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On estimates for maximal operators associated with tangential regions

Synopsis

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Ատենախոսության թեման հաստատվել է Երևանի Պետական Համալսարանում

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Պաշտոնական ընդդիմախոսներ՝	ֆիզ.-մաթ. գիտ. դոկտոր Մ. Գ. Գրիգորյան ֆիզ.-մաթ. գիտ. թեկնածու Ա. Ա. Վաղարշակյան
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The topic of the thesis was approved in Yerevan State University

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The defense will be held on May 8, 2018 at 15 : 00 at a meeting of the specialized council of mathematics 050, operating at the Yerevan State University (0025, 1 Alek Manukyan St, Yerevan).
The thesis can be found at the YSU library.
The synopsis was sent on April 6, 2018.

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Overview

Relevance of the topic. It is well known the theorems of Fatou [9] about non-tangential convergence, which have many applications in different mathematical theories including analytic functions, Hardy spaces, harmonic analysis, differential equations and etc. There are various generalization of these theorems in different aspects. Almost everywhere convergence over some semi-tangential regions investigated by Nagel and Stein [25], Di Biase [7], Di Biase-Stokolos-Svensson-Weiss [8]. Sjögren [32, 33, 34], Rönning [27, 28, 29], Katkovskaya-Krotov [17, 21], Krotov [19, 20], Brundin [5], Mizuta-Shimomura [24], Aikawa [3] studied fractional Poisson integrals with respect to the fractional power of the Poisson kernel and obtained some tangential convergence properties for such integrals.

Littlewood [22] made an important complement to the theorem of Fatou, proving essentiality of non-tangential approach in that theorem. Lohwater and Piranian [23] proved, that in Littlewood's theorem almost everywhere divergence can be replaced to everywhere and the example function can be a Blaschke product. Aikawa [1] obtained a similar everywhere divergence theorem for bounded harmonic functions on the unit disk, giving a positive answer to a problem raised by Barth [[4], p. 551].

In the thesis it is considered convolution type integral operators with general kernels and it is investigated some generalizations of theorems of Fatou and Littlewood.

Besides, it is studied some questions of equivalency of differentiation bases. The classical theorems of Jessen–Marcinkiewicz–Zygmund [15] and Saks [31] determine the optimal Orlicz space for the basis of all rectangles. Due to Zerekidze [38, 39, 40] and Stokolos [36], the same results can be formulated also for the basis of all dyadic rectangles as well as for the basis of rare dyadic rectangles generated from a given sequence of positive integers. In the thesis it is studied the full equivalence of those bases and it is shown that they do differentiate different set of functions, depending on density of the sequence. At the end of the thesis it is also considered quasi-equivalent bases and the set of functions that such bases differentiate.

Goals. Generalize the theorem of Fatou for convolution type integral operators with general approximate identities in different functional spaces. Describe the connections between general approximate identities and optimal convergence regions for such operators. Generalize the theorem of Littlewood for convolution type integral operators with general kernels. Investigate fully and partially equivalent differentiation bases.

Research methods. Methods of theory of functions, harmonic analysis and mathematical analysis.

Scientific novelty. All results are new and are the following:

1. It is found a necessary and sufficient condition on $\lambda(r)$ that ensures almost everywhere $\lambda(r)-$convergence for convolution type integral operators in both spaces of bounded measures and integrable functions. Moreover, in the case of bounded measures, the convergence occurs at any point where the measure is differentiable. In the case of integrable functions, the convergence occurs at any Lebesgue point of the function.

2. It is discovered a necessary and sufficient condition on $\lambda(r)$ that provides almost everywhere $\lambda(r)-$convergence for the same convolution type integral operators in the space of essentially bounded functions. Additionally, the convergence occurs at any Lebesgue point of the function.

3. Under general assumptions, it is constructed a characteristic function such that the convolution with general kernels possesses everywhere divergent property along a given tangential curve. Particularly, it is proved that there exists a bounded harmonic function having everywhere strong divergent property along a given tangential curve.

4. Under general assumptions, it is constructed a bounded function, which is the boundary values of some Blaschke product, such that the convolution with general kernels owns everywhere divergent property along a given tangential curve.

5. It is found a necessary and sufficient condition for the full equivalence of basis of rare dyadic rectangles and the basis of complete dyadic rectangles in $\mathbb{R}^2$.

6. It is proved that two quasi-equivalent bases of some density basis in $\mathbb{R}^n$ differentiate the same set of non-negative functions.

Theoretical and practical value. All the results and developed methods represent theoretical interest and are applied in theories of orthogonal series, harmonic functions and differentiation of integrals.

Approbation of results. Most of the results were reported in the following conferences:

- *On theorems of Fatou and Littlewood*, International Conference, Harmonic Analysis and Approximations VI, September 12-18, 2015, Tsaghkadzor, Armenia.

- *On an equivalency of di erentiation basis of dyadic rectangles*, Armenian Mathematical Union Annual Session dedicated to the 100th anniversary of Professor Haik Badalyan, June 23-25, 2015, Yerevan, Armenia.

- *On theorems of Fatou and Littlewood*, Armenian–Georgian Conference, September, 2014, Tsaxkadzor, Armenia.

Publications. Main results of the thesis are published in 5 works (4 papers and 1 presentation), which are listed at the end of references.

Structure and volume of the thesis. The thesis consists of introduction, three chapters, conclusion and bibliography with 45 items. Total number of pages is 75.

Thesis content

The following remarkable theorems of Fatou [9] play significant role in the study of boundary value problems of analytic and harmonic functions.

Theorem A (Fatou, 1906). *Any bounded analytic function on the unit disc* $D = \{z \in \mathbb{C} : |z| < 1\}$ *has non-tangential limit for almost all boundary points.*

Theorem B (Fatou, 1906). *If a function μ of bounded variation is differentiable at $x_0 \in \mathbb{T}$, then the Poisson integral*

$$\mathcal{P}_r(x, d\mu) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - r^2}{1 - 2r\cos(x - t) + r^2} d\mu(t)$$

converges non-tangentially to $\mu'(x_0)$ as $r \to 1$.

These two fundamental theorems, have many applications in different mathematical theories including analytic functions, Hardy spaces, harmonic analysis, differential equations and etc. There are various generalization of these theorems in different aspects. Almost everywhere convergence over some semi-tangential regions investigated by Nagel and Stein [25], Di Biase [7], Di Biase-Stokolos-Svensson-Weiss [8]. Sjögren [32, 33, 34], Rönning [27, 28, 29], Katkovskaya-Krotov [17, 21], Krotov [19, 20], Brundin [5], Mizuta-Shimomura [24], Aikawa [3] studied fractional Poisson integrals with respect to the fractional power of the Poisson kernel and obtained some tangential convergence properties for such integrals. More precisely they considered the integrals

$$\mathcal{P}_r^{(1/2)}(x,f)=\int_{\mathbb{T}}P_r^{(1/2)}(x-t)f(t)\,dt=\frac{1}{c(r)}\int_{\mathbb{T}}[P_r(x-t)]^{1/2}f(t)\,dt,$$

where

$$P_r(x)=\frac{1-r^2}{1-2r\cos x+r^2},\quad 0<r<1,\quad x\in\mathbb{T}$$

is the Poisson kernel for the unit disk and

$$c(r)=\int_{\mathbb{T}}[P_r(t)]^{1/2}\,dt\asymp(1-r)^{1/2}\log\frac{1}{1-r}$$

is the normalizing coefficient. Here, the notation $A\asymp B$ means double inequality $c_1A\le B\le c_2A$ for some positive absolute constants c_1 and c_2, which might differ in each case.

Theorem C (see [32, 27, 28])**.** *For any $f\in L^p(\mathbb{T})$, $1\le p\le\infty$*

$$\lim_{r\to 1}\mathcal{P}_r^{(1/2)}(x+\theta(r),f)=f(x)\tag{1}$$

almost everywhere $x\in\mathbb{T}$, whenever

$$|\theta(r)|\le\begin{cases}c(1-r)\left(\log\frac{1}{1-r}\right)^p & \text{if}\quad 1\le p<\infty,\\ c_\alpha(1-r)^\alpha,\ \text{for any}\ 0<\alpha<1 & \text{if}\quad p=\infty,\end{cases}\tag{2}$$

where $c_\alpha>0$ is a constant, depended only on α.

The case of $p=1$ is proved in [32], $1<p\le\infty$ is considered in [27], [28]. Moreover, in [27] weak type inequalities for the maximal operator of square root Poisson integrals are established.

Theorem D (Rönning, 1997)**.** *Let $1<p<\infty$. Then the maximal operator*

$$\mathcal{P}^*_{1/2}(x,f)=\sup_{\substack{|\theta|<c(1-r)\left(\log\frac{1}{1-r}\right)^p\\ 1/2<r<1}}\mathcal{P}_r^{(1/2)}(x+\theta,|f|)$$

is of weak type (p,p).

In [17] weighted strong type inequalities for the same operators are established. Related questions were considered also in higher dimensions. Saeki [30] studied Fatou type theorems for non-radial kernels. Korani [18] extended Fatou's theorem for the Poisson-Szegö integral. In [25] Nagel and Stein proved that the Poisson integral on the upper half space of $\mathbb{R}^{n+1}$ has the boundary limit at almost every point within a certain approach region, which is not contained in any non-tangential approach regions. Sueiro [37] extended Nagel-Stein's result for the Poisson-Szegö integral. Almost everywhere convergence over tangential tress (family of curves) were investigated by Di Biase [7], Di Biase-Stokolos-Svensson-Weiss [8]. In [17] and [3] higher dimensional cases of fractional Poisson integrals are studied as well.

In **Chapter 1** we thoroughly investigate the connection between approximate identities and convergence regions. In particular, how the non-tangential convergence is connected to Poisson kernel and bounds (2) to the square root Poisson kernel.

We introduce $\lambda(r)-$convergence, which is a generalization of non-tangential convergence in the unit disc, where $\lambda(r)$ is a function

$$\lambda : (0,1) \to (0,\infty) \quad \text{with} \quad \lim_{r\to 1} \lambda(r) = 0. \tag{3}$$

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the one dimensional torus. For a given $x \in \mathbb{T}$ we define $\lambda(r,x)$ to be the interval $[x-\lambda(r), x+\lambda(r)]$. If $\lambda(r) \geq \pi$ we assume that $\lambda(r,x) = \mathbb{T}$. Let $F_r(x)$ be a family of functions from $L^1(\mathbb{T})$, where r varies in $(0,1)$. We say $F_r(x)$ is $\lambda(r)-$convergent at a point $x \in \mathbb{T}$ to a value A, if

$$\lim_{r\to 1} \sup_{\theta\in\lambda(r,x)} |F_r(\theta) - A| = 0.$$

Otherwise this relation will be denoted by

$$\lim_{\substack{r\to 1 \\ \theta\in\lambda(r,x)}} F_r(\theta) = A. \tag{4}$$

We say $F_r(x)$ is $\lambda(r)-$divergent at $x \in \mathbb{T}$ if (4) does not hold for any $A \in \mathbb{R}$.

Denote by $\mathrm{BV}(\mathbb{T})$ the functions of bounded variation on $\mathbb{T}$. Any given function of bounded variation $\mu \in \mathrm{BV}(\mathbb{T})$ defines a Borel measure on $\mathbb{T}$. We consider the family of integrals

$$\Phi_r(x, d\mu) = \int_{\mathbb{T}} \varphi_r(x-t)\, d\mu(t), \quad \mu \in \mathrm{BV}(\mathbb{T}), \tag{5}$$

where $0 < r < 1$ and kernels $\varphi_r \in L^\infty(\mathbb{T})$ form an *approximate identity*, that is

Φ1. $\int_{\mathbb{T}} \varphi_r(t)\, dt \to 1$ as $r \to 1$,

Φ2. $\varphi_r^*(x) = \sup_{|x|\leq|t|\leq\pi} |\varphi_r(t)| \to 0$ as $r \to 1, \quad 0 < |x| \leq \pi$, Φ3. $\sup\limits_{0<r<1} \|\varphi_r^*\|_1 < \infty$.

In case of μ is absolutely continuous and $d\mu(t) = f(t)dt$ for some $f \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$, then the integral (5) will be denoted as $\Phi_r(x, f)$.

Carlsson [6] obtained some weak type inequalities for non-negative approximate identities:

Theorem E (Carlsson, 2008). *Let $\{\varphi_r(x) \geq 0\}$ be an approximate identity and $\rho(r) = \|\varphi_r\|_q^{-p}$, where $1 \leq p < \infty$ and $q = p/(p-1)$ is the conjugate number of p. Then for any $f \in L^p(\mathbb{T})$*

$$\sup_{\substack{|\theta|<c\rho(r)\\0<r<1}} |\Phi_r(x+\theta, f)| \leq C(M|f|^p(x))^{1/p}, \quad x \in \mathbb{T},$$

where the constant C does not depend on function f.

Although Theorem E gives a general connection, we will see that the regions associated with function $\rho(r)$ are not optimal in general and can be improved. The central question of Chapter 1 is the following:

Question. *For a given approximate identity $\{\varphi_r\}$ what is the necessary and sufficiant condition on $\lambda(r)$ for which*

- $\lim\limits_{\substack{r\to1\\y\in\lambda(r,x)}} \Phi_r(x, d\mu) = \mu'(x)$ *almost everywhere for any $\mu \in \mathrm{BV}(\mathbb{T})$?*
- $\lim\limits_{\substack{r\to1\\y\in\lambda(r,x)}} \Phi_r(x, f) = f(x)$ *almost everywhere for any $f \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$?*

An analogous question can also be formulated for $f \in C(\mathbb{T})$. However, in this case it can be shown that (3) already sufficient for everywhere $\lambda(r)-$convergence.

We prove that the condition

$$\Pi(\lambda, \varphi) = \limsup_{r\to1} \lambda(r)\|\varphi_r\|_\infty < \infty$$

is necessary and sufficient for almost everywhere $\lambda(r)-$convergence of the integrals $\Phi_r(x, d\mu)$, $\mu \in \mathrm{BV}(\mathbb{T})$ as well as $\Phi_r(x, f)$, $f \in L^1(\mathbb{T})$. Moreover, we prove that convergence holds at any point where μ is differentiable for the integrals $\Phi_r(x, d\mu)$ and at any Lebesgue point of $f \in L^1(\mathbb{T})$ for the integrals $\Phi_r(x, f)$.

Definition 1.1. *We say that a given approximate identity $\{\varphi_r\}$ is regular if each $\varphi_r(x)$ is non-negative, decreasing on $[0, \pi]$ and increasing on $[-\pi, 0]$.*

Theorem 1.1. *Let $\{\varphi_r\}$ be a regular approximate identity and $\lambda(r)$ satisfies the condition $\Pi(\lambda,\varphi)<\infty$. If $\mu\in \mathrm{BV}(\mathbb{T})$ is differentiable at x_0, then*

$$\lim_{\substack{r\to 1\\ x\in\lambda(r,x_0)}} \Phi_r(x,d\mu)=\mu'(x_0).$$

An analogous theorem holds as well in the non-regular case of kernels, but at this time the points where (5) converges satisfy strong differentiability condition.

Definition 1.2. *We say a given function of bounded variation μ is strong differentiable at $x_0\in\mathbb{T}$, if there exist a number c such that the variation of the function $\mu(x)-cx$ has zero derivative at $x=x_0$.*

If μ is absolutely continuous and $d\mu(t)=f(t)dt$ then this property means that x_0 is a Lebesgue point for $f(x)$, i.e.

$$\lim_{h\to 0}\frac{1}{2h}\int_{-h}^{h}|f(x)-f(x_0)|dx=0.$$

It is well-known that strong differentiability at x_0 implies the existence of $\mu'(x_0)$, and any function of bounded variation is strong differentiable almost everywhere.

Theorem 1.2. *Let $\{\varphi_r\}$ be an arbitrary approximate identity and $\lambda(r)$ satisfies the condition $\Pi(\lambda,\varphi)<\infty$. If $\mu\in \mathrm{BV}(\mathbb{T})$ is strong differentiable at $x_0\in\mathbb{T}$, then*

$$\lim_{\substack{r\to 1\\ x\in\lambda(r,x_0)}} \Phi_r(x,d\mu)=\mu'(x_0).$$

The following theorem implies the sharpness of the condition $\Pi(\lambda,\varphi)<\infty$ in Theorem 1.1 and Theorem 1.2.

Theorem 1.3. *If $\{\varphi_r\}$ is an arbitrary approximate identity and the function $\lambda(r)$ satisfies the condition $\Pi(\lambda,\varphi)=\infty$, then there exist a function $f\in L^1(\mathbb{T})$ such that*

$$\limsup_{\substack{r\to 1\\ y\in\lambda(r,x)}} \Phi_r(y,f)=\infty$$

for all $x\in\mathbb{T}$.

Thus, the condition $\Pi(\lambda,\varphi)<\infty$ determines the exact rate of $\lambda(r)$ function, ensuring such convergence. It is interesting, that this rate depends only on the values $\|\varphi_r\|_\infty$. Notice that, if the kernel φ_r coincides with the Poisson kernel P_r (which is a regular approximate identity), then $\|P_r\|_\infty\asymp\frac{1}{1-r}$ and the bound $\Pi(\lambda,P)<\infty$ coincides with the well-known condition

$$\limsup_{r\to 1}\frac{\lambda(r)}{1-r}<\infty, \tag{6}$$

guaranteeing non-tangential convergence in the unit disk. So, Theorem 1.1 implies and generalizes Fatou's theorem. Furthermore, if we take the fractional Poisson kernel $P_r^{(1/2)}$ (which is regular as well), then

$$\|P_r^{(1/2)}\|_\infty = \frac{1}{c(r)}\|P_r^{1/2}\|_\infty \asymp \left((1-r)\log\frac{1}{1-r}\right)^{-1}$$

and from Theorem 1.1 we deduce (1) when $p = 1$ with an additional information about the points where the convergence occurs.

Additionally, some weak type inequalities are established for the associated maximal operator Φ_λ^*, which is defined as

$$\Phi_\lambda^*(x, f) = \sup_{\substack{|x-y|<\lambda(r)\\0<r<1}} |\Phi_r(y, f)| = \sup_{\substack{|x-y|<\lambda(r)\\0<r<1}} \left|\int_{\mathbb{T}} \varphi_r(y-t)f(t)\,dt\right|. \tag{7}$$

Theorem 1.4. *Let $\{\varphi_r\}$ be an arbitrary approximate identity and for some $1 \le p < \infty$ the function $\lambda(r)$ satisfies*

$$\tilde{\Pi}_p(\lambda, \varphi) = \sup_{0<r<1} \lambda(r)\|\varphi_r\|_\infty \varphi_*(r)^{p-1} < \infty,$$

where

$$\varphi_*(r) = \sup_{x\in\mathbb{T}} |x\varphi_r^*(x)|.$$

Then for any $f \in L^1(\mathbb{T})$

$$\Phi_\lambda^*(x, f) \le C\left(M|f|^p(x)\right)^{1/p}, \quad x \in \mathbb{T},$$

where the constant C does not depend on function f.

Using the standard methods, it can be shown that these weak type inequalities imply almost everywhere $\lambda(r)-$convergence with the condition

$$\Pi_p(\lambda, \varphi) = \limsup_{r\to 1} \lambda(r)\|\varphi_r\|_\infty \varphi_*^{p-1}(r) < \infty,$$

from which we can conclude (1) when $1 < p < \infty$ as well as Theorem D.

An analogous necessary and sufficient condition will be established also for almost everywhere $\lambda(r)-$convergence of $\Phi_r(x, f)$, $f \in L^\infty(\mathbb{T})$, and this condition looks like

$$\Pi_\infty(\lambda, \varphi) = \limsup_{\delta\to 0}\limsup_{r\to 1} \int_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t)dt = 0,$$

which contains more information about $\{\varphi_r\}$ than $\Pi(\lambda, \varphi)$ does.

Theorem 1.5. *If $\{\varphi_r\}$ is a regular approximate identity consisting of even functions and the function $\lambda(r)$ satisfies $\sqcap_\infty(\lambda,\varphi)=0$, then for any $f\in L^\infty(\mathbb{T})$ the relation*

$$\lim_{\substack{r\to 1\\ y\in\lambda(r,x)}} \Phi_r(y,f)=f(x)$$

holds at any Lebesgue point $x\in\mathbb{T}$.

Theorem 1.6. *If $\{\varphi_r\}$ is a regular approximate identity consisting of even functions and the function $\lambda(r)$ satisfies $\sqcap_\infty(\lambda,\varphi)>0$, then there exists a set $E\subset\mathbb{T}$, such that $\Phi_r(x,\mathbb{I}_E)$ is $\lambda(r)-$divergent at any $x\in\mathbb{T}$.*

One can easily check that in the case of Poisson kernel $P_r(t)$, for a given function $\lambda(r)$ with (3), the value of $\sqcap_\infty(\lambda,P)$ can be either 0 or 1. Besides, the condition $\sqcap_\infty(\lambda,P)=0$ is equivalent to (6), and $\sqcap_\infty(\lambda,P)=1$ coincides with

$$\limsup_{r\to 1}\frac{\lambda(r)}{1-r}=\infty.$$

Now suppose that $\lambda(r)$ satisfies the condition (2) with $p=\infty$. Simple calculations show that for such $\lambda(r)$ and for the square root Poisson kernel $P_r^{(1/2)}(t)$ we have $\sqcap_\infty(\lambda,P^{(1/2)})=0$. Hence, Theorem 1.5 implies (1) when $p=\infty$ with an additional information about the points where the convergence occurs. Taking $\lambda(r)=(1-r)^\alpha$ with a fixed $0<\alpha<1$ we will get $\sqcap_\infty(\lambda,P^{(1/2)})=1-\alpha>0$, and applying Theorem 1.6 we conclude the optimality of the bound (2) in the case $p=\infty$ too.

In the definition of $\lambda(r)-$convergence the range of the parameter r is $(0,1)$ with the limit point 1, that is, we consider the convergence or divergence properties when $r\to 1$. We do this way in order to compare our results with the boundary properties of analytic and harmonic functions in the unit disc. Certainly it is not essential in the theorems. We could take any set $Q\subset\mathbb{R}$ with limit point r_0 which is either a finite number or ∞. We may define an approximate identity on the real line to be a family of functions $\varphi_r\in L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$, $r>0$, which satisfies the same conditions *Φ1 − Φ3* as approximate identity on $\mathbb{T}$ does. We just need to make a little change in the condition *Φ2*, that is to add $\left\|\varphi_r^*\cdot\mathbb{I}_{\{|t|\ge\delta\}}\right\|_1\to 0$ as $r\to 0$ for any $\delta>0$. In this case usually convergence is considered while $r\to 0$. Analogously, all the results Theorem 1.1−Theorem 1.6 can be formulated and proved for the integrals

$$\Phi_r(x,d\mu)=\int_{\mathbb{R}}\varphi_r(x-t)\,d\mu(t),\quad \mu\in \mathrm{BV}(\mathbb{R}),\quad r>0, \tag{8}$$

and they can be done just repeating the proofs with miserable changes.

Any function $\Phi \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\|\Phi\|_1 = 1$ and $\Phi^* \in L^1(\mathbb{R})$ defines an approximate identity by

$$\varphi_r(x) = \frac{1}{r}\Phi\left(\frac{x}{r}\right) \quad \text{as} \quad r \to 0.$$

Operators corresponding to such kernels in higher dimensional case were investigated by Stain ([35], p. 57). Note for such kernels we have

$$\|\varphi_r\|_\infty = \frac{1}{r}\|\Phi\|_\infty, \quad \varphi_*(r) = \sup_{x\in\mathbb{R}} |x\Phi^*(x)| \le \|\Phi^*\|_1$$

and therefore, for $1 \le p < \infty$, the condition $\Pi_p(\lambda, \varphi) < \infty$ takes the form $\lambda(r) \le c{\cdot}r$. The case $p = \infty$ can be done in the same way as we did it for the Poisson kernel. The value $\Pi_\infty(\lambda, \Phi)$ can be either 0 or 1, the condition $\Pi_\infty(\lambda, \Phi) = 0$ is equivalent to $\lambda(r) \le c{\cdot}r$ and the condition $\Pi_\infty(\lambda, \Phi) = 1$ is equivalent to $\limsup_{r\to 0} \lambda(r)/r = \infty$. The bound $\lambda(r) \le c \cdot r$ characterizes the non-tangential convergence in the upper half plane and it turns out to be a necessary and sufficient condition for almost everywhere $\lambda(r)-$convergence of the integrals (8).

In addition, we would like to bring one consequence of our results, that we consider interesting.

Corollary 1.1. *If $\sigma_n(x, f)$ are the Fejer means of Fourier series of a function $f \in L^1(\mathbb{T})$ and $\theta_n = O(1/n)$, then $\sigma_n(x + \theta_n, f) \to f(x)$ at any Lebesgue point $x \in \mathbb{T}$.*

Littlewood [22] made an important complement to the theorem of Fatou, proving essentiality of non-tangential approach in that theorem. The following formulation of Littlewood's theorem fits to the further aim of the thesis.

Theorem F (Littlewood, 1927)**.** *If a continuous function $\lambda : [0, 1] \to \mathbb{R}$ satisfies the conditions*

$$\lambda(1) = 0, \quad \lim_{r\to 1} \frac{\lambda(r)}{1 - r} = \infty, \tag{9}$$

then there exists a bounded analytic function $f(z)$, $z \in D$, such that the boundary limit

$$\lim_{r\to 1} f\left(re^{i(x+\lambda(r))}\right)$$

does not exist almost everywhere on $\mathbb{T}$.

There are various generalization of these theorems in different aspects. A simple proof of this theorem was given by Zygmund [41]. In [23] Lohwater and Piranian proved, that in Littlewood's theorem almost everywhere divergence can be replaced to everywhere and the example function can be a Blaschke product. That is

Theorem G (Lohwater and Piranian, 1957)**.** *If $\lambda(r)$ is a continuous function with (9), then there exists a Blaschke product $B(z)$ such that the limit*

$$\lim_{r\to 1} B\left(re^{i(x+\lambda(r))}\right)$$

does not exist for any $x \in \mathbb{T}$.

In [1] Aikawa obtained a similar everywhere divergence theorem for bounded harmonic functions on the unit disk, giving a positive answer to a problem raised by Barth [[4], p. 551].

Theorem H (Aikawa, 1990)**.** *If $\lambda(r)$ is a continuous function with (9), then there exists a bounded harmonic function $u(z)$ on the unit disc, such that the limit*

$$\lim_{r\to 1} u\left(re^{i(x+\lambda(r))}\right)$$

does not exist for any $x \in \mathbb{T}$.

Related questions were considered also in higher dimensions. Littlewood type theorems for the higher dimensional Poisson integral established by Aikawa [1, 2] and for the Poisson-Szegö integral by Hakim-Sibony [12] and Hirata [14].

In **Chapter 2** we generalize Littlewood's theorem for the integrals $\Phi_r(x, f)$ with more general kernels than approximate identities. Namely, we consider the same integrals $\Phi_r(x, f)$ with a family of kernels $\{\varphi_r\}$ satisfying

Φ1. $\int_{\mathbb{T}} \varphi_r(t)\, dt \to 1 \quad \text{as} \quad r \to 1,$

Φ4. $\varphi_r(x) \geq 0, \quad x \in \mathbb{T}, 0 < r < 1,$

Φ5. $\sup_{0<r<\tau} \|\varphi_r\|_\infty < \infty, \quad 0 < \tau < 1.$

We introduce another quantity

$$\Pi^*(\lambda, \varphi) = \limsup_{\delta\to 0} \liminf_{r\to 1} \int_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t)\, dt \leq \Pi_\infty(\lambda, \varphi)$$

and prove the following theorems.

Theorem 2.1. *Let $\{\varphi_r\}$ be a family of kernels with Φ1, Φ4, Φ5. If a function $\lambda \in C[0,1]$ satisfies the conditions $\lambda(1) = 0$ and $\Pi^*(\lambda, \varphi) > 1/2$, then there exists a measurable set $E \subset \mathbb{T}$ such that*

$$\limsup_{r\to 1} \Phi_r\left(x + \lambda(r), \mathbb{I}_E\right) - \liminf_{r\to 1} \Phi_r\left(x + \lambda(r), \mathbb{I}_E\right) \geq 2\Pi^* - 1.$$

In the case of Poisson kernel under the condition (9) we have $\Pi^* = 1 > 1/2$. Therefore, Theorem 2.1 implies the following generalization of Theorem F and Theorem H, giving additional information about the divergence character.

Corollary 2.1. *For any function $\lambda \in C[0,1]$ satisfying (9), there exists a harmonic function $u(z)$, $z \in D$ on the unit disc with $0 \le u(z) \le 1$, such that*

$$\limsup_{r\to 1} u\left(re^{i(x+\lambda(r))}\right) = 1, \quad \liminf_{r\to 1} u\left(re^{i(x+\lambda(r))}\right) = 0,$$

at any point $x \in \mathbb{T}$.

The higher dimensional case of this corollary was considered by Hirata [14]. We construct also a Blaschke product with Littlewood type divergence condition as in Theorem 2.1, which generalizes Theorem G. In this case a stronger condition $\Pi^*(\lambda, \varphi) = 1$ is required.

Theorem 2.2. *Let a family of kernels $\{\varphi_r\}$ satisfies $\Phi 1$, $\Phi 4$, $\Phi 5$ and for $\lambda \in C[0,1]$ we have $\lambda(1) = 0$ and $\Pi^*(\lambda, \varphi) = 1$. Then there exists a function $B \in L^\infty(\mathbb{T})$, which is the boundary function of a Blaschke product, such that the limit*

$$\lim_{r\to 1} \Phi_r\left(x + \lambda(r), B\right)$$

does not exist for any $x \in \mathbb{T}$.

Note that, as Theorem 1.1–Theorem 1.6, Theorem 2.1 can also be formulated and proved for the integrals

$$\Phi_r(x, f) = \int_{\mathbb{R}} \varphi_r(x-t) f(t)\, dt, \quad f \in L^1(\mathbb{R}), \quad 0 < r < 1, \tag{10}$$

where the kernels $\varphi_r \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ satisfy the conditions $\Phi 1$, $\Phi 4$, $\Phi 5$. Furthermore, notice that for any positive function $\Phi \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\|\Phi\|_1 = 1$ the kernels

$$\varphi_r(x) = \frac{1}{1-r}\Phi\left(\frac{x}{1-r}\right), \quad x \in \mathbb{R}, \quad 0 < r < 1 \tag{11}$$

satisfy the conditions $\Phi 4$ and $\Phi 5$. One can check, that for the Poisson kernel and for (11) the following conditions are equivalent

$$\lim_{r\to 1} \frac{\lambda(r)}{1-r} = \infty \iff \Pi^*(\lambda, \varphi) = 1 \iff \Pi^*(\lambda, \varphi) > 0.$$

Therefore, if the kernels in (10) coincide with (11) and $\lambda(r)$ satisfies (9), then Theorem 2.1 formulated for the integrals (10) implies everywhere strong-type divergence for (10), which covers the one-dimensional case of a theorem obtained by Aikawa in [3].

Now we proceed to the **Chapter 3**.

Let $\mathcal{R}^n$ be the family of half-open (or half-closed) rectangles $\prod_{i=1}^{n}[a_i, b_i)$ in $\mathbb{R}^n$ and $\mathcal{DR}^n$ be the family of dyadic rectangles of the form

$$\prod_{i=1}^{n}\left[\frac{j_i-1}{2^{m_i}}, \frac{j_i}{2^{m_i}}\right), \quad j_i, m_i \in \mathbb{Z}, \quad i = 1, 2, \dots, n. \tag{12}$$

Definition 3.1. *A family $\mathcal{B}$ of bounded, positively measured sets from $\mathbb{R}^n$ is said to be a differentiation basis (or simply basis), if for any point $x \in \mathbb{R}^n$ there exists a sequence of sets $E_k \in \mathcal{B}$ such that $x \in E_k$, $k = 1, 2, \dots$ and $\operatorname{diam}(E_k) \to 0$ as $k \to \infty$.*

Let $\mathcal{B}$ be a differentiation basis and $L_{\mathrm{loc}}(\mathbb{R}^n)$ be the space of locally integrable functions:

$$L_{\mathrm{loc}}(\mathbb{R}^n) = \{f \colon f \in L(K) \text{ for any compact } K \subset \mathbb{R}^n\}.$$

For any function $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$ we define

$$\delta_{\mathcal{B}}(x, f) = \limsup_{\operatorname{diam}(E)\to 0,\, x\in E\in\mathcal{B}} \left|\frac{1}{|E|}\int_E f(t)dt - f(x)\right|.$$

The integral of a function $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be differentiable at a point $x \in \mathbb{R}^n$ with respect to the basis $\mathcal{B}$, if $\delta_{\mathcal{B}}(x, f) = 0$. The integral of a function is said to be differentiable with respect to the basis $\mathcal{B}$, if it is differentiable at almost every point. Consider the following classes of functions

$$\mathcal{F}(\mathcal{B}) = \{f \in L_{\mathrm{loc}}(\mathbb{R}^n) : \delta_{\mathcal{B}}(x, f) = 0 \text{ almost everywhere }\},$$
$$\mathcal{F}^+(\mathcal{B}) = \{f \in L_{\mathrm{loc}}(\mathbb{R}^n) : f(x) \geq 0,\ \delta_{\mathcal{B}}(x, f) = 0 \text{ almost everywhere }\}.$$

Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex function. Denote by $\Psi(L)(\mathbb{R}^n)$ the class of measurable functions f defined on $\mathbb{R}^n$ such that $\Psi(|f|) \in L^1(\mathbb{R}^n)$. If Φ satisfies the Δ_2-condition $\Psi(2x) \leq k\Psi(x)$, then $\Psi(L)$ turns to be an Orlicz space with the norm

$$\|f\|_{\Psi} = \inf\left\{c > 0 : \int_{\mathbb{R}^n} \Psi\left(\frac{|f|}{c}\right) \leq 1\right\}.$$

The following classical theorems determine the optimal Orlicz space, which functions have a.e. differentiable integrals with respect to the entire family of rectangles $\mathcal{R}^n$ is the space

$$L(1 + \log^+ L)^{n-1}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n),$$

corresponding to the case $\Psi(t)=t(1+\log^+ t)^{n-1}$ ([10]).

Theorem I (Jessen–Marcinkiewicz–Zygmund, [15]). $L(1+\log^+ L)^{n-1}(\mathbb{R}^n)\subset \mathcal{F}(\mathcal{R}^n)$.

Theorem J (Saks, [31]). *If the function* Ψ *satisfies*

$$\Psi(t)=o(t\log^{n-1}t) \text{ as } t\to\infty,$$

then $\Psi(L)(\mathbb{R}^n)\not\subset \mathcal{F}(\mathcal{R}^n)$. *Moreover, there exists a positive function* $f\in\Psi(L)(\mathbb{R}^n)$ *such that* $\delta_{\mathcal{R}^n}(x,f)=\infty$ *everywhere.*

Such theorems are valid also for the basis $\mathcal{DR}^n$. The first one trivially follows from embedding $L(1+\log^+ L)^{n-1}(\mathbb{R}^n)\subset\mathcal{F}(\mathcal{R}^n)\subset\mathcal{F}(\mathcal{DR}^n)$. The second can be deduced from the following

Theorem K (Zerekidze, [38] (see also [39, 40])). $\mathcal{F}^+(\mathcal{DR}^n)=\mathcal{F}^+(\mathcal{R}^n)$.

Let $\Delta=\{\nu_k : k=1,2,\dots\}$ be an increasing sequence of positive integers. This sequence generates rare basis $\mathcal{DR}^n_\Delta$ of dyadic rectangles of the form (12) with $m_i\in\Delta$, $i=1,2,\dots,n$. This kind of bases first considered in the papers [36], [11], [13]. Stokolos [36] proved that the analogous of Saks theorem holds for any basis $\mathcal{DR}^n_\Delta$ with an arbitrary Δ sequence. That means $L(1+\log^+ L)^{n-1}(\mathbb{R}^n)$ is again the largest Orlicz space containing in $\mathcal{F}(\mathcal{DR}^n_\Delta)$. Oniani and Zerekidze [26] characterised translation invariant as well as net type bases formed of rectangles that are equivalent to the basis of all rectangles in the class of all non-negative functions. Karagulyan [16] proved some theorems, establishing an equivalency of some convergence conditions for multiple martingale sequences, those in particular imply some results of the papers [36], [11], [13].

In spite of the largest Orlicz spaces corresponding to the bases $\mathcal{DR}^2_\Delta$ and $\mathcal{DR}^2$ coincide, they do differentiate different set of functions, depending on density of the sequence Δ. We prove that the condition

$$\gamma_\Delta=\sup_{k\in\mathbb{N}}(\nu_{k+1}-\nu_k)<\infty$$

is necessary and sufficient for the full equivalency of rare dyadic basis $\mathcal{DR}^2_\Delta$ and complete dyadic basis $\mathcal{DR}^2$.

Theorem 3.1. *If* $\Delta=\{\nu_k\}$ *is an increasing sequence of positive integers with* $\gamma_\Delta<\infty$, *then*

$$\mathcal{F}(\mathcal{DR}^2_\Delta)=\mathcal{F}(\mathcal{DR}^2).$$

Theorem 3.2. *If $\Delta = \{\nu_k\}$ is an increasing sequence of positive integers with $\gamma_\Delta = \infty$, then there exists a function $f \in \mathcal{F}(\mathcal{DR}^2_\Delta)$ such that*

$$\limsup_{\operatorname{len}(R)\to 0,\, x\in R\in\mathcal{DR}^2} \left| \frac{1}{|R|} \int_R f(t)\, dt \right| = \infty$$

for any $x \in \mathbb{R}^n$.

Definition 3.2. *A basis $\mathcal{B}$ is said to be density basis if $\mathcal{B}$ differentiates the integral of any characteristic function $\mathbb{I}_E$ of measurable set E:*

$$\delta_{\mathcal{B}}(x, \mathbb{I}_E) = 0 \text{ at almost every } x \in \mathbb{R}^n.$$

We will say that the basis $\mathcal{B}$ differentiates a class of functions $\mathcal{F}$, if basis $\mathcal{B}$ differentiates the integrals of all functions of $\mathcal{F}$.

Definition 3.3. *Let $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B}$ be subbases. We will say that basis $\mathcal{B}_2$ is quasi-coverable by basis $\mathcal{B}_1$ (with respect to basis $\mathcal{B}$) if for any $R \in \mathcal{B}_2$ there exist $R_k \in \mathcal{B}_1$, $k = 1, 2, \ldots, p$ and $R' \in \mathcal{B}$ such that*

$$\begin{aligned}
&R \subseteq \tilde{R} \subseteq R', \quad \tilde{R} = \bigcup_{k=1}^{p} R_k \\
&\operatorname{diam}(R') \le c \cdot \operatorname{diam}(R), \quad |R'| \le c|R_k|, \quad k = 1, 2, \ldots, p, \\
&\sum_{k=1}^{p} |R_k| \le c|\tilde{R}|, \quad |\tilde{R}| \le c|R|,
\end{aligned}$$

where constant $c \ge 1$ depends only on bases $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}$. We will say two bases are quasi-equivalent if they are quasi-coverable with respect to each other.

We prove that quasi-equivalent subbases $\mathcal{B}_1, \mathcal{B}_2$ of density basis $\mathcal{B}$ differentiate the same class of non-negative functions.

Theorem 3.3. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be subbases of density basis $\mathcal{B}$ formed of open sets from $\mathbb{R}^n$. If the bases $\mathcal{B}_1$ and $\mathcal{B}_2$ are quasi-equivalent with respect to $\mathcal{B}$ then*

$$\mathcal{F}^+(\mathcal{B}_1) = \mathcal{F}^+(\mathcal{B}_2).$$

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1. Safaryan M. H., *On an equivalency of rare differentiation bases of rectangles*, Journal of Contemporary Math. Anal., 2018, vol. 53, no. 1, pp. 57–61.

2. Karagulyan G. A., Safaryan M. H., *On a theorem of Littlewood*, Hokkaido Math J., 2017, vol. 46, no. 1, 87–106.

3. Karagulyan G. A., Karagulyan D. A., Safaryan M. H., *On an equivalence for differentiation bases of dyadic rectangles*, Colloq. Math., 2017, 3506, 295–307.

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ԱՄՓՈՓՈՒՄ

Աշխատանքը բաղկացած է երեք գլխից:

Առաջին գլխում ուսումնասիրվում են ոչ-շոշափողային զուգամիտության վերաբերյալ Ֆատուի թեորեմի որոշ ընդհանրացումներ ընդհանուր կորիզով փաթույթ տիպի ինտեգրալ օպերատորների համար: Ներմուծվում է նոր զուգամիտության տեսակ, $\lambda(r)$−զուգամիտություն, որը հանդիսանում է միավոր շրջանում ոչ-շոշոփողային զուգամիտության ընդհանրացում: Նկարագրվում է ընդհանուր կորիզների և զուգամիտության օպտիմալ տիրույթների կապը: Ստացված արդյունքները ընդհանրացնում են նաև շոշափողային զուգամիտության հետ կապված որոշ արդյունքներ:

1. Ստացվել է անհրաժեշտ և բավարար պայման $\lambda(r)$ ֆունկցիայի համար, որն ապահովում է ընդհանուր կորիզով փաթույթ տիպի ինտեգրալ օպերատորների համարյա ամենուրեք $\lambda(r)$−զուգամիտությունը վերջավոր չափերի և ինտեգրելի ֆունկցիաների տարածություններում: Ավելին, վերջավոր չափի դեպքում զուգամիտությունը տեղի ունի ցանկացած կետում, որտեղ չափը դիֆերենցելի է, իսկ ինտեգրելի ֆունկցիայի դեպքում զուգամիտությունը տեղի ունի ֆունկցիայի ցակացած Լեբեգի կետում:

2. Նմանատիպ անհրաժեշտ և բավարար պայման ստացվել է նաև էապես սահմանափակ ֆունկցիաների տարածության համար: Ընդ որում, զուգամիտությունը տեղի ունի ֆունկցիայի ցակացած Լեբեգի կետում:

Երկրորդ գլխում հետազոտվում են Լիթլվուդի թեորեմի որոշ ընդհանրացումներ ընդհանուր կորիզով փաթույթ տիպի ինտեգրալ օպերատորներ համար: Լիթլվուդի թեորեմը հանդիսանում է Ֆատուի թեորեմի կարևոր լրացում՝ կառուցելով անալիտիկ ֆունկցիա միավոր շրջանի վրա, որն օժտված է համարյա ամենուրեք տարամիտության հատկությամբ տրված շոշափողային կորի երկայնքով: Ձևակերպվում են Լիթլվուդի թեորեմի երկու տիպի ընդհանրացումներ, որոնք օժտված են ամենուրեք տարամիտության հատկությամբ:

3. Նշվում են ընդհանուր պայմաններ, որոնց առկայության դեպքում կառուցվում է այնպիսի բնութագրիչ ֆունկցիա, որի փաթույթ տիպի ինտեգրալ օպերատորը տրված շոշափողային կորի երկայնքով տարամիտում է ամենուրեք: Մասնավորաբար, տրված շոշափողային կորի համար կառուցվում է սահմանափակ հարմոնիկ ֆունկցիա, որը կորի երկայնքով արտահայտում է ամենուրեք ուժեղ տարամիտության հատկություն:

4. Նշվում են ընդհանուր պայմաններ, որոնց առկայության դեպքում կառուցվում է սահմանափակ ֆունկցիա, որն իրենից ներկայացնում է Բլյաշկեի արտադրյալի եզրային արժեքներ և օժտված է փրված շոշափողային կորի երկայնքով ամենուրեք պարամիտության հատկությամբ:

Երրորդ գլուխը նվիրված է $\mathbb{R}^n$ տարածություններում դիֆերենցիալ բազիսների համարժեքության որոշ հարցերին: Դիտարկվում է նոսր երկուական ուղղանկյունների և բոլոր երկուական ուղղանկյունների լրիվ համարժեքությունը $\mathbb{R}^2$ տարածությունում: Երկու դիֆերենցիալ բազիսների միջև սահմանվում է քվազի-համարժեքություն և դիտարկվում են ֆունկցիաներ որոնք դիֆերենցվում են այդ բազիսների նկատմամբ:

5. Ստացվել է անհրաժեշտ և բավարար պայման, որի դեպքում $\mathbb{R}^2$ տարածությունում նոսր երկուական ուղղանկյուններից կազմված բազիսը և բոլոր երկուական ուղղանկյուններից կազմված բազիսը դիֆերենցում են նույն ֆունկցիաների բազմությունը:

6. Ապացուցվում է, որ $\mathbb{R}^n$ տարածությունում միևնույն խտության բազիսի երկու քվազի-համարժեք ենթաբազիսներ դիֆերենցում են նույն ոչ-բացասական ֆունկցիաների բազմությունը:

ЗАКЛЮЧЕНИЕ

Диссертация состоит из трех глав.

В главе 1 исследуются обобщения теоремы Фату для интегральных операторов типа свертки с общими ядрами. Вводится $\lambda(r)-$сходимость, являющаяся обобщением некасательной сходимости в единичном круге. Описаны связи между ядрами и областями оптимальной сходимости для таких операторов в разных функциональных пространствах.

1. Обнаружено необходимое и достаточное условие для $\lambda(r)$, обеспечивающего почти всюду $\lambda(r)-$сходимость для интегральных операторов типа свертки в обоих пространствах ограниченных мер и интегрируемых функций. Более того, в случае ограниченных мер сходимость происходит в любой точке, где мера дифференцируема. В случае интегрируемых функций, сходимость происходит в любой точке Лебега функции.

2. Обнаружено необходимое и достаточное условие для $\lambda(r)$, обеспечивающего почти всюду $\lambda(r)-$сходимость для тех же интегральных операторов типа свертки в пространстве существенно ограниченных функций. Кроме того, сходимость происходит в любой точке Лебега функции.

В главе 2 изучаются некоторые обобщения теоремы Литтлвуда, которая делает важное дополнение к теореме Фату, создавая аналитическую функцию, обладающую почти везде расходящимся свойством вдоль данной касательной кривой. Те же интегральные операторы типа свертки рассматриваются с более общими ядрами. Получены два вида обобщений теоремы Литтлвуда, обладающих всюду расходящимся свойством.

3. При общих предположениях строится такая характеристическая функция, при которой свертка с общими ядрами обладает всюду расходящимся свойством вдоль данной касательной кривой. В частности, доказано существование ограниченной гармонической функции, имеющей всюду сильное расходящееся свойство вдоль данной касательной кривой.

4. При общих предположениях строится ограниченная функция, являющаяся граничным значением некоторого произведения Бляшке, при которой свертка с общими ядрами обладает всюду расходящимся свойством вдоль данной касательной кривой.

Глава 3 посвящена некоторым вопросам эквивалентности дифференциальных базисов в $\mathbb{R}^n$. Исследуется полная эквивалентность базиса редких диадических прямоугольников и базиса полных диадических прямоугольников в $\mathbb{R}^2$. Введена квази-эквивалентность между двумя дифференциальными базисамы в $\mathbb{R}^n$ и рассматривается множество функций, которые дифференцируют такие базисы.

5. Обнаружено необходимое и достаточное условие для полной эквивалентности базиса редких диадических прямоугольников и базиса полных диадических прямоугольников в $\mathbb{R}^2$.

6. Доказано, что два квази-эквивалентных базиса некоторого плотностного базиса в $\mathbb{R}^n$ дифференцируют одно и то же множество неотрицательных функций.