),$$

$$|\beta| + |\gamma| + |\delta| \neq 0,$$

is a non-zero solution of homogeneous problem (4.2.1) and (4.2.4) if and only if

$$P_5(\sigma,\tau) = 0$$

We applied the Mathematica software to find the concrete values of the defect numbers for the different values of σ and τ. The calculation based on computer programming, showed that the defect numbers may be equal to zero or one only.

4.4. Mathematica Codes

```
{{1, k + 1, (z1 / m) ^ (k + 1), (z2 / m) ^ (k + 1)}, {m, (k - 1) * m, (z1 / m) ^k, (z2 / m) ^k},
  {m^k, (-k + 1) * m^k, (z1 / m), (z2 / m)}, {m^ (k + 1), (-k - 1) * m^ (k + 1), 1, 1}};

MatrixForm[%]
```

$$\begin{pmatrix} 1 & 1+k & \left(\frac{z1}{m}\right)^{1+k} & \left(\frac{z2}{m}\right)^{1+k} \\ m & (-1+k)\, m & \left(\frac{z1}{m}\right)^{k} & \left(\frac{z2}{m}\right)^{k} \\ m^k & (1-k)\, m^k & \frac{z1}{m} & \frac{z2}{m} \\ m^{1+k} & (-1-k)\, m^{1+k} & 1 & 1 \end{pmatrix}$$

First Stage

```
k = 1;

Factor[Det[{{1, k + 1, (z1 / m) ^ (k + 1), (z2 / m) ^ (k + 1)}, {m, (k - 1) * m, (z1 / m) ^k, (z2 / m) ^k},
   {m^k, (-k + 1) * m^k, (z1 / m), (z2 / m)}, {m^ (k + 1), (-k - 1) * m^ (k + 1), 1, 1}}]]

0
```

Second Stage

```
k = 2;

Factor[I * Det[{{1, k + 1, (z1 / m) ^ (k + 1), (z2 / m) ^ (k + 1)}, {m, (k - 1) * m, (z1 / m) ^k, (z2 / m) ^k},
    {m^k, (-k + 1) * m^k, (z1 / m), (z2 / m)}, {m^ (k + 1), (-k - 1) * m^ (k + 1), 1, 1}}]]
```

$-2\,i\,(-1+z1)^2\,(z1-z2)\,(-1+z2)^2$

```
k = 3;

Factor[I * Det[{{1, k + 1, (z1 / m) ^ (k + 1), (z2 / m) ^ (k + 1)}, {m, (k - 1) * m, (z1 / m) ^k, (z2 / m) ^k},
    {m^k, (-k + 1) * m^k, (z1 / m), (z2 / m)}, {m^ (k + 1), (-k - 1) * m^ (k + 1), 1, 1}}]]
```

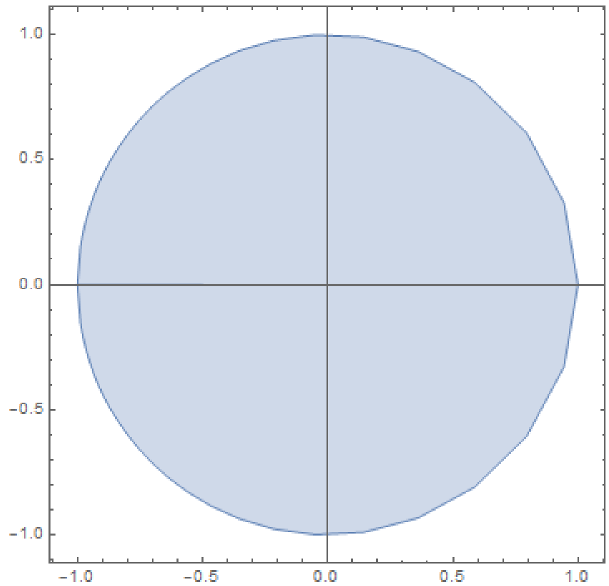
$-2\,i\,(-1+z1)^2\,(z1-z2)\,(-1+z2)^2\,(1+2\,z1+2\,z2+z1\,z2)$

Fourth Stage

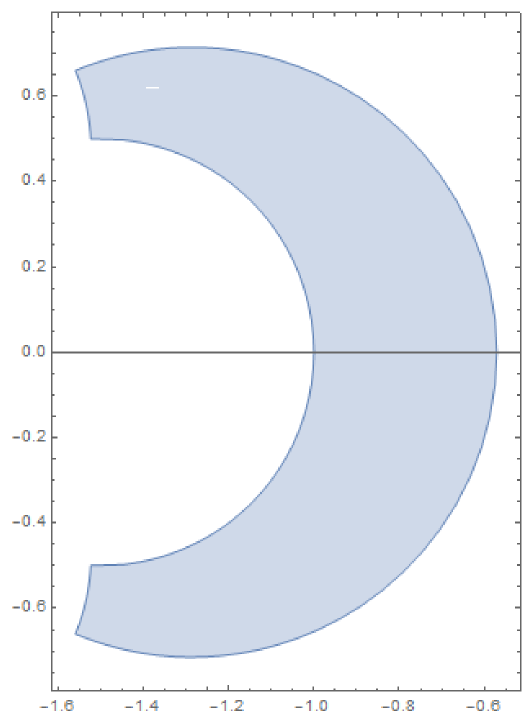
```
k = 4;

Factor[I * Det[{{1, k + 1, v1^ (k + 1), v2^ (k + 1)}, {m, (k - 1) * m, v1^k, v2^k},
    {m^k, (-k + 1) * m^k, v1, v2}, {m^ (k + 1), (-k - 1) * m^ (k + 1), 1, 1}}]]
```

$-2\,i\,(-1+z1)^2\,(z1-z2)\,(-1+z2)^2\,\left(1+2\,z1+3\,z1^2+2\,z2+4\,z1\,z2+2\,z1^2\,z2+3\,z2^2+2\,z1\,z2^2+z1^2\,z2^2\right)$

```
w[z_] := (-1 - 2 z) / (2 + z)
ParametricPlot[{Re[w[z]], Im[w[z]]} /. z → r * Cos[t] + I * r * Sin[t],
 {t, 0, 2 Pi}, {r, 0, 1}, PlotRange → All]
```




```
w[z_] := (-1 - 2 z) / (2 + z)
p2 = ParametricPlot[{Re[w[z]], Im[w[z]]} /. z → x + I * y, {x, 1 / 10, 1}, {y, -Pi, Pi}, PlotRange → All]
```



4.5. Improperly Elliptic Equations

In the final part we review and consider the case of improperly elliptic equation. The consideration are similar to the case of the section (4.2), therefore we omit the same details and we introduce this equation immediately in the form of complex variables as

$$\left(\frac{\partial}{\partial\bar{z}}-\mu_0\frac{\partial}{\partial z}\right)^2\left(\frac{\partial}{\partial\bar{z}}-\mu_1\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial\bar{z}}-\mu_2\frac{\partial}{\partial z}\right)u=0, \qquad (4.5.1)$$

where

$$\mu_k\neq\mu_j\,,0<|\mu_j|<1,$$

with Dirichlet boundary conditions

$$\left.\frac{\partial u}{\partial\bar{z}^{1-j}\,\partial z^j}\right|_{\Gamma}=F_j(x,y)\,,(x,y)\in\Gamma\,,j=0{,}1, \qquad (4.5.2)$$

$$u(1{,}0)=f_0(1),u_r(1{,}0)=f_1(1),u_\theta(1{,}0)=f_0(1).$$

We seek the solution of problem (4.5.1) and (4.5.2) in the class of $C^{(1,\alpha)}(\mathcal{D}\cup\Gamma)\cap C^4(\mathcal{D})$. The general solution of (4.5.1) is as following

$$u=\phi_0(z+\mu_0\bar{z})+\frac{\partial}{\partial\theta}\phi_1(z+\mu_0\bar{z})+\phi_2(z+\mu_1\bar{z})+\phi_3(z+\mu_2\bar{z}), \qquad (4.5.3)$$

where ϕ_0 and ϕ_1 are analytic functions in the domain

$$D(\mu_0)=\{z+\mu_0\bar{z}:z\in D\},$$

also ϕ_2 and ϕ_3 are analytic functions respectively in the domains

$$D(\mu_1)=\{z+\mu_1\bar{z}:z\in D\},$$

$$D(\mu_2)=\{z+\mu_2\bar{z}:z\in D\}.$$

By inserting the last general solution in the boundary condition (4.5.2) we earn

$$\phi_0'(z+\mu_0\bar{z})\mu_0+\left(\frac{\partial}{\partial\theta}-iI\right)\phi_1'(z+\mu_0\bar{z})\mu_0+\phi_2'(z+\mu_1\bar{z})\mu_1 + \phi_3'(z+\mu_2\bar{z})\mu_2=F_0(z,\bar{z})\,,\qquad |z|=1,$$

$$\phi_0'(z+\mu_0\bar{z})+\left(\frac{\partial}{\partial\theta}+iI\right)\phi_1'(z+\mu_0\bar{z})+\phi_2'(z+\mu_1\bar{z}) + \phi_3'(z+\mu_2\bar{z})=F_1(z,\bar{z}) \qquad (4.5.4)$$

For $|z| = 1$, $k = 1, 2, 3$ suppose that

$$\phi_0'(z + \mu_0 \bar{z}) = \varphi_0(z) + \varphi_0(\mu_0 \bar{z}) = \sum_{j=0}^{\infty} A_{0j} z^j + \sum_{j=0}^{\infty} A_{0j} \mu_0^j z^{-j},$$

$$\phi_k'(z + \mu_{k-1} \bar{z}) = \varphi_k(z) + \varphi_k(\mu_{k-1} \bar{z}) = \sum_{j=0}^{\infty} A_{kj} z^j + \sum_{j=0}^{\infty} A_{kj} \mu_{k-1}^j z^{-j}$$

We insert the value of ϕ_0' and ϕ_k' into (4.5.3) and (4.5.4) to get

$$\sum_{j=0}^{\infty} A_{0j} z^j \mu_0 + \sum_{j=0}^{\infty} A_{0j} \mu_0^{j+1} z^{-j} + \sum_{j=0}^{\infty} A_{1j} (ij - i) z^j \mu_0 + \sum_{j=0}^{\infty} A_{1j} (-ij - i) \mu_0^{j+1} z^{-j} + \sum_{j=0}^{\infty} A_{2j} \mu_1 z^j + \sum_{j=0}^{\infty} A_{2j} \mu_1^{j+1} z^{-j} + \sum_{j=0}^{\infty} A_{3j} \mu_2 z^j + \sum_{j=0}^{\infty} A_{3j} \mu_2^{j+1} z^{-j} = \sum_{j=-\infty}^{\infty} d_{j1} z^j,$$

$$\sum_{j=0}^{\infty} A_{0j} z^j + \sum_{j=0}^{\infty} A_{0j} \mu_0^j z^{-j} + \sum_{j=0}^{\infty} A_{1j} (ij + i) z^j + \sum_{j=0}^{\infty} A_{1j} (-ij + i) \mu_0^j z^{-j} + \sum_{j=0}^{\infty} A_{2j} z^j + \sum_{j=0}^{\infty} A_{2j} \mu_1^j z^{-j} + \sum_{j=0}^{\infty} A_{3j} z^j + \sum_{j=0}^{\infty} A_{3j} \mu_2^j z^{-j} = \sum_{j=-\infty}^{\infty} d_{j2} z^j$$

Now from equality of coefficients of same orders of z and $\bar{z}$ we get the following system

$$\begin{cases} A_{0j} + i(j+1) A_{1j} + A_{2j} + A_{3j} = d_{j2} \\ \mu_0 A_{0j} + i(j-1) \mu_0 A_{1j} + \mu_1 A_{2j} + \mu_2 A_{3j} = d_{j1} \\ \mu_0^j A_{0j} - i(j-1) \mu_0^j A_{1j} + \mu_1^j A_{2j} + \mu_2^j A_{3j} = d_{-j2} \\ \mu_0^{j+1} A_{0j} - i(j+1) \mu_0^{j+1} A_{1j} + \mu_1^{j+1} A_{2j} + \mu_2^{j+1} A_{3j} = d_{-j1} \end{cases} \tag{4.5.5}$$

Define Δ_j the determinant of the matrix of coefficients of the system (4.5.5)

$$\Delta_j = \begin{vmatrix} 1 & i(j+1) & 1 & 1 \\ \mu_0 & i(j-1) & \mu_1 & \mu_2 \\ \mu_0^j & -i(j-1)\mu_0^j & \mu_1^j & \mu_2^j \\ \mu_0^{j+1} & -i(j+1)\mu_0^{j+1} & \mu_1^{j+1} & \mu_2^{j+1} \end{vmatrix}$$

First case.
Assume that $|\mu_0| > |\mu_j|, j = 1,2$, by the assumptions

$$\sigma = \frac{\mu_1}{\mu_0}, \tau = \frac{\mu_2}{\mu_0},$$

we will have $0 < |\sigma| < 1, 0 < |\tau| < 1, |\sigma| \neq |\tau|$, then we will rewrite Δ_j as

$$\Delta_j = \mu_0^{2j+2} i \begin{vmatrix} 1 & j+1 & 1 & 1 \\ 1 & j-1 & \sigma & \tau \\ 1 & -j+1 & \sigma^j & \tau^j \\ 1 & -j-1 & \sigma^{j+1} & \tau^{j+1} \end{vmatrix} = -\mu_0^{2j+2} S_j(\sigma,\tau),$$

where S_j is determined from (4.2.20). Thus in this case we get the result similar to the case of properly elliptic equation for homogeneous problem (4.5.1) and (4.5.2).

Let us consider inhomogeneous problem. By dividing both parts of the equations in the system (5.5.5) by corresponding powers of μ_0 and after defining $\tilde{A}_{1_j} = iA_{1_j}$, we get

$$\begin{cases} A_{0j} + (j+1)\tilde{A}_{1j} + A_{2j} + A_{3j} = d_{j2} \\ A_{0j} + (j-1)\tilde{A}_{1j} + \sigma A_{2j} + \tau A_{3j} = d_{j1}\mu_0^{-1} \\ A_{0j} - (j-1)\tilde{A}_{1j} + \sigma^j A_{2j} + \tau^j A_{3j} = d_{-j2}\mu_0^{-j} \\ A_{0j} - (j+1)\tilde{A}_{1j} + \sigma^{j+1} A_{2j} + \tau^{j+1} A_{3j} = d_{-j1}\mu_0^{-j} \end{cases}$$

We apply the Gauss elimination process to reduce the system into the form

$$\begin{cases} A_{0j} + (j+1)\tilde{A}_{1j} + A_{2j} + A_{3j} = d_{j2} \\ \quad -2\tilde{A}_{1j} + (\sigma-1)A_{2j} + (\tau-1)A_{3j} = d_{j1}\mu_0^{-1} - d_{j2} \\ \quad -2j\tilde{A}_{1j} + \left(\sigma^j - 1\right)A_{2j} + \left(\tau^j - 1\right)A_{3j} = d_{-j1}\mu_0^{-j} - d_{j2} \\ \quad -2(j+1)\tilde{A}_{1j} + \left(\sigma^{j+1} - 1\right)A_{2j} + \left(\tau^{j+1} - 1\right)A_{3j} = d_{-j1}\mu_0^{-j} - d_{j2} \end{cases}$$

Now we denote

$$\begin{cases} A_{0j} + (j+1)\tilde{A}_{1j} + A_{2j} + A_{3j} = d_{j2} \\ -2\tilde{A}_{1j} + (\sigma-1)A_{2j} + (\tau-1)A_{3j} = d_{j1}\mu_0^{-1} - d_{j2} \\ \left(\sigma^j - 1 - j(\sigma-1)\right)A_{2j} + \left(\tau^j - 1 - j(\tau-1)\right)A_{3j} = D_j \\ \left(\sigma^{j+1} - 1 - (j+1)(\sigma-1)\right)A_{2j} + \left(\tau^{j+1} - 1 - (j+1)(\tau-1)\right)A_{3j} = E_j \end{cases} \tag{4.5.6}$$

where

$$D_j = d_{-j2}\mu_0^{-j} - d_{j2} - j\left(d_{j1}\mu_0^{-1} - d_{j2}\right),$$
$$E_j = d_{-j1}\mu_0^{-j} - d_{j2} - (j+1)\left(d_{j1}\mu_0^{-1} - d_{j2}\right) \tag{4.5.7}$$

We solve the last two equations of the system (4.5.6) and derive

$$A_{2j} = \frac{1}{Z_j}\left(D_j\left(\tau^{j+1} - 1 - (j+1)(\tau-1)\right) - E_j\left(\tau^j - 1 - j(\tau-1)\right)\right),$$
$$A_{3j} = \frac{1}{Z_j}\left(E_j\left(\sigma^j - 1 - j(\sigma-1)\right) - D_j\left(\sigma^{j+1} - 1 - (j+1)(\sigma-1)\right)\right), \tag{4.5.8}$$

where

$$Z_j = \begin{vmatrix} \sigma^j - 1 - j(\sigma-1) & \tau^j - 1 - j(\tau-1) \\ \sigma^{j+1} - 1 - (j+1)(\sigma-1) & \tau^{j+1} - 1 - (j+1)(\tau-1) \end{vmatrix} \tag{4.5.9}$$

Taking into account that $|\sigma| < 1$ and $|\tau| < 1$, we gain for $j \to \infty$

$$Z_j \sim \sigma - \tau \neq 0$$

Finally we have

$$A_{2j} \sim c_j d_{-j_k}\mu_0^{-j},$$

$$A_{3j} \sim c_j d_{-j_k}\mu_0^{-j}, \qquad k = 1,2,$$

and the same estimation holds for A_{0j} and A_{1j}, so if we suppose the function F_k with first order derivatives satisfies Hölder condition in closed ring $|\mu_0| < |z| < 1$, then the function ϕ_k will belong to the class $C^{(1,\alpha)}(\overline{D})$ and, therefore the function u will be in the prescribed class. Now we can summarize previous considerations.

Definition. We denote $A^{(1,\alpha)}(r)$ class of the functions, which satisfy Hölder condition with first order derivatives in the closed ring $r \leq |z| \leq 1$.

Theorem 4.4.1. Assume that $|\mu_0| > |\mu_j|, j = 1, 2$, and the boundary functions F_j belong to the class $A^{(1,\alpha)}(|\mu_0|)$, then the problem (4.5.1) and (4.5.2) is uniquely solvable if and only if the following conditions hold.

$$P_j(\sigma,\tau) = \sum_{k=1}^{j-1}\sum_{l=0}^{k-1}(k-l)(\sigma\tau)^l \sum_{m=0}^{k-l-1} \tau^m \sigma^{k-l-m-1} \ , j = 3,4,..., \tag{4.5.11}$$

where

$$\sigma = \frac{\mu_1}{\mu_0}, \qquad \tau = \frac{\mu_2}{\mu_0}, \qquad |\sigma| \neq |\tau|.$$

If the conditions (4.5.11) fail, then homogenous problem has finite number of linearly independent solutions, and for $F_j \in A^{(1,\alpha)}(|\mu_0|)$ inhomogeneous problem has a solution if and only if the finite number of linearly independent solvability conditions to boundary functions F_j hold. These defect numbers are equal to the quantity of numbers j, for which the conditions (4.5.11) fail.

Now we consider the case $|\sigma| = |\tau|$, that is

$$\tau = e^{i\alpha}\sigma, \ \ \alpha \in \mathbb{R}$$

In this case the polynomial P_j maybe reduced to the form of

$$P_j(\sigma,\tau) = \sum_{k=1}^{j-1}\sum_{l=0}^{k-1}(k-l)\sigma^{k+l-1}e^{i\alpha l}\sum_{m=0}^{k-l-1} e^{i\alpha m}$$

$$= \sum_{k=1}^{j-1}\sum_{l=0}^{k-1}(k-l)\sigma^{k+l-1}e^{i\alpha l}\,\frac{e^{i(k-l)\alpha}-1}{e^{i\alpha}-1}$$

$$= \sum_{k=1}^{j-1}\sum_{l=0}^{k-1}(k-l)\sigma^{k+l-1}\,\frac{e^{i\alpha k}-e^{i\alpha l}}{e^{i\alpha}-1}$$

Now define

$$z = \sigma e^{i\frac{\alpha}{2}},$$

then P_j is represented in the form of

$$\Lambda_j(z) \equiv \frac{\left(e^{i\alpha}-1\right)}{2ie^{\frac{i\alpha}{2}}}P_j = \sum_{k=1}^{j-1}\sum_{l=0}^{k-1}(k-l)z^{k+l-1}\sin\left(\frac{k-l}{2}\right)\alpha,$$

or

$$\Lambda_j(z) \equiv \sin\left(\frac{\alpha}{2}\right) P_j = \sum_{k=1}^{j-1} \sum_{l=0}^{k-1} (k-l) z^{k+l-1} \sin\left(\frac{k-l}{2}\right) \alpha.$$

Also define

$$p = k - l, \qquad q = k + l,$$

then

$$k = \frac{p+q}{2}, \qquad l = \frac{q-p}{2},$$

where

$$1 \le \frac{p+q}{2} \le j-1, \qquad 0 \le \frac{q-p}{2} \le \frac{p+q}{2} - 1,$$

and

$$q \ge p\,, p \ge 1, \qquad 1 \le p \le j-1,$$

$$(k,l): (p,0), (p+1,1), \dots, (j-1, j-1-p),$$

$$k+l: p+0, p+2, p+4, \dots, p+2(j-1-p),$$

$$\Lambda_j(z) = \sum_{p=1}^{j-1} \sum_{s=0}^{j-1-p} p \sin\left(\frac{p\alpha}{2}\right) z^{p+2s-1}$$

$$= \sum_{p=1}^{j-1} p \sin\left(\frac{p\alpha}{2}\right) z^{p-1} \sum_{s=0}^{j-1-p} z^{2s}$$

$$= \sum_{p=1}^{j-1} p \sin\left(\frac{p\alpha}{2}\right) z^{p-1} \frac{z^{2j-2p} - 1}{z^2 - 1}$$

$$= \sum_{p=1}^{j-1} p \sin\left(\frac{p\alpha}{2}\right) \frac{z^{2j-p-1} - z^{p-1}}{z^2 - 1}$$

$$= z^{j-2} \sum_{p=1}^{j-1} p \sin\left(\frac{p\alpha}{2}\right) \frac{z^{j-p} - z^{p-j}}{z - z^{-1}}.$$

Thus we get the following result.

Proposition. We consider the problem (4.5.1) and (4.5.2) when $\mu_2 = e^{i\alpha}\mu_1$ and $|\mu_0| > |\mu|$. In this this case we get the result similar to theorem (4.4.1), where instead of P_j from (4.5.11) we have

$$Q_j(z) = z^{j-2} \sum_{p=1}^{j-1} p \sin\left(\frac{p\alpha}{2}\right) \frac{z^{j-p} - z^{p-j}}{z - z^{-1}}\ ,\ \ z = \sigma e^{\frac{i\alpha}{2}}$$

Second Case.

Now we suppose that $|\mu_0| < |\mu_1| < |\mu_2|$, and we repeat the previous considerations to get the determinant S_j, but in this case we have the conditions $|\sigma| > 1, |\tau| > 1$. Taking into account that the determinant S_j satisfies the equality

$$S_j(\sigma, \tau) = -(\sigma\tau)^{j+1} S_j\left(\frac{1}{\sigma}, \frac{1}{\tau}\right)$$

We have for homogeneous problem the result identical to the theorem 4.4.1. Let us consider inhomogeneous problem, and the system (4.4.8), by invoking the inequalities $|\sigma| > 1, |\tau| > 1$, we derive the estimation of

$$Z_j \sim (\sigma\tau)^j (\tau - \sigma),$$

therefore we get asymptotic estimations for the coefficients A_{2j}, A_{3j} as

$$A_{2j} \sim \frac{D_j \tau^{j+1}}{(\sigma\tau)^j (\tau - \sigma)} - \frac{E_j \tau^j}{(\sigma\tau)^j (\tau - \sigma)} = \left(D_j \tau - E_j\right) \frac{1}{\sigma^j (\tau - \sigma)},$$

$$A_{3j} \sim \frac{E_j \sigma^j}{(\sigma\tau)^j (\tau - \sigma)} - \frac{D_j \sigma^{j+1}}{(\sigma\tau)^j (\tau - \sigma)} = \left(E_j - D_j \sigma\right) \frac{1}{\tau^j (\tau - \sigma)},$$

now by applying the relations $\sigma = \frac{\mu_1}{\mu_0}, \tau = \frac{\mu_2}{\mu_0}$ we will earn

$$A_{2j} \sim \left(d_{-j2}\tau - d_{-j1}\right)\mu_1^{-j} \frac{1}{\tau - \sigma},$$

$$A_{3j} \sim \left(d_{-j1} - d_{-j2}\sigma\right)\mu_2^{-j} \frac{1}{\tau - \sigma},$$

and also the same estimations hold for A_{0j} and A_{1j}. Finally we will get the following theorem which is similar to the theorem 4.5.1.

Theorem 4.4.2. Assume that $|\mu_0| < |\mu_1| \leq |\mu_2|$, $(\mu_1 \neq \mu_2)$, and the boundary functions F_j belong to the class of $A^{(1,\alpha)}(|\mu_1|)$, then the problem (4.4.1) and (4.5.2) is uniquely solvable if and only if the conditions (4.5.11) hold. If the conditions (4.5.11) fail, then homogeneous problem has finite number of linearly independent solutions, and for $F_j \in A^{(1,\alpha)}(|\mu_1|)$ inhomogeneous problem has a solution if and only if the finite number of linearly independent solvability conditions for boundary conditions F_j hold. Hence defect numbers are equal to the quantity of numbers j, which the conditions (4.5.11) fail.

Third case.
Assume that $|\mu_0| = |\mu_1| = |\mu_2|$, and we consider

$$\sigma = e^{i\alpha}, \tau = e^{i\beta},$$

thus the determinant Δ_j is reduced to

$$\Delta_j = \mu_0^{2j+2} i \begin{vmatrix} 1 & j+1 & 1 & 1 \\ 1 & j-1 & e^{i\alpha} & e^{i\beta} \\ 1 & -j+1 & e^{ij\alpha} & e^{ij\beta} \\ 1 & -j-1 & e^{i(j+1)\alpha} & e^{i(j+1)\beta} \end{vmatrix}$$

$$= \mu_0^{2j+2} i \left(1 - e^{i\alpha}\right)^2 \left(1 - e^{i\beta}\right)^2 \sum_{k=0}^{j-1} \sum_{m=0}^{k-1} (k-m)\left(e^{im\alpha} e^{ik\beta} - e^{ik\alpha} e^{im\beta}\right)$$

We are trying to simplify the summation

$$U = \sum_{k=0}^{j-1} \sum_{m=0}^{k-1} (k-m)(\sigma^m \tau^k - \sigma^k \tau^m)$$

$$= \sum_{k=0}^{j-1} \sum_{m=0}^{k-1} (k-m)(\sigma\tau)^{\frac{k+m}{2}} \left(\sigma^{\frac{m-k}{2}} \tau^{\frac{k-k}{2}} - \sigma^{\frac{k-m}{2}} \tau^{\frac{m-k}{2}}\right)$$

In the other hand

$$(\sigma\tau)^{\frac{m+k}{2}} = e^{i(\alpha+\beta)\frac{m+k}{2}}$$

$$(\sigma\tau^{-1})^{\frac{m-k}{2}} - (\sigma\tau^{-1})^{\frac{k-m}{2}} = e^{i(\beta-\alpha)\frac{k-m}{2}} - e^{i(\alpha-\beta)\frac{k-m}{2}}$$

$$= 2i\sin(\beta - \alpha)\frac{k-m}{2},$$

then

$$U = \sum_{k=0}^{j-1}\sum_{m=0}^{k-1} 2i(k-m)\, sin(\beta-\alpha)\frac{k-m}{2} e^{i(\alpha+\beta)\frac{m+k}{2}} \tag{4.5.12}$$

Suppose that

$$p = k - m, q = k + m,$$

then

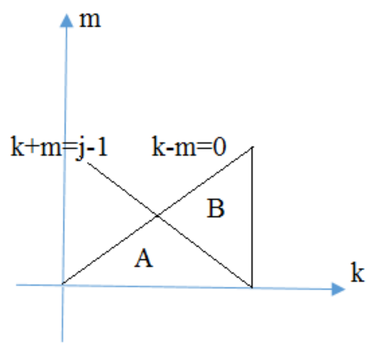
$$k = \frac{p+q}{2}, m = \frac{q-p}{2},$$

In *A*:

$$0 \le q \le j-1, 0 \le p \le q.$$

In *B*:

$$j-1 \le q \le 2(j-1), 0 \le p \le 2(j-1) - q,$$



We will continue the simplification process of U as

$$\frac{1}{2i}U = \sum_{q=0}^{j-1}\sum_{p=0}^{q} p\sin p\frac{\beta-\alpha}{2} e^{iq\frac{\alpha+\beta}{2}} + \sum_{q=j-1}^{2(j-1)}\sum_{p=0}^{2(j-1)-q} p\sin p\frac{\beta-\alpha}{2} e^{iq\frac{\alpha+\beta}{2}}$$

Let us assume

$$r = 2(j-1) - q,$$

then

$$\frac{1}{2i}U = \sum_{q=0}^{j-1}\sum_{p=0}^{q} p\sin p\frac{\beta-\alpha}{2}e^{iq\frac{\alpha+\beta}{2}} + \sum_{r=0}^{j-1}\sum_{p=0}^{r} p\sin p\frac{\beta-\alpha}{2}e^{i(2j-r-2)\frac{\alpha+\beta}{2}},$$

also

$$U = 2i\sum_{q=0}^{j-1}\sum_{p=0}^{q} p\sin p\frac{\beta-\alpha}{2}\left(e^{iq\frac{\alpha+\beta}{2}} + e^{i(2j-q-2)\frac{\alpha+\beta}{2}}\right)$$

$$= 2ie^{i(j-1)\frac{\alpha+\beta}{2}}\sum_{q=0}^{j-1}\sum_{p=0}^{q} p\sin p\frac{\beta-\alpha}{2}\left(e^{-i(j-q-1)\frac{\alpha+\beta}{2}} + e^{i(j-q-1)\frac{\alpha+\beta}{2}}\right),$$

finally

$$U = -4\sum_{q=0}^{j-1} cos(j-q-1)\frac{\alpha+\beta}{2}\sum_{p=0}^{q} p\,sin\,p\frac{\beta-\alpha}{2} \qquad (4.5.13)$$

By applying the value of U from last equality we could rewrite Δ_j as

$$\Delta_j = 4i\mu_0^{2j+2}\left(1-e^{i\alpha}\right)^2\left(1-e^{i\beta}\right)^2\sum_{q=0}^{j-1}\cos(j-q-1)\frac{\alpha+\beta}{2}\sum_{p=0}^{q} p\sin p\frac{\beta-\alpha}{2}$$

$$= 4i\mu_0^{2j+2}\left(1-e^{i\alpha}\right)^2\left(1-e^{i\beta}\right)^2\Lambda_j(\alpha,\beta),$$

where

$$\Lambda_j(\alpha,\beta) = \sum_{q=0}^{j-1} cos(j-q-1)\frac{\alpha+\beta}{2}\sum_{p=0}^{q} p\,sin\,p\frac{\beta-\alpha}{2} \qquad (4.5.14)$$

Let's consider the case when in (4.5.13)

$$\alpha = \frac{2\pi}{l}, \qquad \beta = \frac{2\pi}{n}, \qquad l,n = 2,3,\ldots,j, \qquad l \neq n. \qquad (4.5.15)$$

We denote $j_0 = [l,n]$, least common multiple of the numbers l and n, then if $j = kj_0$, we have $e^{i\alpha j} = e^{i\beta j} = 1$, and therefore

$$\Delta_j = \mu_0^{2j+2} i \begin{vmatrix} 1 & j+1 & 1 & 1 \\ 1 & j-1 & e^{i\alpha} & e^{i\beta} \\ 1 & -j+1 & e^{ij\alpha} & e^{ij\beta} \\ 1 & -j-1 & e^{i(j+1)\alpha} & e^{i(j+1)\beta} \end{vmatrix}$$

$$= \mu_0^{2j+2} i \begin{vmatrix} 1 & j+1 & 1 & 1 \\ 1 & j-1 & e^{i\alpha} & e^{i\beta} \\ 1 & -j+1 & 1 & 1 \\ 1 & -j-1 & e^{i\alpha} & e^{i\beta} \end{vmatrix} = 0 \qquad (4.5.16)$$

First we consider the homogeneous problem (4.5.1), (4.5.2), this problem is reduced to homogeneous system (4.5.5). Taking into account equality (4.5.16), we see that for $j = kj_0$, $k = 1,2, \dots$ the homogeneous system (4.5.5) has non-zero solutions $A_{0j}, A_{1j}, A_{2j}, A_{3j}$. There is only linearly independent such solution, because the rank of the system (4.5.5) is three. Now we suppose that $A_{ml} = 0$ for $l \neq j$, and for $l = j$, $\{\tilde{A}_{mj}\}_{m=0}^{3}$ is this linearly independent solution. Then we derive

$$\phi_0'(z + \mu_0 \bar{z}) =$$

$$\frac{1}{2}\tilde{A}_{0j}\left(\left(z + \mu_0\bar{z} + \sqrt{(z+\mu_0\bar{z})^2 - 4\mu_0}\right)^j + \left(z + \mu_0\bar{z} - \sqrt{(z+\mu_0\bar{z})^2 - 4\mu_0}\right)^j\right),$$

$$\phi_k'(z + \mu_{k-1}\bar{z}) = \qquad (4.5.17)$$

$$\frac{1}{2}\tilde{A}_{kj}\left(\left(z + \mu_{k-1}\bar{z} + \sqrt{(z+\mu_{k-1}\bar{z})^2 - 4\mu_{k-1}}\right)^j + \left(z + \mu_{k-1}\bar{z} - \sqrt{(z+\mu_{k-1}\bar{z})^2 - 4\mu_{k-1}}\right)^j\right)$$

After integrating we substitute in (4.5.1), (4.5.2) we find $u_j(z,\bar{z})$ nontrivial solution of the homogeneous problem which is a polynomial of order $j+1$. Thus in this case we get following theorem.

Theorem 4.4.3. If $\mu_1 = e^{i\alpha}\mu_0$, $\mu_2 = e^{i\beta}\mu_0$, when α, β is determined from (4.5.15), then the homogeneous problem (4.5.1), (4.5.2) has infinitely many linearly independent solutions. These solutions are polynomials of order $kj_0 + 1$ ($j_0 = [l, n]$), which determined by the formula (4.5.3), and ϕ_j is stated as (4.5.17).

Now let's pass to inhomogeneous problem (4.5.1), (4.5.2). This problem is reduced to the inhomogeneous system

$$\begin{cases} A_{0j} + i(j+1)A_{1j} + A_{2j} + A_{3j} = d_{j2} \\ \mu_0 A_{0j} + i(j-1)\mu_0 A_{1j} + \mu_0 e^{i\alpha} A_{2j} + \mu_0 e^{i\beta} A_{3j} = d_{j1} \\ \mu_0^j A_{0j} - i(j-1)\mu_0^j A_{1j} + \mu_0^j e^{ij\alpha} A_{2j} + \mu_0^j e^{ij\beta} A_{3j} = d_{-j2} \\ \mu_0^{j+1} A_{0j} - i(j+1)\mu_0^{j+1} A_{1j} + \mu_0^{j+1} e^{i(j+1)\alpha} A_{2j} + \mu_0^{j+1} e^{i(j+1)\beta} A_{3j} = d_{-j2} \end{cases}$$

For $j = kj_0 = k[l,n]$, this system is reduced to the form of

$$\begin{cases} A_{0j} + i(j+1)A_{1j} + A_{2j} + A_{3j} = d_{j2} \\ A_{0j} + i(j-1)A_{1j} + e^{i\alpha} A_{2j} + e^{i\beta} A_{3j} = d_{j1}\mu_0^{-1} \\ A_{0j} - i(j-1)A_{1j} + A_{2j} + A_{3j} = d_{-j2}\mu_0^{-j} \\ A_{0j} - i(j+1)A_{1j} + e^{i\alpha} A_{2j} + e^{i\beta} A_{3j} = d_{-j1}\mu_0^{-j-1} \end{cases}$$

We use the equalities $e^{ij\alpha} = e^{ij\beta} = 1$ (see (4.5.15)). This system has a solution if the condition

$$\frac{d_{-j2}\mu_0^{-j} - d_{j2}}{j} = \frac{d_{-j1}\mu_0^{-j-1} - d_{j1}\mu_0^{-1}}{j+1} \tag{4.5.18}$$

holds. Thus we prepare the following theorem.

Theorem 4.4.4. We coincide the inhomogeneous problem (4.5.1), (4.5.2) if

$$\mu_1 = e^{i\alpha}\mu_0, \qquad \mu_2 = e^{i\beta}\mu_0,$$

and the conditions (4.5.15) hold. Then for solvability of this problem, it is necessary infinitely many conditions (4.5.18). Here $j = kj_0 = k[l,n]$, and d_{jk} are the Fourier coefficients of the boundary functions F_k.

4.6. Conclusion

In this chapter we considered special kind of boundary value problems, that is, the Dirichlet problem for the fourth order elliptic equation with constant coefficients in the unit disc. We assumed two cases, first if double root in the upper half-plane and the simple roots in lower-half plane, that is, the equation is properly elliptic, and second case if all the roots belong to the upper-half plane, that is improperly elliptic case. We gained the new formula for the determination of the defect numbers. The solvability conditions and the solutions of homogeneous and inhomogeneous problems were investigated in explicit form. The numerical results showed that the defect numbers may be only zero and one. For improperly elliptic case the set of boundary functions provided, and the normal solvability of the Dirichlet problem was specified. Also we researched that for inhomogeneous problem if $\mu_1 = e^{i\alpha}\mu_0$, $\mu_2 = e^{i\beta}\mu_0$, within the conditions (4.5.15), the solvability of problem depends on infinitely many conditions (4.5.18).

Conclusion

In the thesis the following results are obtained:

1. Mathematical modeling of heat transfer (Transport Phenomena) in three-dimension for the especial furnaces with cylindrical shapes is derived, where these Furnaces in the business area are called Garnissage furnace. In the present work we obtained the mathematical modeling based on the stream functions and cylindrical coordinates. We prepared the details of the modeling by invoking the conservation laws, and the Stefan condition is applied to describe the behavior of the free boundary.

2. We defined stream function in two dimension and then we gained the mathematical modeling of heat transfer according to stream function. The modeling based on stream function has its sufficient advantages and we prepared complete description about the modeling process, variational approach and weak formulation, and numerical solution of the transport system in two-dimension.

3. We achieved the mathematical modeling in the cylindrical coordinate system according to Garnissage tank shape and its symmetric properties. We got the conservation equations in the cylindrical coordinate system, so the Stefan condition, and then we followed the variational approach to convert the system of equations into the weak formulation. For expressing the system in the variational formulation we will exert sufficient smooth test functions with small support.

4. To complete the finite element method, we discrete the domain to special mesh cubes and we divided them into 24 tetrahedrons. We computed the piecewise continuous test function according to 24 tetrahedrons, and then we determined values of coefficients in the linear and nonlinear system. We solved the system numerically by the Newton’s method and we exhibited the final results by the sufficient graphs.

5. We focused on the Dirichlet problem for the fourth order properly elliptic equation with constant coefficients in the unit disc. The characteristic equation has one double root in upper half-plane and two different roots in the lower half-plane. The solution must be found in the class of functions Hölder continuous with first order derivatives up to the boundary, then we obtained the new formula for the determination of the defect numbers. The solvability of homogeneous and inhomogeneous problems are investigated in explicit form. We demonstrated that the defect numbers may be only zero and one by the numerical approach by applying Mathematica programing.

6. Results of the thesis were published in the following papers:

   Results of chapter 1 were published in [83].

   Results of chapter 2 were published in [81]

   Results of chapter 3 were published in [82], [85], and [86].

   Results of chapter 4 were published in [84].

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