ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Տիգրան Կարենի Բաքարյան

Ձևափոխված Ֆուրիեի բազիսով մոտարկումների զուգամիտության արագացում

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Convergence Acceleration of Approximations by the Modified Fourier System

Synopsis

Dissertation for the degree of candidate of
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A.01.01 – "Mathematical analysis"

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Ատենախոսության թեման հաստատվել է ՀՀ ԳԱԱ Մաթեմատիկայի ինստիտուտում

Գիտական ղեկավար՝	ֆիզ.-մաթ. գիտ. թեկնածու, դոցենտ Ա. Վ. Պողոսյան
Պաշտոնական ընդդիմախոսներ՝	ֆիզ.-մաթ. գիտ. դոկտոր, պրոֆեսոր Հ. Վ. Մելաձե ֆիզ.-մաթ. գիտ. թեկնածու Լ. Դ. Պողոսյան
Առաջատար կազմակերպություն՝	Հայ-Ռուսական (Սլավոնական) Համալսարան

Պաշտպանությունը կկայանա 2018թ. հունիսի 12-ին, ժ. 15^{00}-ին ԵՊՀ-ում գործող ԲՈՀ-ի 050 "Մաթեմատիկա" մասնագիտական խորհրդի նիստում (0025, Երևան, Ալեք Մանուկյան 1):
Ատենախոսությանը կարելի է ծանոթանալ ԵՊՀ-ի գրադարանում:
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Մասնագիտական խորհրդի
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ֆիզ.-մաթ. գիտ. դոկտոր Տ. Ն. Հարությունյան

The topic of the thesis was approved in Institute of Mathematics of NAS RA

Scientific advisor	candidate of phys.-math. sciences A. V. Poghosyan
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Leading organization	Russian-Armenian (Slavonic) University

The defense will be held on June 12, 2018 at 15 : 00 at a meeting of the specialized council of mathematics 050, operating at the Yerevan State University (0025, 1 Alex Manoogian St, Yerevan).
The thesis can be found at the YSU library.
The synopsis was sent on May 10, 2018.

Scientific secretary
of specialized council,
doctor of phys.-math. sciences T. N. Harutyunyan

Overview

Relevance of the topic.

It is well known [1] that approximation of a 2-periodic and smooth function by the truncated Fourier series or trigonometric interpolation is highly effective. It leads to a wide range of well-known applications, from number theory to electrical engineering, from theoretical computer science to signal and image processing [2]. However, existence of jumps severely degrades the quality of approximations and interpolations. This is true even though the approximated function is infinitely differentiable on $[-1, 1]$, but non-periodic.

Different approaches were considered in literature for approximation and interpolation of smooth, but non-periodic functions on $[-1, 1]$.

One such approach is expansions by the modified trigonometric system, which was originally proposed by Krein [3]. Expansions by the modified trigonometric system were studied recently in a series of papers [4–11]. Fourier coefficients by the modified trigonometric system decay faster compared to the classical coefficients, which leads to better accuracy of the corresponding expansions. However, non-periodicity still impacts the quality of approximations and interpolations.

An efficient approach for convergence acceleration of trigonometric expansions is a polynomial subtraction method which involves a polynomial representing the jumps was suggested by Krylov [12] and Lanczos [13]. This approach was very popular in the last decades and mentioned by a series of researchers in different frameworks ([14–20]). Different authors [11] suggested to apply this method to the modified expansions.

Another well-cited approach for convergence acceleration of trigonometric expansions is application of trigonometric-rational correction functions (see [21–28]), which lead to rational approximations by classical or modified trigonometric functions.

Goals. Thesis is devoted to approximations and interpolations by the modified trigonometric system. The first goal is convergence acceleration of the expansions by the modified trigonometric system with rational corrections and investigation of the convergence of the corresponding rational approximations in different frameworks. The second goal is investigation of the convergence of interpolations by the modified trigonometric system. The third goal is implementation of the corresponding algorithms for validation of their applicability in practical problems.

Research methods. Methods of theory of functions, numerical analysis and mathematical analysis.

Scientific novelty. All results are new and are the following:

1. We consider convergence acceleration of the expansions by the modified trigonometric system by rational corrections which lead to modified-trigonometric-rational (MTR-) approximations. Rational corrections contain unknown parameters which determination is crucial for the convergence properties of the MTR-approximations. We suggest two different approaches for their determination. The first approach leads to the modified Fourier-Pade (MFP-) approximations. The second approach leads to the optimal MTR-approximations.

2. We explore the convergence of the MFP-approximations in different frameworks: pointwise convergence and convergence in the L_2-norm. We derive the exact constants of the asymptotic errors for $|x| < 1$, at $x = \pm 1$ and in the L_2-norm.

3. We explore the convergence of the optimal MTR-approximations in different frameworks. In each case we find the exact constants of the corresponding asymptotic errors. It helps to find the optimal values of parameters that vanish or minimize the main terms of the asymptotic errors.

4. We introduce interpolations by the modified trigonometric system with a uniform grid on $[-1, 1]$. We explore the convergence of the modified interpolations in different frameworks: the pointwise convergence and convergence in the L_2-norm. In all cases, we derive the exact constants of the corresponding asymptotic errors.

Theoretical and practical value. All results and developed methods represent both theoretical and practical interest for the theory of function approximation and interpolation, and numerical analysis.

Approbation of the results. The results were reported in the seminars of the Institute of Mathematics and in the conference:

- *On a modified Fourier interpolation*, Armenian Mathematical Union Annual Session dedicated to the 110th anniversary of Professor Artashes Shahinyan, 2016, Yerevan, Armenia.

Publications. The main results of the thesis are published in 4 papers.

Structure and volume of the thesis. Thesis consists of introduction, the main chapter with five sections, conclusion, notations, and references with 58 references. The total number of pages is 80.

Content

Thesis considers approximations and interpolations by the modified trigonometric system

$$\mathcal{H} = \{\cos \pi n x : n \in \mathbb{Z}_+\} \cup \{\sin \pi (n - \tfrac{1}{2})x : n \in \mathbb{N}\},\ x \in [-1, 1]. \tag{1}$$

Set $\mathcal{H}$ is an orthonormal basis of $L_2[-1, 1]$, as it consists of the eigenfunctions of the Sturm-Liouville operator $\mathcal{L} = -d^2/dx^2$ with Neumann boundary conditions $u'(1) = u'(-1) = 0$ (see [29]). Let $M_N(f, x)$ be the truncated modified Fourier series

$$M_N(f, x) = \frac{1}{2} f_0^c + \sum_{n=1}^{N} [f_n^c \cos \pi n x + f_n^s \sin \pi (n - \tfrac{1}{2})x], \tag{2}$$

where

$$f_n^c = \int_{-1}^{1} f(x) \cos \pi n x dx,\ \ f_n^s = \int_{-1}^{1} f(x) \sin \pi (n - \tfrac{1}{2})x dx. \tag{3}$$

Obviously, for even functions on $[-1, 1]$, expansions by the modified Fourier system coincide with the expansions by the classical Fourier system

$$\mathcal{H}_{class} = \{\cos \pi n x : n \in \mathbb{Z}_+\} \cup \{\sin \pi n x : n \in \mathbb{N}\},\ x \in [-1, 1]. \tag{4}$$

Next theorems show that the expansions by the modified trigonometric system have better convergence properties for smooth and odd functions on $[-1, 1]$ compared to the classical expansions ([1]). It is connected with faster decay of f_n^s compared to the corresponding classical coefficients, which follows from the following asymptotic estimates

$$f_n^c = (-1)^n \sum_{k=0}^{q} \frac{A_{2k+1}(f)}{(\pi n)^{2k+2}} + o(n^{-2q-2}),\ n \to \infty, \tag{5}$$

and

$$f_n^s = (-1)^{n+1} \sum_{k=0}^{q} \frac{B_{2k+1}(f)}{(\pi(n - \frac{1}{2}))^{2k+2}} + o(n^{-2q-2}),\ n \to \infty, \tag{6}$$

where

$$A_{2k+1}(f) = \left(f^{(2k+1)}(1) - f^{(2k+1)}(-1)\right)(-1)^k, \quad k = 0, \dots, q-1, \tag{7}$$

$$B_{2k+1}(f) = \left(f^{(2k+1)}(1) + f^{(2k+1)}(-1)\right)(-1)^k, \quad k = 0, \dots, q-1, \tag{8}$$

and $f \in C^{2q+2}[-1,1]$. We see that $f_n^s = O(n^{-2})$, $n \to \infty$ when f is enough smooth, but non-periodic on $[-1,1]$. More important is the fact, that for rapid decay of the modified Fourier coefficients, the approximated function must obey the first q derivative conditions

$$f^{(2r+1)}(\pm 1) = 0, \; r = 0, 1 \dots, q-1. \tag{9}$$

Let

$$R_N(f,x) = f(x) - M_N(f,x). \tag{10}$$

Theorem A. [I] *Let $f \in C^{2q+1}[-1,1]$, $q \geq 0$, $f^{(2q+1)} \in BV[-1,1]$ and f obeys the first q derivative conditions* (9). *Then, the following estimate holds*

$$\lim_{N \to \infty} N^{2q+\frac{3}{2}} ||R_N(f,x)||_{L_2} = \frac{1}{\pi^{2q+2}\sqrt{4q+3}} \sqrt{A_{2q+1}^2(f) + B_{2q+1}^2(f)}. \tag{11}$$

Theorem B. [11] *Let $f \in C^{2q+2}(-1,1)$, $q \geq 0$, $f^{(2q+2)} \in BV[-1,1]$ and f obeys the first q derivative conditions* (9). *Then, the following estimate holds $x \in (-1,1)$ as $N \to \infty$*

$$\begin{aligned} R_N(f,x) = {} & \frac{(-1)^{N+1}}{2\pi^{2q+2} N^{2q+2} \cos\frac{\pi x}{2}} \\ & \times \left(A_{2q+1}(f) \cos \pi(N+1/2)x - B_{2q+1}(f) \sin \pi N x\right) + o(N^{-2q-2}). \end{aligned} \tag{12}$$

Theorem C. [8] *Let $f \in C^{2q+2}[-1,1]$, $q \geq 0$, $f^{(2q+2)} \in BV[-1,1]$ and f obeys the first q derivative conditions* (9). *Then, the following estimate holds*

$$R_N(f, \pm 1) = \frac{1}{\pi^{2q+2}(2q+1)N^{2q+1}} \left(A_{2q+1}(f) \pm B_{2q+1}(f)\right) + o(N^{-2q-1}). \tag{13}$$

Sections 1-3 consider rational approximations by the modified trigonometric system. They reproduce the results of papers [I-III]. Consider a finite sequence of real numbers $\theta = \{\theta_k\}_{k=1}^p$, $p \ge 1$ and by $\Delta_n^k(\theta, \hat{f})$, $\hat{f} = \{f_n\}_{n=1}^\infty$ denote the following generalized finite differences

$$\begin{aligned} &\Delta_n^0(\theta, \hat{f}) = f_n, \\ &\Delta_n^k(\theta, \hat{f}) = \Delta_n^{k-1}(\theta, \hat{f}) + \theta_k \Delta_{n-1}^{k-1}(\theta, \hat{f}), \ k \ge 1. \end{aligned} \tag{14}$$

Consider two sequences of real numbers $\theta^c = \{\theta_k^c\}_{k=1}^p$ and $\theta^s = \{\theta_k^s\}_{k=1}^p$. Let $\hat{f}^s = \{f_n^s\}_{n=1}^\infty$ and $\hat{f}^c = \{f_n^c\}_{n=0}^\infty$. Let $\mu_j(k,\theta)$ be defined by the following identities

$$\prod_{j=1}^k (1 + \theta_j x) = \sum_{j=0}^k \mu_j(k,\theta) x^j, \ k = 1, \dots, p. \tag{15}$$

Thesis explores the following modified-trigonometric-rational (MTR-) approximations

$$\begin{aligned} M_{N,p}(f, \theta^c, \theta^s, x) = M_N(f,x) - &\sum_{k=1}^p \frac{\theta_k^c \, \Delta_N^{k-1}(\theta^c, \hat{f}^c)}{\prod_{r=1}^k (1 + 2\theta_r^c \cos \pi x + (\theta_r^c)^2)} \\ &\times \sum_{j=0}^k \mu_j(k, \theta^c) \cos \pi (N + 1 - j) x - \\ - \sum_{k=1}^p \frac{\theta_k^s \, \Delta_N^{k-1}(\theta^s, \hat{f}^s)}{\prod_{r=1}^k (1 + 2\theta_r^s \cos \pi x + (\theta_r^s)^2)}& \\ &\times \sum_{j=0}^k \mu_j(k, \theta^s) \sin \pi (N + \tfrac{1}{2} - j) x, \end{aligned} \tag{16}$$

where

$$\begin{aligned} R_{N,p}(f, \theta^c, \theta^s, x) &= f(x) - M_{N,p}(f, \theta^c, \theta^s, x) \\ &= R_{N,p}^{cos}(f, \theta^c, x) + R_{N,p}^{sin}(f, \theta^s, x), \end{aligned} \tag{17}$$

and

$$\begin{aligned} R_{N,p}^{cos}(f, \theta, x) = &\frac{1}{2 \prod_{k=1}^p (1 + \theta_k e^{i\pi x})} \sum_{n=N+1}^\infty \Delta_n^p(\theta, \hat{f}^c) e^{i\pi n x} \\ &+ \frac{1}{2 \prod_{k=1}^p (1 + \theta_k e^{-i\pi x})} \sum_{n=N+1}^\infty \Delta_n^p(\theta, \hat{f}^c) e^{-i\pi n x}, \end{aligned} \tag{18}$$

$$\begin{aligned} R_{N,p}^{sin}(f, \theta, x) = &\frac{e^{-\frac{i\pi x}{2}}}{2i \prod_{k=1}^p (1 + \theta_k e^{i\pi x})} \sum_{n=N+1}^\infty \Delta_n^p(\theta, \hat{f}^s) e^{i\pi n x} \\ &- \frac{e^{\frac{i\pi x}{2}}}{2i \prod_{k=1}^p (1 + \theta_k e^{-i\pi x})} \sum_{n=N+1}^\infty \Delta_n^p(\theta, \hat{f}^s) e^{-i\pi n x}. \end{aligned} \tag{19}$$

A crucial step for realization of the rational approximations is determination of parameters θ^c and θ^s. Different approaches are known for solution of this problem (see [21–28]). In general, appropriate determination of these parameters should lead to rational approximations with improved accuracy compared to the classical ones in case of smooth f. However, the rational approximations are essentially non-linear in the sense that

$$M_{N,p}(f+g,\theta^c,\theta^s,x) \neq M_{N,p}(f,\theta^c,\theta^s,x) + M_{N,p}(g,\theta^c,\theta^s,x) \tag{20}$$

as for each approximation we need to determine its own vectors θ^c and θ^s.

In [I,II], those parameters are determined from the following systems of equations

$$\Delta_n^p(\theta^c,\hat{f}^c) = 0,\ n = N, N-1, \ldots, N-p+1, \tag{21}$$

and

$$\Delta_n^p(\theta^s,\hat{f}^s) = 0,\ n = N, N-1, \ldots, N-p+1, \tag{22}$$

which lead to the Fourier-Pade type approximations ([21]) with better convergence for smooth functions compared to the classical expansions by the modified trigonometric system. We call those approximations as modified Fourier-Pade (MFP-) approximations. It is a complex approach as parameters θ^c, θ^s depend on N and systems (21) and (22) must be solved for each N.

Section 1 considers convergence of the modified Fourier-Pade approximations in different frameworks. Theorem 1 (see Theorem 1.1 of the thesis) explores the pointwise convergence for $|x| < 1$.

Theorem 1. *Assume $f \in C^{(2q+2p+2)}[-1,1]$ and $f^{(2q+2p+2)} \in BV[-1,1]$, $q \geq 0$, $p \geq 1$, and let systems (21), (22) have unique solutions. If f obeys the first q derivative conditions (9) then, the following estimates are valid for $x \in (-1,1)$*

$$\begin{aligned} R_{N,p}^{cos}(f,\theta^c,x) = A_{2q+1}(f)\frac{(-1)^{N+1}(2q+p+1)!p!}{2^{2p+1}\pi^{2q+2}N^{2q+2p+2}(2q+1)!}\frac{\cos\frac{\pi x}{2}(2N-2p+1)}{\cos^{2p+1}\frac{\pi x}{2}} \\ + o(N^{-2q-2p-2}), \end{aligned} \tag{23}$$

and

$$\begin{aligned} R_{N,p}^{sin}(f,\theta^s,x) = B_{2q+1}(f)\frac{(-1)^{N}(2q+p+1)!p!}{2^{2p+1}\pi^{2q+2}N^{2q+2p+2}(2q+1)!}\frac{\sin\frac{\pi x}{2}(2N-2p)}{\cos^{2p+1}\frac{\pi x}{2}} \\ + o(N^{-2q-2p-2}). \end{aligned} \tag{24}$$

Theorem 1.2 of the thesis proves similar result at $x = \pm 1$. It shows convergence rate $O(N^{-2q-1})$ as $N \to \infty$. Comparison with Theorem C shows that the expansions by the modified Fourier system and the MFP-approximations have the same convergence rates at the endpoints $x = \pm 1$. However, comparison of the corresponding constants $h_{p,q}$ (see the estimate of Theorem 1.2) and $h_{0,q} = 1$, which corresponds to the classical estimate, shows that the MFP-approximations are much more accurate than the classical expansions (see Table 1.1 of the thesis) at $x = \pm 1$.

Section 1 considers also the L_2-convergence of the MFP-approximations. Theorem 1.3 of the thesis shows the exact constant of the asymptotic L_2-error. Comparison with Theorem A shows that the classical expansions and the MFP-approximations have the same convergence rates $O(N^{-2q-3/2})$ in the L_2-norm. However, comparison of the corresponding constants $c_{p,q}$ (see the estimate of Theorem 1.3) and $c_{0,q} = 1$, which corresponds to the classical estimate, shows that the MFP-approximations are asymptotically more accurate (see Table 1.3 of the thesis).

Papers [II,III] consider simpler alternative approach for smooth functions, assuming that θ^s and θ^c are determined as follows

$$\theta_k^c = 1 - \frac{\tau_k^c}{N}, \ \theta_k^s = 1 - \frac{\tau_k^s}{N}, \ \tau_k^c \neq 0, \tau_k^s \neq 0, \ k = 1, \ldots, p, \tag{25}$$

with $\tau^c = \{\tau_1^c, \ldots, \tau_p^c\}$ and $\tau^s = \{\tau_1^s, \ldots, \tau_p^s\}$ independent of N. Actually, it takes into consideration only the first two terms of the asymptotic expansions of $\theta_k = \theta_k(N)$ in terms of $1/N$. Although, parameters θ^c and θ^s in (25) depend on N, we need only to determine τ^c and τ^s which are independent of N. Hence, this approach is less complex than the modified Fourier-Pade approximations.

Sections 2 and 3 consider the convergence of the MTR-approximations with parameters θ^c and θ^s defined by (25). The standard approach of these sections is derivation of the exact estimates for the main terms of asymptotic errors without specifying parameters τ^c and τ^s. Then, determination of the optimal values of parameters which vanish or minimize the main terms of the asymptotic errors and lead to approximations with substantially better convergence rates.

Sections show that in case of the pointwise convergence, the optimal values of parameters τ_k^c and τ_k^s, $k = 1, \ldots, p$ are the roots of some polynomials depending on p and q, where q indicates the number of zero derivatives in (9). Moreover, the choice of optimal parameters depends on the parity of p and also on the location of x, whether $|x| < 1$ or $x = \pm 1$. Section 2 considers convergence on $x \in (-1, 1)$ and Section 3 at $x = \pm 1$.

Next theorem (see Theorem 2.1 of the thesis) reveals the asymptotic behavior of the MTR-approximations for $|x| < 1$ without specifying the selection of parameters τ^c and τ^s in (25).

Theorem 2. *Assume $f \in C^{2q+p+2}[-1,1]$, $f^{(2q+p+2)} \in BV[-1,1]$, $q \geq 0$, $p \geq 1$, and f obeys the first q derivative conditions (9). Let θ_k, $k = 1, \ldots, p$ be defined by (25). Then, the following estimates hold for $|x| < 1$ as $N \to \infty$*

$$\begin{aligned} R_{N,p}^{cos}(f,\theta,x) &= A_{2q+1}(f)\frac{(-1)^{N+p+1}}{N^{2q+p+2}2^{p+1}\pi^{2q+2}(2q+1)!} \\ &\times \frac{\cos\frac{\pi x}{2}(2N-p+1)}{\cos^{p+1}\frac{\pi x}{2}}h_{p,2q+1}(\tau) + o(N^{-2q-p-2}), \end{aligned} \tag{26}$$

and

$$\begin{aligned} R_{N,p}^{sin}(f,\theta,x) &= B_{2q+1}(f)\frac{(-1)^{N+p}}{N^{2q+p+2}2^{p+1}\pi^{2q+2}(2q+1)!} \\ &\times \frac{\sin\frac{\pi x}{2}(2N-p)}{\cos^{p+1}\frac{\pi x}{2}}h_{p,2q+1}(\tau) + o(N^{-2q-p-2}), \end{aligned} \tag{27}$$

where

$$h_{p,m}(\tau) = \sum_{k=0}^{p}(-1)^k\gamma_k(\tau)(m+p-k)!. \tag{28}$$

Estimates of Theorem 2 leads to optimal choice of parameters τ^s and τ^c. Improvement could be achieved if parameters are chosen such that $\tau^s = \tau^c = \tau$ and

$$h_{p,2q+1}(\tau) = 0. \tag{29}$$

Definition of $h_{p,2q+1}(\tau)$ shows that condition (29) can be achieved, for example, if

$$\gamma_k(\tau) = \binom{p}{k}\frac{(2q+1+p)!}{(2q+1+p-k)!}\mathcal{Q}_r(k), \tag{30}$$

where $\mathcal{Q}_r(k)$ is a polynomial of order $r \leq p-1$

$$\mathcal{Q}_r(k) = \sum_{j=0}^{r} c_j k^j,\ c_0 = 1, \tag{31}$$

with unknown coefficients c_j, $j = 1, \ldots, r$. Next theorem (see Theorem 2.3 of the thesis) determines the values of c_j, $j = 1, \ldots, r$ for improved convergence of the rational approximations with (25) when p is odd.

Theorem 3. *Let parameter $p \geq 1$ be odd, $f \in C^{2q+p+\frac{p+1}{2}+2}[-1,1]$, $q \geq 0$, $f^{(2q+p+\frac{p+1}{2}+2)} \in BV[-1,1]$ and f obeys the first q derivative conditions (9). Let θ_k, $k = 1, ..., p$ be defined by (25), where τ_k be the roots of the generalized Laguerre polynomial $L_p^{(2q+1)}(x)$. Then, the following estimates hold for $|x| < 1$ as $N \to \infty$*

$$
\begin{aligned}
R_{N,p}^{cos}(f,\theta,x) = A_{2q+1}(f) & \frac{(-1)^{N+1}}{N^{2q+p+\frac{p+1}{2}+2}\pi^{2q+2}2^{p+1}} \\
& \times \Bigg(\frac{\cos\frac{\pi x}{2}(2N-p+1)}{\cos^{p+1}\frac{\pi x}{2}} \sigma_{q,2q+\frac{p+1}{2},0}(0) \\
& \qquad + \frac{\cos\frac{\pi x}{2}(2N-p)}{2\cos^{p+2}\frac{\pi x}{2}} \sigma_{q,2q+\frac{p+1}{2},0}(1) \Bigg) \\
& + o(N^{-2q-p-\frac{p+1}{2}-2}),
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
R_{N,p}^{sin}(f,\theta,x) = B_{2q+1}(f) & \frac{(-1)^{N}}{N^{2q+p+\frac{p+1}{2}+2}\pi^{2q+2}2^{p+1}} \\
& \times \Bigg(\frac{\sin\frac{\pi x}{2}(2N-p)}{\cos^{p+1}\frac{\pi x}{2}} \tilde{\sigma}_{q,2q+\frac{p+1}{2},0}(0) \\
& \qquad + \frac{\sin\frac{\pi x}{2}(2N-p-1)}{2\cos^{p+2}\frac{\pi x}{2}} \tilde{\sigma}_{q,2q+\frac{p+1}{2},0}(1) \Bigg) \\
& + o(N^{-2q-p-\frac{p+1}{2}-2}),
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
\sigma_{s,t,j}(w) &= (2q+p+1)! \sum_{k=0}^{p} \binom{p}{k} \beta_{k,s,t}(w) \frac{(p-k+t+1)!}{(2q+p+1-k)!} k^j, \\
\tilde{\sigma}_{s,t,j}(w) &= (2q+p+1)! \sum_{k=0}^{p} \binom{p}{k} \tilde{\beta}_{k,s,t}(w) \frac{(p-k+t+1)!}{(2q+p+1-k)!} k^j,
\end{aligned}
\tag{34}
$$

and

$$
\begin{aligned}
\beta_{k,s,t}(w) &= \sum_{\ell=w}^{t-2s} k^{t-2s-\ell} \frac{\alpha_{w+p-k,\ell+p-k}}{(t-2s-\ell)!(p-k+\ell)!}, \\
\tilde{\beta}_{k,s,t}(w) &= \sum_{\ell=w}^{t-2s} \left(k+\frac{1}{2}\right)^{t-2s-\ell} \frac{\alpha_{w+p-k,\ell+p-k}}{(t-2s-\ell)!(p-k+\ell)!},
\end{aligned}
\tag{35}
$$

where

$$
\alpha_{k,j} = \sum_{s=0}^{k} \binom{k}{s} (-1)^s s^j, \ j \geq 0. \tag{36}
$$

Theorem 2.4 of the thesis proves similar result when p is even and $|x| < 1$.

Theorem 2.5 of the thesis explores the L_2-error of the MTR-approximations without specifying the choice of the corresponding parameters τ^c and τ^s. First, it derives the exact constant of the asymptotic L_2-error. Then, parameters are selected such to minimize (numerically) the mentioned asymptotic constant. We call these approximations as L_2-minimal MTR-approximations. Table 2.1 of the thesis shows that the latests have better asymptotic L_2-accuracy compared to the MFP-approximations.

Theorems 3.1, 3.2, 3.3 and 3.4 of the thesis explore the pointwise convergence at $x = \pm 1$. Theorem 3.1 shows convergence rate $O(N^{-2q-1})$ without specifying the choice of parameters. Theorem 3.2 shows how the optimal values should be chosen. Theorem 3.3 finds the optimal values of parameters for odd p. It proves that the best accuracy could be achieved when parameters $\tau_k^s = \tau_k^c$ are the roots of the generalized Laguerre polynomial $L_p^{(2q)}(x)$. Theorem 3.4 proves similar results for even p.

Sections 4 and 5 study interpolations by the modified trigonometric system. They reproduce the results of paper [IV]. These sections explore the convergence of the modified interpolations in different frameworks: pointwise and L_2-convergence. In each case, we derive exact constants of the asymptotic errors and provide comparisons with the classical trigonometric interpolation which shows better convergence properties of the modified interpolation for odd functions.

The modified interpolation was introduced in [IV]. We write the modified trigonometric system more compactly

$$\mathcal{H} = \{\varphi_n(x) : n \in \mathbb{Z}_+\}, \tag{37}$$

where

$$\varphi_0(x) = \frac{1}{\sqrt{2}}, \ \varphi_n(x) = \frac{1}{2}\left((-1)^n e^{\frac{i\pi n x}{2}} + e^{-\frac{i\pi n x}{2}}\right), \ n \in \mathbb{N}. \tag{38}$$

Then, we write the interpolation as follows

$$\mathcal{I}_N(f, x) = \sum_{n=0}^{2N} \check{f}_n^m \varphi_n(x), \tag{39}$$

$$\check{f}_n^m = \frac{2}{2N+1} \sum_{k=-N}^{N} f(x_k)\overline{\varphi_n}(x_k), \quad x_k = \frac{2k}{2N+1}, \ |k| \le N. \tag{40}$$

Let

$$r_N(f, x) = f(x) - \mathcal{I}_N(f, x). \tag{41}$$

Both, the condition of interpolation and exactness on $\mathcal{H}$ follow from the discrete orthogonality of the modified trigonometric system for the grid x_k

$$\frac{2}{2N+1}\sum_{n=0}^{2N}\varphi_n(x_k)\overline{\varphi_n}(x_s)=\delta_{k,s},\ |k|,|s|\le N, \tag{42}$$

and

$$\frac{2}{2N+1}\sum_{k=-N}^{N}\varphi_n(x_k)\overline{\varphi_m}(x_k)=\delta_{n,m},\ 0\le m,n\le 2N. \tag{43}$$

Section 4 studies the L_2-convergence of the modified interpolation (see Theorem 4.1 of the thesis).

Theorem 4. *Let f be odd function on $[-1,1]$. Assume that $f\in C^{2q+1}[-1,1]$ and $f^{(2q+1)}\in BV[-1,1]$, $q\ge 0$. Let f obeys the first q derivative conditions* (9). *Then, the following estimate holds*

$$\lim_{N\to\infty} N^{2q+\frac{3}{2}}||r_N||_{L_2}=|B_{2q+1}(f)|\frac{\sqrt{a(q)}}{\pi^{2q+2}}, \tag{44}$$

where

$$a(q)=\frac{1}{4q+3}+\int_0^1\left(\sum_{s\neq 0}\frac{(-1)^s}{(2s+x)^{2q+2}}\right)^2dx. \tag{45}$$

Section 5 explores the pointwise convergence of the modified interpolation. Theorem 5 (see Theorem 5.1 of the thesis) shows the exact constant of the asymptotic error when $|x|<1$ and Theorem 6 (see Theorem 5.2 of the thesis) at $x=\pm 1$.

Theorem 5. *Let f be an odd function on $[-1,1]$. Assume that $f\in C^{2q+3}[-1,1]$ and $f^{(2q+3)}\in BV[-1,1]$, $q\ge 0$. Let f obeys the first q derivative conditions* (9). *Then, the following estimate holds for $|x|<1$ as $N\to\infty$*

$$r_N(f,x)=B_{2q+1}(f)\frac{(-1)^N}{N^{2q+3}}\frac{\pi|E_{2q+2}|}{2^{2q+5}(2q+1)!}\frac{\sin\pi(N+\frac{1}{2})x}{\cos^2\frac{\pi x}{2}}+o(N^{-2q-3}), \tag{46}$$

where E_k is the k-th Euler number.

Theorem 6. *Let f be an odd function on $[-1,1]$. Assume that $f\in C^{2q+2}[-1,1]$ and $f^{(2q+2)}\in BV[-1,1]$, $q\ge 0$. Let f obeys the first q derivative conditions* (9). *Then, the following estimate holds as $N\to\infty$*

$$r_N(f,\pm 1)=\pm B_{2q+1}(f)\frac{(-1)^{N+1}}{N^{2q+1}}\frac{|E_{2q}|}{2^{2q+1}\pi(2q+1)!}+o(N^{-2q-1}), \tag{47}$$

where E_k is the k-th Euler number.

Conclusion

• Section 1 explores the convergence of the MFP-approximations:

- Theorem 1.1 of the thesis explores the pointwise convergence for $|x| < 1$ and shows the exact constant of the asymptotic error. The convergence rate is $O(N^{-2q-2p-2})$ as $N \to \infty$ if f obeys the first q derivative conditions (9). Compared to Theorem 0.4 (see the introduction of the thesis), the improvement in convergence rate is by factor $O(N^{2p})$. However, it is important to note that, in all theorems regarding the modified expansions, it is required less smoothness from approximated functions than for the rational approximations.

- Theorem 1.2 studies the convergence at $x = \pm 1$ and derives the exact constant of the asymptotic error. The convergence rate is $O(N^{-2q-1})$ as $N \to \infty$. Comparison with Theorem 0.5 shows that the expansions by the modified Fourier system and the MFP-approximations have the same convergence rates at the endpoints $x = \pm 1$. However, comparison of the corresponding constants $h_{p,q}$ and $h_{0,q} = 1$ shows that the MFP-approximations are much more accurate than the classical expansions (see Table 1.1).

- Theorem 1.3 shows the exact constant of the asymptotic L_2-error. Comparison with Theorem 0.3 shows that the classical modified expansions and the MFP-approximations have the same convergence rates $O(N^{-2q-3/2})$ in the L_2-norm. However, comparison of the corresponding constants $c_{p,q}$ and $c_{0,q} = 1$ shows that the MFP-approximations are asymptotically more accurate (see Table 1.3).

• Section 2 considers the pointwise convergence of the MTR-approximations on $(-1, 1)$ with parameters θ^c and θ^s defined by (25):

- Theorem 2.1 derives the exact constant of the asymptotic error for pointwise convergence for $|x| < 1$ without specifying the selection of parameters τ_k^c and τ_k^s, $k = 1, \ldots, p$. It shows that MTR-approximations have convergence rate $O(N^{-2q-p-2})$ as $N \to \infty$ if an approximated function has enough smoothness and obeys the first q derivative conditions (9). Compared to the modified Fourier expansions (see Theorem 0.4), the improvement is by factor $O(N^p)$ as $N \to \infty$.

- We see that the MTR-approximations with parameters (25) are less accurate than the MFP-approximations. However, the latests are more complex in their realization as systems (21) and (22) must be solved for each N.

- Theorem 2.3 provides the optimal choice of parameters τ_k when $|x| < 1$ and p is odd and f obeys the first q derivative conditions (9). If $\tau_k^c = \tau_k^s$, $k = 1, \ldots, p$

are the roots of the generalized Laguerre polynomial $L_p^{(2q+1)}(x)$ then, the rational approximations have convergence rate $O(N^{-2q-p-\left[\frac{p+1}{2}\right]-2})$ with improvement by factor $O(N^{\left[\frac{p+1}{2}\right]})$ compared to non-optimal choice of parameters (Theorem 2.1). The improvement is by factor $O(N^{\left[\frac{p+1}{2}\right]+p})$ compared to the expansions by the modified Fourier system (Theorem 0.4).

- Theorem 2.4 provides the optimal choice of parameters when $|x| < 1$ and p is even. It shows that the set of optimal parameters is wider compared to odd p. If polynomial

$$\sum_{k=0}^{p} \binom{p}{k} \frac{1 + c_1(p-k)}{(2q+1+k)!} (-1)^k x^k \tag{48}$$

has only nonzero and real-valued roots $x = z_k$, $k = 1, \dots, p$ then, selection $\tau_k^s = \tau_k^c = z_k$ provides with better convergence rate $O(N^{-2q-p-\left[\frac{p}{2}\right]-2})$ compared to the estimate of Theorem 2.1 and improvement is by factor $O(N^{\left[\frac{p}{2}\right]})$. Improvement is by factor $O(N^{\left[\frac{p}{2}\right]+p})$ compared to the expansions by the modified Fourier system. The problem is to find the values of c_1 in (48) for which it will have only real-valued and nonzero roots. In two cases it is obvious. When $c_1 = 0$, the roots of (48) coincide with the roots of $L_p^{(2q+1)}(x)$. When $c_1 = -1/(2q+p+1)$, the roots coincide with the ones of $L_p^{(2q)}(x)$. In both cases all roots are positive.

- Theorem 2.5 explores the L_2-error of the MTR-approximations without specifying the choice of the corresponding parameters. First, it derives the exact constant of the asymptotic L_2-error. Then, parameters are selected such to minimize (numerically) the mentioned asymptotic constant. We call these approximations as L_2-minimal MTR-approximations. Table 2.1 shows that the latests have better asymptotic L_2-accuracy compared to the MFP-approximations.

• Section 3 considers the convergence of the optimal MTR-approximations at the endpoints $x = \pm 1$ with parameters θ^c and θ^s defined by (25):

- Theorem 3.1 reveals the convergence rate of the MTR-approaximations at $x = \pm 1$ without specifying parameters τ^c and τ^s. It derives the exact constant of the asymptotic error, which helps to determine the optimal values of parameters for better convergence. Theorem 3.1 shows the convergence rate $O(N^{-2q-1})$ as $N \to \infty$. Comparison with Theorem 0.5 shows the same convergence rate.

- Using the explicit form of the exact constant, Theorem 3.3 finds the optimal values of parameters for odd p for better convergence rate at $x = \pm 1$. It proves that the best accuracy could be achieved when parameters $\tau_k^s = \tau_k^c$ are the roots of the generalized Laguerre polynomial $L_p^{(2q)}(x)$. For that choice, the convergence rate is $O(N^{-2q-[\frac{p+1}{2}]-1})$ and improvement is by factor $O(N^{[\frac{p+1}{2}]})$ compared to the modified Fourier expansions.

- When p is odd, the optimal choices for $|x| < 1$ and $x = \pm 1$ are different. The choice of polynomial $L_p^{(2q)}(x)$ will provide with the minimal uniform error on $[-1, 1]$, but for $|x| < 1$, the convergence rate will be worse by factor $O(N)$ compared to the optimal choice $L_p^{(2q+1)}(x)$.

- Theorem 3.4 outlines the set of optimal parameters for even p. It shows that the optimal choice is $\tau_k^c = \tau_k^s = z_k$, $k = 1, \ldots, p$, where z_k are real-valued and non-zero roots of

$$\sum_{k=0}^{p} \binom{p}{k} \frac{1 + d_1(p-k)}{(2q+k)!} (-1)^k x^k \tag{49}$$

It provides convergence rate $O(N^{-2q-[\frac{p}{2}]-1})$ with improvement by factor $O(N^{[\frac{p}{2}]})$ compared to the modified Fourier expansions. When $d_1 = 0$ or $d_1 = -1/(2q+p)$, polynomial (49) has only real-valued and non-zero roots. For the first choice, the roots coincide with the ones of $L_p^{(2q)}(x)$ and for the second choice, with the roots of $L_p^{(2q-1)}(x)$. The choice of $L_p^{(2q)}(x)$ is better as it will provide with optimal approximations both for $|x| < 1$ and $x = \pm 1$.

- The optimal values of parameters τ^c and τ^s depend only on p and q and are independent of f and N. It means that if functions f, g and $f + g$ have enough smoothness and obey the same derivative conditions, the optimal approach leads to linear rational approximations in the sense that

$$M_{N,p}(f + g, \theta^c, \theta^s, x) = M_{N,p}(f, \theta^c, \theta^s, x) + M_{N,p}(g, \theta^c, \theta^s, x)$$

with the same parameters θ^c and θ^s for all included functions.

- Section 4 explores the convergence of the modified interpolation in the L_2-norm:

  - Theorem 4.1 reveals the convergence rate in the L_2-norm. It shows that the convergence rate is $O(N^{-2q-3/2})$ as $N \to \infty$ if f obeys the first q derivative conditions (9). The modified interpolation has the same convergence rate as expansions by the modified trigonometric system (see Theorem 0.3).

  - When $q = 0$, Theorem 4.1 shows convergence rate $O(N^{-\frac{3}{2}})$ in the L_2-norm. The classical interpolation with the same uniform grid has convergence rate $O(N^{-\frac{1}{2}})$ in the L_2-norm for odd functions on $[-1, 1]$. Hence, the improvement is by factor $O(N)$. Recall that for even functions on $[-1, 1]$, the modified interpolation is identical to the classical interpolation.

- Section 1.5 explores the pointwise convergence of the modified interpolation:

- Theorem 5.1 explores the pointwise convergence on $|x| < 1$ and derives the exact constant of the asymtotic error for a fixed $x \in (-1, 1)$. The convergence rate is $O(N^{-2q-3})$ which is better than the convergence rate of the expansions by the modified trigonometric system and improvement is by factor $O(N)$ (see Theorem 0.4).

- When $q = 0$, Theorem 5.1 implies the convergence rate $O(N^{-3})$ as $N \to \infty$. The classical interpolation has convergence rate $O(N^{-1})$ for the same uniform grid on $[-1, 1]$. Hence, the improvement is by factor $O(N^2)$ for odd functions.

- Theorem 5.2 reveals the exact constant of the asymptotic error when $x = \pm 1$. It shows convergence rate $O(N^{-2q-1})$, which is the same as for the convergence rate of the expansions by the modified trigonometric system (see Theorem 0.5).

- When $q = 0$, Theorem 5.2 shows convergence rate $O(1/N)$. In this case, as $f(1) \neq f(-1)$, the classical interpolation doesn't converge at the endpoints. Hence, the modified interpolations have better convergence rate at the endpoints with improvement by factor $O(N)$.

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ԱՄՓՈՓՈՒՄ

Աշխատանքում ուսումնասիրվում են վերլուծություններ և ինտերպոլիացիաներ ձևափոխված եռանկյունաչափական համակարգով

$$\mathcal{H} = \{\cos \pi n x : n \in \mathbb{Z}_+\} \cup \{\sin \pi (n - \tfrac{1}{2}) x : n \in \mathbb{N}\},\ x \in [-1, 1],$$

որը ներմուծել է Կրեյնը 1936թ. [3]՝ առանց ուսումնասիրելու նրա հատկությունները: $\mathcal{H}$ բազմության տարրերը՝ Շտուրմ-Լիուվիլի օպերատորով եզրային խնդրի սեփական ֆունկցիաներն են, և այդ պատճառով, հանդիսանում են $L_2[-1,1]$ տարածության օրթոնորմալ բազիս:

Ուսումնասիրություններն այս ասպարեզում վերսկսվել են 2008թ՝ [4-11] աշխատանքներում: Այդտեղ դիտարկվել են վերլուծություններ ըստ ձևափոխված համակարգի և ուսումնասիրվել է նրանց զուգամիտության հետ կապված հարցեր: Ցույց է տրվել, որ այդ վերլուծություններն ունեն մեկ կարգով ավելի մեծ զուգամիտության արագություն՝ [-1,1] հատվածի վրա բավականաչափ ողորկ կենտ ֆունկցիաների համար, քան վերլուծություններն ըստ դասական եռանկյունաչափական համակարգի:

Մենք շարունակել ենք այս հետազոտությունները և ատենախոսության առաջին մասում դիտարկել ենք մոտարկումներ ձևափոխված եռանկյունաչափական համակարգի տարրերով կառուցված ռացիոնալ ֆունկցիաներով: Ռացիոնալ ֆունկցիաները պարունակում են անհայտ պարամետրեր, որոնց որոշումը էական ազդեցություն է թողնում ալգորիթմների իրականացման բարդության և զուգամիտության հատկությունների վրա:

Ատենախոսության առաջին պարագրաֆում դիտարկվում է մոտեցում, որը հանգեցնում է ձևափոխված Ֆուրիե-Պադե մոտարկումների: Սա մոտեցումներից ամենաբարդն է, բայց փոխարենն ապահովում է զուգամիտության ամենամեծ կետային արագությունը: Ուսումնասիրվել է այս մոտեցման կետային զուգամիտությունը և զուգամիտությունը L_2 իմաստով: Համեմատությունները դասական վերլուծությունների հետ հաստատում են ռացիոնալ մոտարկումների զուգամիտության լավ հատկությունները՝ բավարար ողորկ ֆունկցիաների համար:

Ատենախոսության հաջորդ երկու պարագրաֆներում դիտարկվում է պարամետրերի որոշման ավելի պարզ տարբերակ: Պարամետրերն ընտրվում են այնպես, որ փոքրացնեն կամ զրոյացնեն ասիմպտոտական սխալի առաջին մի քանի անդամները: Արդյունքում ստացվում են օպտիմալ ռացիոնալ մոտարկումներ: Ուսումնասիրվել է նրանց կետային զուգամիտությունը և զուգամիտությունը L_2 իմաստով: Ցույց է տրվել, որ

տարբեր դեպքերում օպտիմալ պարամետրերի ընտրությունը տարբեր է: Օրինակ, կետային զուգամիտության դեպքում՝ պարամետրի օպտիմալ արժեքները համնկնում են Լագերի բազմանդամների արմատների հետ:

Այս մոտեցումը ավելի փոքր զուգամիտության արագություն ունի, քան ձևափոխված Ֆուրիե-Պադե մոտեցումը, բայց ավելի պարզ է իրականացման տեսանկյունից: Համեմատությունը դասական վերլուծությունների հետ կրկին ցույց է տալիս, որ այն ունի ավելի լավ զուգամիտության հատկություններ:

Ատենախոսության վերջին մասում դիտարկվում են ինտերպոլիացիաներ ձևափոխված եռանկյունաչափական համակարգով՝ [-1,1] հատվածի վրա տրված հավասարաչափ ցանցով: Ուսումնասիրվել է դրանց կետային զուգամիտությունը և զուգամիտությունը L_2 իմաստով: Ստացվել են ասիմպտոտական սխալի ճշգրիտ գնահատականներ: Համեմատությունը նույն ցանցով իրականացվող դասական ինտերպոլիացիայի հետ հաստատում է ձևափոխված ինտերպոլիացիայի զուգամիտության լավ հատկությունները՝ ողորկ և կենտ ֆունկցիաների համար:

ЗАКЛЮЧЕНИЕ

В диссертации изучаются разложения и интерполяции по модифицированной тригонометрической системе

$$\mathcal{H} = \{\cos \pi n x : n \in \mathbb{Z}_+\} \cup \{\sin \pi(n - \tfrac{1}{2})x : n \in \mathbb{N}\},\ x \in [-1, 1]$$

который был предложен Крейном [3] в 1936г. без рассмотрения ее свойств. Множество $\mathcal{H}$ является ортогональным базисом в $L_2[-1, 1]$, так как состоит из собственных функции граничной задачи Неймана с оператором Штурма-Лиувилля.

Разложения по модифицированной тригонометрической системе исследованы в ряде работ [4-11] начиная с 2008г. Там же рассмотрены разложения по модифицированной системе и изучена их сходимость. Исследования показали, что для гладкой и нечетной на [-1,1] функции, скорость сходимости разложений по модифицированной системе на порядок больше, чем у классических разложений.

В диссертации продолжены эти исследования и в первой части рассмотрены аппроксимации рациональными функциями по модифицированной системе. Рациональные функции зависят от неопределенных параметров, которые определяют свойства сходимости разложений и сложность их алгоритмической реализации.

В первом параграфе рассматривается подход, который приводит к модифицированным аппроксимациям Фурье-Паде. Этот подход самый сложный, с точки зрения реализации, но является и самым точным, с точи зрения точечной сходимости. Изучены точечная сходимость и сходимость в L_2 норме. Показаны более хорошие свойства сходимости для достаточно гладких функции по сравнению с классическими разложениями по модифицированной тригонометрической системе.

В следующих двух параграфах рассматривается более простой подход определения параметров. Они определяются из условия минимизации нескольких первых членов асимптотической ошибки, что приводит к оптимальным рациональным аппроксимациям. Изучена точечная сходимость и сходимость в L_2 норме и показан, что оптимальный выбор зависит от формы сходимости. Например, в случае точечной сходимости, значения оптимальных параметров совпадают с корнями полиномов Лагерра. Этот подход менее точен в смысле точечной сходимости, чем модифицированный Фурье-Паде, но более простой, с точки зрения реализации. И в этом случае, сравнение с классическими разложениями показывает более хорошие свойства сходимости.

В последней части диссертации рассматривается интерполяция по модифициривaнной тригонометрической системе и изучается сходимость в разных формах – точечная сходимость и сходимость в L_2 норме. Получены точные оценки для асимптотической ошибки. Сравнение с классической интерполяцией показывает лучшую сходимость модифицированных интерполяций для нечетных функций во всех случаях.