ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Անահիտ Գևորգի Գուլյան

ՄԱՐՏԻՆԳԱԼԱՅԻՆ ՄՈՏԵՑՈՒՄ ԲՈՆՈՒՍ-ՄԱԼՈՒՍ ՀԱՄԱԿԱՐԳԵՐԻՆ

ՍԵՂՄԱԳԻՐ

Ը.00.08-"Մաթեմատիկական տնտեսագիտություն" մասնագիտությամբ Ֆիզիկամաթեմատիկական գիտությունների թեկնածուի գիտական աստիճանի հայցման ատենախոսության

ԵՐԵՎԱՆ-2016

YEREVAN STATE UNIVERSITY

Anahit G. Gulyan

MARTINGALE APPROACH TO BONUS-MALUS SYSTEMS

ABSTRACT

of dissertation submitted for the degree of candidate of phys-math sciences
Specialty: 08.00.08-“Mathematical Economics”

YEREVAN-2016

Ատենախոսության թեման հաստատվել է Երևանի պետական համալսարանում

Գիտական ղեկավար՝ Ֆիզ-մաթ. գիտությունների թեկնածու, դոցենտ Ռ.Ն.Չիթչյան

Պաշտոնական ընդդիմախոսներ՝ Տեխնիկական գիտությունների դոկտոր Գ.Ի. Բելյավսկի (Դոնի Ռոստով, ՌԴ)

Ֆիզ-մաթ. գիտությունների դոկտոր

Վ.Կ. Օհանյան

Առաջատար կազմակերպություն՝ Մոսկվայի պետական համալսարան

Պաշտպանությունը կկայանա 2016 թ.-ի Հոկտեմբերի 17-ին ժ.15:00 Երևանի պետական համալսարանում գործող ԲՈՀ-ի 050 մասնագիտական խորհրդի հատուկ նիստում (0025, Երևան, Ալեք Մանուկյան 1):

Ատենախոսությանը կարելի է ծանոթանալ Երևանի պետական համալսարանի գրադարանում:

Սեղմագիրն առաքված է 2016թ.-ի Սեպտեմբերի 16-ին:

Մասնագիտական խորհրդի գիտական քարտուղար՝ Տ.Ն. Հարությունյան

The topic of dissertation was approved by the Yerevan State University

Scientific adviser: Candidate of Phys.-math. sciences, associate professor R.N.Chitchyan

Official opponents: Doctor of Technical sciences

G.I. Belyavski (Rostov-on-Don, Russia)

Doctor of Phys.-math. sciences

V.K.Ohanyan

Leading institution: Moscow State University

Dissertation defense will take place on 17th of October, 2016 at 15:00, at the special meeting of the Specialized Council 050 of the Supreme Certifying Commission at YSU (1 Alex Manoogian, Yeravan 0025, Armenia).

The dissertation is available in the library of Yerevan State University.

The abstract of dissertation was distributed on 16th of September, 2016.

Scientific secretary of specialized council T.N. Harutyunyan

General description

Topicality: There are many types of Bonus-Malus Systems (BMS) used in the world. They have various applications. Insurance companies operating in some countries are highly recommended to apply the same BMS for a particular class of insurance such as Compulsory Third Party Liability (CMTPL) insurance in Armenia. The most developed countries in this area, mainly European ones, entitled full or partial freedom to insurance companies by developing a highly competitive market. The common characteristic of these systems is that the transition of a policyholder from one BMS class to another is described by the number of claims incurred by that policyholder. In applying this rule many problems are arise which cannot be solved by the experience rating methods used up to now. The application of martingale theory in this field gives new opportunities to introduce more efficient systems. The topicality of the dissertation is conditioned by the study of the current issues of BMS's and finding new ways to solve them.

Objective and Issues: The main aim of the dissertation is the construction of new BMSs, which will include the number of claims and aggregate claim amount components as a posteriori risk classification. The systems constructed here must be financially balanced and the future malus of a policyholder must be proportional to the loss incurred by insurance company because of him. On the basis of the given dissertation the following problems have to be studied:

- Analyzing currently operating BMSs; considering the approaches of solving or eliminating their current issues
- Suggesting new systems satisfying to optimal BMS definition
- Analyzing suggested systems' applicability by using Armenian CMTPL insurance market data

Research Methods and Informational Backgrounds: Two new BMS models are presented in the following dissertation. On the basis of the first model's construction stands the martingale approach. For the second model the Markov process have been applied where the parameters of the model were estimated with the Expectation Maximization (EM) algorithm.

The data used in the dissertation has been gained from "IngoArmenia" Insurance CJSC, from the official site of the "Armenian Motor Insurance Bureau" and from the professional literature sources.

The databases were analyzed with the help of Easy-Fit, SPSS and MS Excel software packages.

Scientific Novelty:

1. An alternative model for BMS was proposed where a necessary and sufficient condition was found out for the premiums of the insurance policies' portfolio to form a martingale series.
2. It was shown that the proposed model can reach to a stationary state.

3. An upper bound for the ruin probability in the alternative BMS was found out with the help of martingales and supermartingales.
4. It was stated the claim amount below which the "bonus hunger" phenomenon arises in the alternative BMS model
5. A generalized BMS model was proposed where the transition of the policyholder among BMS classes described by his/her current class, by number of claims and by aggregate claim amount
6. Estimates for generalized BMS model parameters were stated with the help of "hidden" Markov models (HMM) and change of measure
7. For the claims number and aggregate claim amount random variables hypothesis for distributions were made on the basis of data received from insurance company.
8. Comparative analyzes of the current BMS with BMSs proposed in the dissertation were done.

The all results presented in the dissertation are new.

Practical and Theoretical Significance: The main results of this research have theoretical and practical character as well. The models presented in this work can be used by insurance companies as well as by the supervisory and decision-making participants of the insurance market for the research, strategic and commercial purposes.

Approbation: The main results of the dissertation have been presented in the scientific seminars held in the department of Actuarial Mathematics and Risk Management of YSU, in the IX International Academic Congress "Contemporary Science and Education in Americas, Africa and Eurasia", Rio De Janeiro, Brazil, 18-20 July, 2015. The results have been discussed at the "IngoArmenia" insurance company and at the Central Bank of Armenia.

The main results of the dissertation are presented in 4 (four) articles, 3 (three) of which were published in the journals accepted by Supreme Certifying Commission (SCC) of Armenia and one was included in the SCOPUS database. The references can be found at the end of this booklet.

The Structure and the Content of the Dissertation: The dissertation is stated in 109 pages (the appendix is excluded); consists of an introduction, four main chapters, conclusion, appendix and the list of 134 cited references.

Overview and main results

The first chapter of the dissertation is devoted to the international experience of functioning BMS's, to the ways of their study, to their current issues and to the problem-solving methods and approaches applied up to now.

Bonus-Malus System is a tool used by insurance companies to "penalize" the policyholders responsible for one or more claims by a premium surcharge (malus) and to "reward" the policyholders who had a claim-free year by awarding discount of the premium (bonus).

Describing BMS one should mention their main characteristics, which are:

- BMS classes (with the help of them one recognizes the amount of future premium)
- The beginner's class (new policyholder join to the system from that class)
- The transition rules (the pre-defined conditions in case of which the policyholder moves from its current class to another)

The BMSs were used from 1950's. They can be applied in different areas of insurance but they appear mainly in motor transport insurance (CASCO) and in motor third party liability insurance for road vehicle (MTPL).

The Section of International Actuarial Association for Actuarial STudies In Non-life insurance (ASTIN) was created in 1957. In those years at ASTIN's conferences great attention was paid to the problem of "fairly constructed premium". To solve this problem mathematically the policies with no-claim-discount system were considered. Actually the no-claim-discount system is the special case of BMS.

Many of the BMS in practice follow a Markov chain consisting of a finite number of classes any of which corresponds to some percentage of the base premium. The premium can be reviewed upward or downward depending on a policyholder's past record of reported accidents and in accordance with transition rules (see for instance [1], [2], [3] and [4]). To get the next class occupied by a policyholder it is enough to have information on its current class and the number of claims made by him during the current period. This come to show that the BMS can be described as Markov chain: the future (the class occupied in $t+1$ year) depends on present (on the current class and on the number of claims during the current period) and does not depend on past (the full history on claims number and occupied classes in $1,2,\dots,t-1$ periods). To analyze BMS with Markov chain transition matrix on a long time period one should discuss its stationarity problem. The solution of this problem is used by the actuaries of insurance companies for the long term forecasting. Several algorithms have been proposed in order to compute the stationary distribution of the policyholders' in a given BMS. Dufresne [5] proposed a very nice technique requiring independence between the annual numbers of accidents per policyholder. We should note that the Dufresne's method would not be applicable to BMS with non-uniform penalties per claim while the technique described in [6] remains applicable for all BMS. Dufresne adapted the reasoning to the mixed Poisson case in his work [7], but at the cost of many numerical difficulties. On the basis of the model offered in [5], the authors of [6] search the stationary distribution with the non-parametric NPMLE method. This method was used also in [2] for the TPL insurance case. It is possible that the system could not reach to its stationary state at all or the rate of convergence to stationary state may be slow in comparison to the typical sojourn time of a customer in the portfolio. This problem was discussed in [8], [9], [10] considering the Bayesian view of premium calculation which can be found in [11], [12], [13]. There are BMSs which are not Markovian. One of such systems is Belgian BMS which is a "dying" one. From 2004 Belgian companies have complete freedom of using their own BMSs. The Markov property disturbed due to the special bonus rule sending the policyholder in the malus zone to initial class after four claim-free years. The works referred to this special bonus rule and to Belgian BMS are [3], [14], [15], [16]. Lemaire [3] proposed to split the classes from 16 to 21 into subclasses. This method gives an opportunity to get free from the special bonus rule and consider the model as a Markov chain again.

From practical point of view it is well known that the existing BMSs possess several considerable disadvantages which are difficult or even impossible to handle within the traditional theory of experience rating [17]. Therefore it is necessary to examine them from different point of view. The existing systems are based on the following characteristic: the claim amounts are omitted as a posterior tariff criterion. This characteristic leads to the following disadvantages:

i. Regarding an occurred claim, the future loss of bonus will in many cases exceed the claim amount.
ii. In many cases it gives the policyholder a filling of unfairness especially when a policyholder makes a small claim and the other one a large; they have the same penalty within the same risk group.
iii. The consequence of (i and ii) is the well-known bonus hunger behavior of policyholders.
iv. Bonus hunger behavior leads to asymmetric information between policyholders, insurers and regulators.

Many authors have focused on the disadvantages mentioned above. To diminish some of the disadvantages (i-iv) Holtan [17] suggested the use of very high deductibles that may be borrowed by the policyholder to the insurance company. In [18] he suggested using deductibles depending on the BMS class of the policyholder. Among the disadvantages (i-iv) the huge share belongs to the problem which in 1960 Philipson [19] called “hunger for bonus”. Grenander [20] derives equations to determine a rule of the form “pay the damage if its amount is smaller than a critical value and claim it otherwise”. Martin-Lof [21] shows that a decision rule of the form formulated in [20] is optimal in the sense that it minimizes total expected costs. The decision rule is derived by applying the general theory of Markov decision processes, which find an optimal control iteratively by using dynamic programming. As a consequence of bonus hunger, adverse selection and moral hazard phenomena, information nonconformity among insured, insurers and their regulators arises. Rothschild and Stiglitz [22] and Stiglitz [23] discussed how to design an optimal insurance contract to deal with adverse selection and moral hazard. Dionne and Lasserre [24], Cooper and Hayes [25] discussed the multi-period insurance contract, and pointed out the experience ratemaking and risk classification can solve information asymmetry.

The main reason for BMS disadvantages (i) and (ii) is the application of big maluses for claims with small severity. A reliance or “sense of fair-dealing” to the BMS will arise when the “punishment” of a policyholder as a malus is proportional to the loss incurred by insurance company because of him. This leads to BMS construction with taking into account the claim severity as well.

One of the first models of BMS designed to take severity into consideration is Picard [26]. Picard generalized the Negative Binomial model in order to take into account the subdivision of claims into two categories, small and large losses. In order to separate large from small losses, two options could be used:

- ✓ The losses under a limiting amount are regarded as small and the remainder as large.
- ✓ Subdivision of accidents in those that caused property damage and those that cause bodily injury, penalizing more severely the policyholders who had a bodily injury accident.

Pinquet [27] designed an optimal BMS which makes allowance for the severity of the claims in the following way: starting from a rating model based on the analysis of number of claims and of costs of claims, two heterogeneity components are added. They represent unobserved factors that are relevant for the explanation of the severity variables. The costs of claims are supposed to follow Gamma or Lognormal distribution. The rating factors, as well as the heterogeneity components are included in the scale parameter of the distribution. Considering that the heterogeneity also follows a Gamma or Lognormal distribution, a credibility expression is obtained which provides a predictor for the average cost of claim for the following period. Frangos and Vrontos [28] assumed that the number of claims is distributed according the Negative Binomial distribution and the losses of the claims are distributed according to the Pareto distribution, and they have expanded the frame that Lemaire [3] used to design an optimal BMS based on the number of claims. Applying Bayes' theorem the posterior distribution of the mean claim frequency and the posterior distribution of the mean claim size given the information about the claim frequency history and the claim size history for each policyholder for the time period he is in the portfolio have been found out. For more on this subject see Vrontos [29]. In [28] the development of a generalized BMS is presented, which integrates the a priori and the a posteriori information on an individual basis. In this generalized BMS the premium is a function of the years that the policyholder is in the portfolio, of his number of accidents, of the size of loss that each of these accidents incurred, and of the significant a priori rating variables for the number of accidents and for the size of loss that each of these claims incurred. Pitrebois at al. [30] suggested introduction of claim amount in the model via premium adjustment factor, which is calculated according to credibility techniques. Bonsdorff [31] discussed some asymptotic properties of Bonus-Malus systems based on the number and on the size of the claims. Analyzing BMS current issues in the following dissertation suggested to leave traditional methods discussed up to now and construct systems, which are basically different from those ones. The systems constructed here are financially balanced and the future malus of a policyholder is proportional to the loss incurred by insurance company because of him.

In the second chapter a new BMS model, constructed on the basis of "optimal" BMS principle, is suggested. One of the main actuarial principles is the presupposition of financially balanced insurance product. Generalizing the principle of financially balanced BMSs Lemaire [3] defines the concept of optimal BMS, which has two conditions. BMS is called optimal if it is:

- Financially balanced for the insurer, i.e. the total amount of bonuses is equal to the total amount of maluses
- Fair for the policyholder, i.e. each policyholder pays a premium proportional to the risk that he imposes to the pool.

In the view of probability theory, the bonuses and maluses provided by the insurer are random variables, while still the detailed definition of "the total amount of bonuses (or maluses)" remains uncertain. Therefore we offer the following statement of the "optimal" BMS: ***Financially balanced for the insurer, i.e. the expected value of total amount of bonuses is equal to the expected value of total amount of maluses."***

In other words, this statement can be interpreted as "the expectation of the BMS total premiums collected by an insurance company remains constant".

One of the processes satisfying to this condition is the martingale series widely known in probability theory.

Let us consider a portfolio of an insurance product. Suppose that a series of independent and identically distributed random variables $Y_1, Y_2, \dots$ are yearly aggregate claim losses of that portfolio, given on a $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathrm{P})$ filtered probability space where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma\{Y_1, Y_2, \dots Y_n\}$. And suppose that $Y_1, Y_2, \dots$ random variables are so that $EY < \infty$ condition is satisfied. Let's denote $P_0, P_1, \dots$ as random variables, which describe yearly aggregate premium charge for that portfolio, where $P_0 = const$ is given and the other members of that series are defined by the following formula:

$$P_n = (1 - \alpha_n)P_{n-1} + \beta_n Y_n \qquad n \geq 1 \qquad (1)$$

where P_n-is an aggregate premium collected for n-th year of the portfolio.

Y_n –is an aggregate claim loss for the given portfolio within $(n-1; n)$ time interval. It is necessary to note that Y_n is independent of P_{n-1} for all n, $(n \geq 1)$.

$\alpha = (\alpha_n, \mathcal{F}_{n-1})_{n\geq 1}$ is a predictable series with $\alpha_n \in (0,1)$, which will be called *a series of bonus factors.*

$\beta = (\beta_n, \mathcal{F}_{n-1})_{n\geq 1}$ is also a predictable series with $\beta_n \in (0,1)$, which will be called *a series of malus factors.*

Lemma 1: *The series* $\mathrm{P} = (P_n, \mathcal{F}_n)$ *constructed by formula (1), where* α_n *and* β_n *are* $\mathcal{F}_{n-1}$-*measurable, is a martingale if and only if:*

$$\frac{\alpha_n}{\beta_n} = \frac{EY}{P_{n-1}}$$

Suppose that the distribution function of aggregate claim is given $Y \sim F_Y(x)$. Then with the help of quantile method α_n and β_n expressed as $\beta_n = \frac{Y_c - P_{n-1}}{F_Y^{-1}(\varepsilon) - EY}$ and $\alpha_n = \frac{Y_c - P_{n-1}}{F_Y^{-1}(\varepsilon) - EY} \cdot \frac{EY}{P_{n-1}}$ where Y_c is a critical value of aggregate claim amount stated by an insurance company and $F_Y^{-1}(\varepsilon)$ is the inverse distribution function.

The results are presented in [40].

Lemma 2: *The coefficients* α_n *and* β_n *have finite limits as* $n \to \infty$.

Lemma 2 leads us to a conclusion that starting from some time t the bonus and malus coefficients will not depend on time and we can consider the following model

$$P_k = (1-\alpha)P_{k-1} + \beta Y_k, \qquad k \geq t \qquad (2)$$

Consider a BMS portfolio where the capital amount at a time i is decreased by the total amount of claims for time interval $(i-1, i)$ and increased by premiums collected at the time i. In addition to the assumptions of independence and identical distribution for Y_k, here we assume also that $VarY < \infty$.

The surplus process of the portfolio is then defined by

$$U_n = u + \sum_{k=1}^{n} P_k - \sum_{k=1}^{n} Y_k$$

where $u = U_0$ is the initial capital of the portfolio, P_k is the aggregate premium defined with (2) and Y_k is the aggregate claim amount of the portfolio for time interval $(k-1, k)$.

<u>Definition 1:</u> *The event that U ever falls below zero is called ruin:*

$Ruin = \{U_n < 0 \; for \; some \; n\}$.

<u>Definition 2:</u> *The time $\tau(u)$ when the process falls below zero for the first time is called ruin time:*

$\tau(u) = inf\{n > 0; U_n < 0\}$.

The probability of ruin is then given by

$$\psi(u) = P\left(\bigcup_{n\geq 0}\{U_n < 0\} \,|\, U_0 = u\right) = P\left(\inf_{n\geq 0} U_n < 0 \,|\, U_0 = u\right) = P(\tau(u) < \infty)$$

Write

$$Z_k = Y_k - P_k = Y_k - (1-\alpha)P_{k-1} - \beta Y_k = (1-\beta)Y_k - (1-\alpha)P_{k-1}$$

This variable shows the net loss of the portfolio at time k.

The total net loss of the portfolio up to time n is defined as

$$S_n = Z_1 + \cdots + Z_n, \quad n \geq 1, \quad S_0 = 0$$

So, for the probability of ruin we have the following equivalent expression:

$$\psi(u) = P\left(\inf_{n\geq 1}(-S_n) \leq -u\right) = P\left(\sup_{n\geq 1} S_n > u\right)$$

<u>Definition 3:</u> *(Net Profit Condition): The process Z satisfies to the net profit condition (NPC), if*

$EZ = (1-\beta)EY - (1-\alpha)P_0 < 0$

Taking the expectations in (2) with some rearrangements, recalling $P_0 = EP_k$ martingale property and using the i.i.d. property of Y_k's we have the following NPC

$\alpha < \beta$.

Making some rearrangements we get:

$$S_n = \sum_{k=1}^{n} Y_k - \sum_{k=1}^{n} P_k = -C_n - \left(\frac{\beta}{\alpha} - 1\right)M_n + \frac{\beta}{\alpha}G_n$$

Where $C_n = P_0\frac{(1-\alpha)(1-(1-\alpha)^n)}{\alpha}$; $M_n = Y_1 + \cdots + Y_n$; $G_n = (1-\alpha)^n Y_1 + \cdots + (1-\alpha)Y_n$

Denote $\varphi_X(t) = Ee^{tX}$ the moment generating function of the random variable X.

Lemma 3: *For any $\gamma > 0$, and for any $\alpha, \beta \in (0,1)$ satisfying the NPC, the sequence*

$$M_n' = \frac{e^{-\gamma\left(\frac{\beta}{\alpha}-1\right)M_n}}{\varphi_Y^n\left(-\gamma\left(\frac{\beta}{\alpha}-1\right)\right)}$$

is a martingale.

Lemma 4: *For any $\gamma > 0$, such that $\varphi_Y(\gamma) < \infty$ and for any $\alpha, \beta \in (0,1)$ satisfying the NPC, the sequence*

$$G_n' = \frac{e^{\gamma\frac{\beta}{\alpha}G_n}}{\varphi_Y^n\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}$$

is a supermartingale.

Lemma 5: *For any $\gamma > 0$, such that $\varphi_Y(\gamma) < \infty$ and for any $\alpha, \beta \in (0,1)$ satisfying the NPC, the sequence*

$$S_n' = \frac{e^{\gamma S_n}}{\varphi_Y^n\left(-\gamma\left(\frac{\beta}{\alpha}-1\right)\right)\varphi_Y^n\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)} \qquad (3)$$

is a supermartingale.

Theorem 1: *If for some $\gamma > 0$, the process S_n' given by (3) is a supermartingale, where $S_n \to -\infty$ as $n \to \infty$, then*

$$\psi(u) \leq \frac{e^{-\gamma u}}{E\left(\frac{e^{-\gamma U_{\tau(u)}}}{\varphi_Y^{\tau(u)}\left(-\gamma\left(\frac{\beta}{\alpha}-1\right)\right)\varphi_Y^{\tau(u)}\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}\middle|\tau(u) < \infty\right)}$$

In the third chapter of the dissertation another new BMS model is introduced, where the next BMS class of a policyholder is determined by the number of his claims and by the aggregate claim amount.

Consider a set of L policyholders. Each policyholder belongs to one of a finite number C of classes (tariff groups) sorted by order; class 1 being the one with lowest premiums etc. That is, each premium depends on the class to which a policyholder belongs. Each year the class of a policyholder is determined on the basis of the class of the previous year, on the number of claims and on the aggregate claim loss reported during that year. If no claim has been reported, then the policyholder gets a bonus expressed in the lowering to a class with a lower premium or stay at the lowest premium class. Otherwise the policyholder may stay in the same class or gets maluses by being shifted to a higher class with possibly higher premium. New policyholders are assigned to a certain class.

Let x_n^i, be the number of policyholders in a class i at time n, where $i = 1,2,\dots,C$. Then $X_n = (x_n^1, \dots, x_n^C)$ will be the distribution of policyholders among classes at time n. It is obvious that the state space $S_X = \{(x_n^1, \dots, x_n^C)\}$ of X_n process is finite.

We split the positive half of the real line into a convenient set of disjoint intervals $I_1, I_2, \dots, I_K$ and discuss the aggregate claim of each policyholder on those intervals. We will denote by H_n^i the number of policyholders whose aggregate claim in the n-th year falls in interval I_i, $i = 1,2\dots,K$. So, $H_n = (H_n^1, \dots, H_n^K)$ row vector will show the distribution of policyholders among the reported aggregate claim intervals.

Consider the number of reported claims. Its state space will be the space of natural numbers {0, 1, 2 …}. Without any distortion we can suppose that it is limited by some number N. We will denote by N_n^i the number of policyholders who have reported i, $i = 0,1,\dots,N$ claims during the n-th year. Then $N_n = (N_n^0, \dots, N_n^N)$ row vector will be the distribution of policyholders among the reported claim numbers.

Assumptions underlying the model:

- The processes X, H and N are Markov chains which, for technical reasons and without loss of generality[1], accordingly live on the standard basis $\{e_1, \dots, e_{|S_X|}\}$, $\{f_1, \dots, f_{|S_H|}\}$ and $\{h_1, \dots, h_{|S_N|}\}$ in $\mathbb{R}^{|S_X|}$, $\mathbb{R}^{|S_H|}$ and respectively $\mathbb{R}^{|S_N|}$, where the i-th component of each vector e_i, f_i and h_i is 1 and others are 0 (see [32], pg. 5). $|S_X|$, $|S_H|$ and $|S_N|$ are the sizes of S_X, S_H and S_N sets accordingly.
- It is assumed that the movement between classes is based on the current class of the policyholder, on the number of claims and on the aggregate claim reported in the year. So the movement of process X_n between its states depends on levels of X_{n-1}, $H_{n-1}$and N_{n-1} and the transition matrix is not time-dependent.
- The next assumption refers to aggregate claim process. It is assumed that the aggregate claims are not independent, so the aggregate claim of a policyholder in any year depends on the aggregate claim and reported claims number of the previous year. We can conclude that the movement between states of process H_n is based on the values of H_{n-1} and N_{n-1} as well. The transition matrix is time-dependent and is not known in advance.
- The yearly reported claim numbers for each policyholder are also suggested dependent. It is assumed that reported claims number of a policyholder depends on the claims number reported last year and on the policyholder's current class of BMS. In other words, the movement from one state for process N_n to another is founded

---

[1] Consider the process $\mathcal{X}_k$ with $S_{\mathcal{X}} = \{s_1, \dots, s_K\}$ finite state space. Let consider the function $\phi_k(s_i) = \begin{cases} 0, & \text{if } k \neq i \\ 1, & \text{if } k = i \end{cases}$. We will construct the process $X_k = \left(\phi_1(\mathcal{X}_k), \dots, \phi_K(\mathcal{X}_k)\right)$. It is easy to note that for any k only one component of X_k is 1 and others are 0. So, instead of process $\mathcal{X}_k$ we will consider the process X_k which is generated by it and which state space is $S_X = \{e_1, \dots, e_K\}$ the space of unit column vectors.

on N_{n-1} and on X_{n-1}. For this process also the transition matrix is assumed a stochastic one.

The time-dependence of transition matrices can be explained by change of policyholders' behavior year by year. They can make conclusions based on their insurance history and be more professional. As an example in motor insurance, the driver can be more careful and make fewer claims if he has many claims in the previous year. On the other hand, he can prefer to cover some small claims himself if he is on the higher bonus class.

Let $\mathfrak{I}_n = \sigma\{X_k, H_{k-1}, N_{k-1}, k \leq n\}$ be the complete filtration generated by processes X, H and N up to the n-th year. Write

$P(X_n = e_j | X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m) \triangleq p_{j,ikm}$

Here $B = (p_{j,ikm})_{j,i=1}^{|S_X|}\,_{k=1}^{|S_H|}\,_{m=1}^{|S_N|}$ is a $|S_X| \times |S_X||S_H||S_N|$ stochastic matrix of tensor mapping $\mathbb{R}^{|S_X||S_H||S_N|}$ into $\mathbb{R}^{|S_X|}$ and has the form $B = \begin{pmatrix} p_{1,111} & \cdots & p_{1,|S_X||S_H||S_N|)} \\ \vdots & \ddots & \vdots \\ p_{|S_X|,111} & \cdots & p_{|S_X|,|S_X||S_H||S_N|} \end{pmatrix}$, where $\sum_{j=1}^{|S_X|} p_{j,ikm} = 1$

Lemma 6: *The process X has the following semimartingale representation (or Doob decomposition): $X_n = BX_{n-1} \otimes H_{n-1} \otimes N_{n-1} + V_n$, where the $|S_X| \times 1$ column vector V_n is an $\mathfrak{I}_n$ martingale difference.*

The same analysis as for Markov chain X shows that the Markov chains H and N have similar representations but with time-dependent transition matrices Q_n and A_n. For matrices Q_n and A_n we develop processes $\breve{Q}_n$ and $\breve{A}_n$, for which transition matrices are not time-dependent. So, we have the following BMS model, which is a HMM:

$$\begin{cases} X_n = BX_{n-1} \otimes H_{n-1} \otimes N_{n-1} + V_n \\ H_n = Q_n H_{n-1} \otimes N_{n-1} + W_n \\ N_n = A_n N_{n-1} \otimes X_{n-1} + L_n \\ \breve{Q}_n = D\breve{Q}_{n-1} + R_n \\ \breve{A}_n = K\breve{A}_{n-1} + T_n \end{cases} \quad (4)$$

It must be noted that the constructed model is a revised and an extended one described in [33]. It has the following peculiarities:

- The model, developed here is "policyholder-oriented", that is the considered group of policyholders is divided into subgroups from 3 different points of view:
  - ✓ partition by levels of BMS
  - ✓ partition by groups of aggregate claim amounts
  - ✓ partition by number of reported claims

  In [33], the partition is applied to two different events: first of all the group of policyholders is divided on subgroups by BMS levels and the other partition applied to the set of reported claims, which are sub grouped by claim amount.
- The process Z_n is the distribution of claim numbers by claim amount intervals in [33], so $\sum_{j=1}^{K} Z_n^j$ represents the total number of claims and it means that a policyholder, who makes more than one claim during the entire year, can appear in

different groups of claim amounts simultaneously. In the model presented in this paper, H_n is the distribution of policyholders among aggregate claim amount groups, so $\sum_{j=1}^{K} H_n^j$ represents the number of policyholders and it means that policyholder's location within the aggregate claim amount intervals can be identified uniquely.

- In comparison with [33], where the transition between BMS levels depends on the reported claim numbers, in the model, presented in this paper, the above-mentioned transition depends on the aggregate claim amount, reported by policyholder as well.

Define $\Lambda_n = \prod_{t=1}^{n} \lambda_t$, where

$$\lambda_t = \prod_{i,j=1}^{|S_X|} \prod_{k=1}^{|S_H|} \prod_{m=1}^{|S_N|} \left(\frac{p_{j,ikm}}{\tilde{p}_{j,ikm}}\right)^{\langle X_t, e_j\rangle\langle X_{t-1}, e_i\rangle\langle H_{t-1}, f_k\rangle\langle N_{t-1}, h_m\rangle} \times \prod_{r,s=1}^{|S_H|} \prod_{l=1}^{|S_N|} \left(\frac{q_{s,rl}(n)}{\tilde{q}_{s,rl}(n)}\right)^{\langle H_t, f_s\rangle\langle H_{t-1}, f_r\rangle\langle N_{t-1}, h_l\rangle} \times \prod_{u,v=1}^{|S_N|} \prod_{w=1}^{|S_X|} \left(\frac{a_{u,vw}(n)}{\tilde{a}_{u,vw}(n)}\right)^{\langle N_t, h_u\rangle\langle N_{t-1}, h_v\rangle\langle X_{t-1}, e_w\rangle}$$

where $\langle\ ,\ \rangle$ denotes the usual scalar product and $\Lambda_0 = \lambda_0 = 1$.

Let $\mathcal{G}_n$ be the complete filtration generated by $\{X_k, H_{k-1}, N_{k-1}, \tilde{Q}_k, \tilde{A}_k, k \le n\}$.

Lemma 7: *The sequence of random variables $\{\Lambda_n\}_{n\ge 0}$ is a $\{\mathcal{G}_n, \tilde{P}\}$ martingale with expectation $\tilde{E}(\Lambda_n) = 1$.*

According to Radon-Nicodym theorem and Kolmogorov's extension theorem (see [32], Appendix A), with the help of measure $\tilde{P}$ we can define the "real world" measure P as follows:

$$\left.\frac{dP}{d\tilde{P}}\right|_{\mathcal{G}_n} \triangleq \Lambda_n \qquad (5)$$

Theorem 2: *Under probability measure P, as defined from $\tilde{P}$ via (5) the dynamics (4) hold.*

Theorem 3: *Joint distribution of processes H_n and N_n expressed via their marginal distributions is as follows:*

$$P(H_n = f_s, N_n = h_u | \mathcal{G}_{n-1}, X_{n-1} = e_i, H_{n-1} = f_k, N_{n-1} = h_m) = q_{s,km}(n) a_{u,mi}(n).$$

Theorem 4: *The unnormalized probability $g_n(\omega, \varpi)$ satisfies the recursion:*

$$g_n(\omega, \varpi) = \langle \frac{B}{\tilde{B}} X_{n-1} \otimes H_{n-1} \otimes N_{n-1}, X_n \rangle \times \sum_{s=1}^{|S_H|} \langle \breve{Q}_\omega H_{n-1} \otimes N_{n-1}, f_s \rangle \times \sum_{u=1}^{|S_N|} \langle \breve{A}_\varpi N_{n-1} \otimes X_{n-1}, h_u \rangle \times \sum_{v=1}^{Q|S_H||S_N|} \sum_{\vartheta=1}^{A|S_X||S_N|} d_{\omega v} k_{\varpi \vartheta} g_{n-1}(v, \vartheta),$$

where $g_0(\omega, \varpi)$ is the initial joint probability of $\breve{Q}_n$ and $\breve{A}_n$.

Lemma 8: *The unnormalized joint conditional probability distribution of reported claim numbers and aggregate claim amounts has the form:*

$$\tilde{E}(H_n N_n \Lambda_n | \mathfrak{I}_n) = \langle \frac{B}{\bar{B}} X_{n-1} \otimes H_{n-1} \otimes N_{n-1}, X_n \rangle \times \sum_{s=1}^{|S_H|} \sum_{u=1}^{|S_N|} \sum_{\omega=1}^{Q|S_H||S_N|} \sum_{\varpi=1}^{A|S_X||S_N|} f_s \langle \check{Q}_\omega H_{n-1} \otimes N_{n-1}, f_s \rangle h_u \langle \check{A}_\varpi N_{n-1} \otimes X_{n-1}, h_u \rangle g_n(\omega, \varpi)$$

Theorem 5: *The m step prediction for joint probability distribution of the processes H_n and N_n given the information $\mathfrak{I}_n$ has the form*

$$\tilde{E}\left(\Lambda_{n+m} \prod_{t=0}^{m} \langle H_{n+t}, f_{s_t} \rangle \langle N_{n+t}, h_{u_t} \rangle \middle| \mathfrak{I}_n \right)$$
$$= \sum_{\omega_0, \omega_1, \dots \omega_m = 1}^{Q|S_H||S_N|} q_{s_1 s_0 u_0}(\omega_1) \sum_{\varpi_0, \varpi_1, \dots \varpi_m = 1}^{A|S_X||S_N|} \sum_{j_1, j_2, \dots j_m = 1}^{|S_X|} \prod_{i=1}^{|S_X|} \left(p_{j_1, i s_0 u_0} a_{u_1, u_0 i}(\varpi_1) \right)^{\langle X_n, e_i \rangle}$$
$$\times \prod_{t=1}^{m} p_{j_{t+1}, j_t s_t u_t} \cdot q_{s_{t+1}, s_t u_t}(\omega_{t+1}) \cdot a_{u_{t+1}, u_t j_t}(\varpi_{t+1}) d_{\omega_t \omega_{t-1}} k_{\varpi_t \varpi_{t-1}}$$
$$\cdot \tilde{E}\left(\langle H_n, f_{s_0} \rangle \langle N_n, h_{u_0} \rangle \middle| \mathfrak{I}_n \right) g_n(\omega_0, \varpi_0).$$

The results are presented in [41].

Consider the Extended model of BMS (4). The system is described with the following set of parameters:

$$\theta := \begin{Bmatrix} p_{j,ikm}, & i,j = \overline{1,|S_X|}, & k = \overline{1,|S_H|}, & m = \overline{1,|S_N|}, \\ q_{s,rl}(n), & s,r = \overline{1,|S_H|}, & l = \overline{1,|S_N|}, & \\ a_{u,vw}(n), & u,v = \overline{1,|S_N|}, & w = \overline{1,|S_X|}, & \\ d_{\omega v}, & \omega, v = \overline{1,Q|S_H||S_N|}, & & \\ k_{\varpi\vartheta}, & \varpi, \vartheta = \overline{1,A|S_X||S_N|} & & \end{Bmatrix}$$

Our purpose is the estimation of the model parameters. It is presented here in two methods.

1. Estimation with EM algorithm. One of the best methods of HMM's coefficient estimation is the EM algorithm which is described detailed in [32].

Define

$$\begin{aligned} {}^X\mathcal{T}_n^{j,ikm} &\triangleq \textstyle\sum_{t=1}^{n} \langle X_t, e_j \rangle \langle X_{t-1}, e_i \rangle \langle H_{t-1}, f_k \rangle \langle N_{t-1}, h_m \rangle \\ {}^H\mathcal{T}_n^{s,rl} &\triangleq \textstyle\sum_{t=1}^{n} \langle H_t, f_s \rangle \langle H_{t-1}, f_r \rangle \langle N_{t-1}, h_l \rangle \\ {}^N\mathcal{T}_n^{u,vw} &\triangleq \textstyle\sum_{t=1}^{n} \langle N_t, h_u \rangle \langle N_{t-1}, h_v \rangle \langle X_{t-1}, e_w \rangle \\ {}^{\check{Q}}\mathcal{T}_n^{\omega v} &\triangleq \textstyle\sum_{t=1}^{n} \langle \check{Q}_t, b_\omega \rangle \langle \check{Q}_{t-1}, b_v \rangle \\ {}^{\check{A}}\mathcal{T}_n^{\varpi\vartheta} &\triangleq \textstyle\sum_{t=1}^{n} \langle \check{A}_t, m_\varpi \rangle \langle \check{A}_{t-1}, m_\vartheta \rangle \end{aligned} \qquad (6)$$

Each figure in (6), for the corresponding process mentioned on the left top angle, represents the number of jumps of the process from one state to another up to time n.

Each figure in the next set of notations shows the number of occasions up to time n for which the corresponding Markov chain was in the mentioned state:

$$
\begin{aligned}
&{}^{X}\mathcal{O}_n^{ikm} \triangleq \sum_{t=1}^{n} \langle X_{t-1}, e_i\rangle\langle H_{t-1}, f_k\rangle\langle N_{t-1}, h_m\rangle \\
&{}^{H}\mathcal{O}_n^{rl} \triangleq \sum_{t=1}^{n} \langle H_{t-1}, f_r\rangle\langle N_{t-1}, h_l\rangle \\
&{}^{N}\mathcal{O}_n^{vw} \triangleq \sum_{t=1}^{n} \langle N_{t-1}, h_v\rangle\langle X_{t-1}, e_w\rangle \qquad (7) \\
&{}^{\check{Q}}\mathcal{O}_n^{v} \triangleq \sum_{t=1}^{n} \langle \check{Q}_{t-1}, b_v\rangle \\
&{}^{\check{A}}\mathcal{O}_n^{\vartheta} \triangleq \sum_{t=1}^{n} \langle \check{A}_{t-1}, m_\vartheta\rangle
\end{aligned}
$$

It is obvious that the figures in (6) and (7) are random variables.

Remark: *For each process holds the relationship* $\sum_{i=1}^{|S_*|} {}^{*}\mathcal{J}_n^{i*} = {}^{*}\mathcal{O}_n^{*}$.

To replace parameters θ by $\hat{\theta}$ in (4) we define the following likelihood function:

$$
\begin{aligned}
\Gamma_n = &\prod_{t=1}^{n}\prod_{i,j=1}^{|S_X|}\prod_{k=1}^{|S_H|}\prod_{m=1}^{|S_N|}\left(\frac{\hat{p}_{j,ikm}(n)}{p_{j,ikm}}\right)^{\langle X_t,e_j\rangle\langle X_{t-1},e_i\rangle\langle H_{t-1},f_k\rangle\langle N_{t-1},h_m\rangle} \\
&\times \prod_{r,s=1}^{|S_H|}\prod_{l=1}^{|S_N|}\left(\frac{\hat{q}_{s,rl}(n)}{q_{s,rl}(n)}\right)^{\langle H_t,f_s\rangle\langle H_{t-1},f_r\rangle\langle N_{t-1},h_l\rangle} \\
&\times \prod_{u,v=1}^{|S_N|}\prod_{w=1}^{|S_X|}\left(\frac{\hat{a}_{u,vw}(n)}{a_{u,vw}(n)}\right)^{\langle N_t,h_u\rangle\langle N_{t-1},h_v\rangle\langle X_{t-1},e_w\rangle} \\
&\times \prod_{\omega,v=1}^{Q|S_H||S_N|}\left(\frac{\hat{d}_{\omega v}(n)}{d_{\omega v}}\right)^{\langle \check{Q}_t,b_\omega\rangle\langle \check{Q}_{t-1},b_v\rangle} \\
&\times \prod_{\varpi,\vartheta=1}^{A|S_X||S_N|}\left(\frac{\hat{k}_{\varpi\vartheta}(n)}{k_{\varpi\vartheta}}\right)^{\langle \check{A}_t,m_\varpi\rangle\langle \check{A}_{t-1},m_\vartheta\rangle}
\end{aligned}
$$

where in the case of $\theta = 0$ we take $\hat{\theta} = 0$ and $\frac{\theta}{\theta} = 1$;

It is not difficult to show that Γ_n is a martingale-measure, so according to the Radon-Nycodim theorem there exists a measure $P_{\hat{\theta}}$ so that $\left.\frac{dP_{\hat{\theta}}}{dP_{\theta}}\right|_{\mathcal{G}_n} = \Gamma_n$ holds.

Lemma 9: *Under the measure* $P_{\hat{\theta}}$ *the analogue of the system (4) holds for parameter set* $\hat{\theta}$.

Theorem 6: *The new estimates of parameter set* $\hat{\theta}$ *given the observations up to time* n *are given by*

$$
\hat{p}_{j,ikm} = \frac{{}^{X}\hat{\mathcal{J}}_n^{j,ikm}}{{}^{X}\hat{\mathcal{O}}_n^{ikm}}, \qquad \hat{q}_{s,rl}(n) = \frac{{}^{H}\hat{\mathcal{J}}_n^{s,rl}}{{}^{H}\hat{\mathcal{O}}_n^{rl}}, \qquad \hat{a}_{u,vw}(n) = \frac{{}^{N}\hat{\mathcal{J}}_n^{u,vw}}{{}^{N}\hat{\mathcal{O}}_n^{vw}},
$$

$$
\hat{d}_{\omega v}(n) = \frac{{}^{Q}\hat{\mathcal{J}}_n^{\omega v}}{{}^{\check{Q}}\hat{\mathcal{O}}_n^{v}}, \qquad \hat{k}_{\varpi\vartheta}(n) = \frac{{}^{A}\hat{\mathcal{J}}_n^{\varpi\vartheta}}{{}^{\check{A}}\hat{\mathcal{O}}_n^{\vartheta}}
$$

where ${}^{*}\hat{\mathcal{J}}_n^{*} = E({}^{*}\mathcal{J}_n^{*}|.)$ *And* ${}^{*}\hat{\mathcal{O}}_n^{*} = E({}^{*}\mathcal{O}_n^{*}|.)$.

2. Parameter Estimation with Recursion. For getting the parameter estimation in the previous part we have to do some prior assumptions on the probability distribution of parameter set θ, but if we have the initial distributions the recursive estimation of the parameters can be done. The new estimate in this case is presented as the previous estimate corrected with the new information.

We assume that θ takes values in some set $\Theta \in \mathbb{R}^l$. Suppose we have the measure $\tilde{P}$ under which the processes of the system (4) are i.i.d.. According to Theorem 2 we define the "real world" measure P under which the system (4) holds. Consider the unnormalized joint conditional density:

$$\alpha_n(j,\theta) = \tilde{E}\left(\Lambda_n\langle X_n, e_j\rangle I(\theta \in d\theta)\middle|\mathfrak{I}_n\right)$$

where $I(A)$ is the indicator function of A.

The normalized joint conditional density is:

$$f_n(j,\theta) = \frac{\alpha_n(j,\theta)}{\sum_{j=1}^{|S_X|}\int_\Theta \alpha_n(j,u)du}$$

Theorem 7: *The unnormalized joint conditional density $\alpha_n(j,\theta)$ satisfies the recursion:*

$$\alpha_n(j,\theta) = \sum_{i=1}^{|S_X|}\sum_{k=1}^{|S_H|}\sum_{m=1}^{|S_N|}\left(\frac{p_{j,ikm}}{\tilde{p}_{j,ikm}}\right)\alpha_{n-1}(i,\theta)$$

Write $g_n(\omega,\varpi,\theta)$ for the unnormalized joint conditional density of processes $\breve{Q}_n$ and $\breve{A}_n$:

$$g_n(\omega,\varpi,\theta) = \tilde{E}\left(\Lambda_n\langle\breve{Q}_n, b_\omega\rangle\langle\breve{A}_n, m_\varpi\rangle I(\theta \in d\theta)\middle|\mathfrak{I}_n\right)$$

The normalized joint conditional density is:

$$f_n(\omega,\varpi,\theta) = \frac{g_n(\omega,\varpi,\theta)}{\sum_{\omega=1}^{Q|S_H||S_N|}\sum_{\varpi=1}^{A|S_X||S_N|}\int_\Theta g_n(\omega,\varpi,u)du}$$

Theorem 8: The unnormalized joint conditional density $g_n(\omega,\varpi,\theta)$ satisfies the recursion

$$g_n(\omega,\varpi,\theta) = \langle\frac{B}{\tilde{B}}X_{n-1}\otimes H_{n-1}\otimes N_{n-1}, X_n\rangle \times \sum_{s=1}^{|S_H|}\langle\breve{Q}_\omega H_{n-1}\otimes N_{n-1}, f_s\rangle$$
$$\times \sum_{u=1}^{|S_N|}\langle\breve{A}_\varpi N_{n-1}\otimes X_{n-1}, h_u\rangle \times \sum_{\upsilon=1}^{Q|S_H||S_N|}\sum_{\vartheta=1}^{A|S_X||S_N|} d_{\omega\upsilon}k_{\varpi\vartheta}g_{n-1}(\upsilon,\vartheta,\theta)$$

The results are presented in [42].

The last chapter is devoted to the BMS analysis and testing of new models presented in the dissertation by a comparative analysis with the BMS used in Armenian CMTPL insurance. To determine the BMS behavior and forecast premium amount to be collected, it is very important to model the number of claimed accidents to insurance company and the amount lost by the company due to a policyholder. Claims number and claim amount are random variables for insurance company. For modeling each of them some probability distributions are used. Those distributions are listed and characterized for instance in [11], [34] and [35]. To accept or reject any distribution one must state a statistical hypothesis. The statistical hypotheses were checked with Kolmogorov-Smirnov and χ^2 statistics, which description and applications are described for example in [36], [37], [38], [39]. Calculations of the statistics were made with the help of computer packages Easy-Fit, SPSS and MS Excel. For the claims number variable the Poisson, Negative Binomial, Binomial, Geometric and Hypergeometric distributions were checked. All distributions were rejected with 10% of significance except of Negative Binomial one. It was accepted with the same level of confidence for 2013, 2014, 2015 and aggregate data. For the aggregate claim amount variable 42 different types of distributions were tested with the help of EasyFit package. Some of them were rejected with the 10% of significance. The others accepted with 90% of confidence are presented in Table 4.5 of the dissertation.

The analysis of the Alternative BMS shows that the estimates of α and β with the help of distributions of the aggregate claim amount are not invariant to distributions: for different distributions we find different values. But the model gives opportunity to developers constructing flexible systems by fixing the loss ratio beforehand. Table1 shows the comparative analysis of the collected premiums according to current BMS and Alternative BMS suggestions. Presented data include all policyholders with one year insurance policy disregarding the fact of claiming. It is obvious that the aggregate premium collected by the insurance company according to the current BMS is less than the Base premium with 1.2%-2.3%. This fact means that the introduction of BMS leads to some additional losses to the company. The results got from the Alternative BMS are better, than from the current BMS as the gap between next premium and base premium is less than 0.54%. This fact comes to confirm that the Alternative BMS is constructed according to the “optimal” BMS statements. From the comparative analysis presented detailed in [43] we can conclude that for the most part of the policyholders the Alternative BMS is preferable than the current BMS. In addition to that, the Alternative BMS is not incurring additional losses to the company and the company can state its loss ratio at the start of the insurance year.

	2013		2014		2015	
	Current BMS	Alternative BMS	Current BMS	Alternativ e BMS	Current BMS	Alternat ive BMS
Year premium/Base premium	98.81%	99.68%	98.79%	99.73%	100.04 %	100.03 %
Next premium/Base premium	98.22%	99.56%	97.68%	99.46%	97.87%	99.33%
Year premium/ Previous year premium	99.41%	99.88%	98.88%	99.73%	97.83%	99.30%

Table1

The BMS presented in the Chapter 3 of the dissertation differs from the current one by the additional term, which is the aggregate claim amount. So, the next level of BMS for a policyholder is determined by the number of claims and aggregate claim amount incurred to the company during the current year. To discuss a more flexible system we take transition rules between BMS classes presented in the Table2

Number of claims	Aggregate Claim Interval	Level (+/-)
0		-1
1	(0-100,000]	+1

1	(100,000-500,000]	**+4**
1	(500,000-1,500,000]	+7
1	(1,500,000+)	+10
2	(0-100,000]	+2
2	(100,000-500,000]	+5
2	(500,000-1,500,000]	**+8**
2	(1,500,000+)	+11
3+	(0-100,000]	+3
3+	(100,000-500,000]	+6
3+	(500,000-1,500,000]	+9
3+	(1,500,000+)	**+12**

Table2

Table3 helps to identify the percentage of policyholders to which the Extended BMS has harder or smoothed terms compared with the current BMS.

Number of Claims	Aggregate Claim Amount (AMD)			
	(0-100000]	(100000-500000]	(500000-1500000]	(1500000+)
1	49%	**43%**	7%	1%
2	14%	69%	**14%**	3%
3+	2%	66%	28%	**4%**

Table3

For the policyholders making one claim the terms of Extended BMS are smoother for 49%, are the same as current BMS for 43% and are harder for 8%. For the policyholders with two claims the Extended BMS is smoother for 83%, and the current BMS for 3%. From the last row of the Table3 it is obvious that the Extended BMS is the same for only 4% of the policyholders with three and more claims, but for the others it is smoother than the current BMS. So, enlarging this analysis to the group of all policyholders with any claim we conclude that two models have the same malus for 39.5% of them. The Extended BMS has harder terms than the current one for 7.5% of policyholders having any claim. On the other hand, the current BMS is harder than suggested Extended BMS for 53% of claimed policyholders. Although, Extended BMS is smoother for most of policyholders, forecast amount of premiums to be collected for 2016 is 100.11% of the base premium in opposite to the current BMS's 97.87%.

Bibliography

[1] **Brouhns, N., Guillen,M., Denuit, M. and Pinquet,J.** "Bonus-malus scales in segmented tariffs with stochastic migration between segments". *Journal of Risk and Insurance.* 2003, Vol. 70, 4, pp. 577-599.

[2] **Denuit, M. and Dhaene,J.** "Bonus-Malus Scales Using Exponential Loss Functions". August 2000, Belgium.

[3] **Lemaire, J.** *“Bonus-Malus Systems in automobile insurance”.* Boston : Kluwer Academic Publisher, 1995.

[4] **Dionne, G. and Vanasse, C.** "A Generalization of Actuarial Automobile Insurance Rating Models: the Negative Binomial Diistribution with a Regression Component" . *ASTIN Bulletin.* 1989, 19, pp. 199-212.

[5] **Dufresne, F.** ″Distribution stationnaire d’un systeme bonus-malus et probabilite de ruine″. *ASTIN Bulletin.* 1988, 18, pp. 31-46.

[6] **Walhin, J. F. and Paris, J.** “Using Mixed Poisson Processes in Connection With Bonus-Malus Systems”. *ASTIN BULLETIN.* 1999, Vol. 29, 1, pp. 81-99.

[7] **Dufresne., F.** ″The efficiency of the Swiss Bonus-Malus system″. *Bulletin of the Swiss Association of Actuaries.* 1995, 1, pp. 29-41.

[8] **Asmussen, S.** “Modeling and Performance of Bonus-Malus Systems: Stationarity versus Age-Correction”. *Risks.* 2014, Vol. 2, pp. 49-73.

[9] **Norberg, R.** ″A credibility theory for automobile bonus system″. *Scandinavian Actuarial Journal.* 1976, pp. 92-107.

[10] **Borgan, Ø., Hoem, J.M. and Norberg, R.** "A nonasymptotic criterion for the evaluation of automobile bonus systems". *Scandinavian Actuarial Journal.* 1981, pp. 165–178.

[11] **Denuit, M., Marechal, X., Pitrebois, S., Walhin, J.F.** *“Actuarial Modelling of Claim Counts”.* s.l. : John Wiley & Sons, 2007.

[12] **Bühlmann, H.** "Mathematical Methods in Risk Theory". *Springer-Verlag.* 1970.

[13] **Bühlmann, H. and Gisler, A.** "A Course in Credibility Theory and its Applications". *Springer-Verlag.* 2005.

[14] **Lemaire, J.** "Construction of the new Belgian motor third party tariff structure". *ASTIN Bulletin.* 1988, Vol. 18, 1, pp. 99–112.

[15] **Centeno, M. and Silva, J.M.A.** ″Optimal Bonus Scales Under Path-Dependent Bonus Rules″. *Scandinavian Actuarial Journal.* 2002, 24, pp. 615-627.

[16] **Pitrebois, S., Denuit, M. and Walhin, J.F.** "Fitting the Belgian Bonus-Malus System". *BELGIAN ACTUARIAL BULLETIN.* 2003, Vol. 3, 1, pp. 58-62.

[17] **Holtan, J.** "Bonus Made Easy". *ASTIN Bulletin.* 1994, 24, pp. 61-74.

[18] **Pitrebois, S., Walhin, J.F., Denuit, M.** "Bonus-Malus Systems with varying deductibles". *ASTIN Bulletin.* 2005, Vol. 35, 1, pp. 261-274.

[19] **Philipson, C.** ″The Swedish system of bonus″. *AUSTIN Bulletin.* 1960, 1, pp. 134-141.

[20] **Grenander, U.** ″Some remarks on bonus systems in automobile insurance″. *Skandinavisk Aktuarietidskrift.* 1957, 40, pp. 180-198.

[21] **Martin-Lof, A.** "A method for finding the optimal decision rule for a policy holder of an insurance with a bonus system". *Scandinavian Actuarial Journal.* 1973, 56, pp. 23-29.

[22] **Rothschild, M. and Stiglitz, J. E.** ″Equilibrium in competitive insurance markets: an essay on the economics of imperfect information″. *Quarterly Journal of Economics.* 1976, Vol. 90, pp. 629-649.

[23] **Stiglitz, J. E.** ″Risk, incentives and insurance: the pure theory of moral hazard″. *The Geneva Papers on Risk and Insurance.* 1983, Vol. 26, pp. 4-33.

[24] **Dionne, G. and Lasserre, P.** ″Adverse selection, repeated insurance contracts and announcement strategy″. *Review of Economic Studies.* 1985, Vol. 52, pp. 719-723.

[25] **Cooper R. and Hayes, B.** ″Multi-period insurance contracts″. *International Journal of Industrial Organization.* 1987, Vol. 5, pp. 211-231.

[26] **Picard, P.** ″Generalisation de l'etude sur la survenance des sinistres en assurance automobile″. *Bulletin Trimestriel de l'Institute des Actuaires Francais.* 1976, pp. 204-267.

[27] **Pinquet, J.** "Allowance for Costs of Claims in Bonus-Malus Systems". *ASTIN Bulletin.* 1997, Vols. 33-57, 27.

[28] **Frangos, N.E. and Vrontos, S.D.** "Design of optimal bonus-malus systems with a frequency and a severity component on an individual basis in automobile insurance". *ASTIN Bulletin.* 2001, Vol. 31, 1, pp. 1-22.

[29] **Vrontos, S.** "Design of an Optimal Bonus-Malus System in Automobile Insurance". *Msc. Thesis, Department of Statistics, Athens University of Economics and Business, ISBN: 960-7929-21-7.* 1998.

[30] **Pitrebois, S., Denuit, M. and Walhin, J.-F.** "Setting a Bonus-Malus Scale in the Presence of Other Rating Factors: Taylor's Work Revisited". *ASTIN Bulletin.* 2003, Vol. 33, 2, pp. 419-436.

[31] **Bonsdorff, H.** ″On asymptotic properties of Bonus-Malus systems based on the number and on the size of the claims″. *Scandinavian Actuarial Journal.* 2005, Vol. 4, pp. 309-320.

[32] **Elliott, R. J., Aggoun, L. and Moore, J. B.** "Hidden Markov models: Estimation and Control". *Springer-Verlag New York.* 1995.

[33] **Aggoun, L. and Benkherouf, L.** "An Insurance Model with Bonus-Malus System". *International Mathematical Forum, 1.* 2006, 26, pp. 1255 – 1272.

[34] **Panjer, H.H., and Willmot, G.E.** *"Insurance Risk Models".* Schaunburg III. s.l. : Society of Actuaries, Schaunburg III, 1992.

[35] **Mikosch, T.** *"Non-life Insurance Mathematics".* s.l. : Springer, 2009.

[36] NIST/SEMATECH e-Handbook of Statistical Methods, http://www.itl.nist.gov/div898/handbook

[37] **Айвазян, С.А., Мхитарян, В.С.** "Теория Вероятностей и Прикладная Статистика". *Юнити, Москва.* 2001.

[38] **Huber-Carol, C., Balakrishnan,N., Nikulin,M. S., Mesbah,M.** "Goodness-of-Fit Tests and Model Validity". *Statistics for Industry and Technology.* 2002.

[39] **Cox, D.R., Donnelly, C.A.** "Principles of Applied Statistics". *Cambridge University Press.* 2011.

Author's publications related to the dissertation

[40] **Gulyan, A.G.** "An Alternative Model for Bonus-Malus System". *Proceedings of the Yerevan State University.* 2015, 1, pp. 15-19.

[41] **Chitchyan, R.N. and Gulyan, A.G.** "A Bonus-Malus System with Aggregate Claim Amount Component". *Journal of Contemporary Mathematical Analysis. 2015.*Vol. 50, 5, pp. 74-88.

[42] **Gulyan, A.G. and Chitchyan, R.N.** "Evaluation Methods of the Coefficients of Transition Matrices in Extended Bonus-malus System". *Contemporary Science and Education in*

Americas, Africa and Eurasia. The IX International Academic Congress (Technology and engineering). 2015, Vol. 3, pp. 215-221.

[43] **Chitchyan, R.N. and Gulyan, A.G.** "Comparative Analysis of the Bonus-Malus Systems". *Proceedings of Engineering Academy of Armenia (Economics and management).* 2016, Vol. 13, 1, pp. 5-16.

Ամփոփագիր

Բոնուս-մալուս համակարգի (ԲՄՀ) մի դասից մյուսին անցումը պայմանավորված է ապահովադրի կատարած հայտերի քանակով: Այս կանոնի կիրառությունից առաջ են գալիս մի շարք խնդիրներ, որոնք հնարավոր չէ լուծել փորձի գնահատման արդեն իսկ կիրառվող եղանակներով: Մարտինգալների տեսության կիրառությունն այս բնագավառում նոր հնարավորություններ է ընձեռում առավել արդյունավետ համակարգերի ներմուծման համար:

Ատենախոսության հիմնական նպատակն է նոր ԲՄՀ-երի կառուցումը, որոնք որպես ռիսկի հետահայաց դասակարգում կպարունակեն ապահովադրի կողմից կատարված հայտերի քանակի և հայտերի մեծության բաղադրիչները: Կառուցվող մոդելները պետք է լինեն ֆինանսապես հավասարակշռված և ապահովադրի մալուսը պետք է լինի համամասնական իր կողմից ապահովագրական ընկերությանը պատճառված վնասին: Աշխատանքի հիմքում դրվել են հետևյալ խնդիրները՝

- Ուսումնասիրել գործող ԲՄՀ-երը և դրանց արդի հիմնախնդիրների լուծման կամ զսպման մեխանիզմները
- Առաջարկել օպտիմալ ԲՄՀ սահմանմանը բավարարող նոր համակարգեր
- Կատարել առաջարկված ԲՄՀ-երի վերլուծություն ՀՀ ԱՊՊԱ համակարգի տվյալների հիման վրա

Ատենախոսությունում ստացվել են հետևյալ արդյունքները.

- ✓ Առաջարկվել է ԲՄՀ այլընտրանքային մոդել և դուրս բերվել անհրաժեշտ և բավարար պայման, որի դեպքում ապահովագրական պայմանագրերի պայուսակի ապահովագրավճարները կազմում են մարտինգալային շարք:
- ✓ Ցույց է տրվել, որ առաջարկված մոդելը կարող է հասնել ստացիոնար վիճակի
- ✓ Առաջարկված այլընտրանքային մոդելի համար մարտինգալների և սուպերմարտինգալների միջոցով դուրս է բերվել սնանկացման հավանականության վերին սահմանը
- ✓ Ստացված է ապահովագրական պահանջի այն մեծությունը, որի դեպքում ապահովադրի մոտ առաջանում է "բոնուսային քաղցի" պահվածք
- ✓ Առաջարկվել է Մարկովյան ԲՄՀ ընդլայնված մոդել, որտեղ ապահովադրի ԲՄՀ մի դասից մյուսի անցումը պայմանավորված է նրա ընթացիկ դասով, տարվա ընթացքում նրա կատարած հայտերի քանակով և այդ հայտերի գումարյալ հատուցման մեծությամբ
- ✓ Ընդլայնված ԲՄՀ մոդելի համար "Թաքնված" Մարկովյան մոդելների և չափի փոփոխման միջոցով կատարվել է մոդելի պարամետրերի գնահատում
- ✓ Ապահովագրական ընկերության տվյալների հիման վրա կատարվել է ՀՀ ԱՊՊԱ համակարգում հայտերի քանակի և հայտերի մեծության բաշխման վերաբերյալ վարկածների ստուգում
- ✓ Կատարվել է համեմատական վերլուծություն գործող ԲՄՀ և ատենախոսությունում առաջարկվող այլընտրանքային և ընդլայնված ԲՄՀ միջև

Արդյունքները ներկայացված են [40]-[43] աշխատանքներում:

Аннотация

Переход от одного класса к другому в системе бонус-малус (СБМ) определяется количеством заявок, представленных страхователем. Применение этого правила приводит к ряду проблем, которые не могут быть решены методами оценок, используемых до сих пор. Применение теории мартингалов в этой области дает новые возможности для внедрения более эффективных систем. Основной целью диссертации является построение новых СБМ, которые как апостериорная классификация риска будут содержать количество заявок и суммарный иск от страхователя. Модели, построенные в диссертации, должны быть финансово сбалансированные ималус страхователя должен быть пропорционально ущербе причиненного страховой компании. В основе диссертации изучены следующие проблемы:

- Исследовать действующие СБМ и методы решения или смягчения текущих проблем.
- Предложить новые системы удовлетворяющие условию оптимальной СБМ.
- Провести анализ предлагаемых СБМ на основе данных системы ОСАГО (обязательное страхование ответственности, возникающее вследствие эксплуатации транспортных средств) в Армении.

В диссертации получены следующие результаты:

- ✓ Предложена альтернативная модель СБМ, учитывающая эквивалентность финансовых обязательств страхователя и страховой компании, а также получено необходимое и достаточное условие, при котором премии портфеля страховых договоров составляют ряд мартингалов.
- ✓ Показано, что для предложенной модели может быть достигнуто состояние стационарности.
- ✓ С помощью оценок, использующих свойства мартингалов и супермартингалов, получен верхний предел вероятности разорения для представленной альтернативной модели СБМ.
- ✓ Получен размер страхового ущерба, при котором у страхователя возникает поведение “бонусного голода”.
- ✓ Предложена расширенная модель Марковской СБМ, где переход страхователя из одного класса СБМ в другой определяется его текущим классом, числом заявок и суммарным иском в течение года.
- ✓ Для расширенной модели СБМ с помощью “скрытых” моделей Маркова и изменением мер получены оценки параметров.
- ✓ На основе эмпирических данных о системе ОСАГО страховой компании «ИНГОАРМЕНИЯ» была проведена проверка гипотез для вероятностного распределения числа заявок и суммарного иска.
- ✓ Проведен сравнительный анализ между текущей СБМ, применяемой в настоящее время в Армении, и системами БМ, предложенными в диссертации.

Результаты диссертации опубликованы в работах [40]-[43].