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Structural Optimization and Control of Vibrations of Deformable Systems with Variable Parameters

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INTRODUCTION

Almost in every field of natural science we are concerned with this or that type of control problem: human beings want to subject to himself and control everything from time immemorial. Sometimes we need to ensure demanded transition from current state into required one, to choose the right trajectory for evader escaping the pursue of predator, to determine parameters of circuit leading to essential increase of its lifetime, to suggest a structure for design extremely resistible to bending, to develop a heating regime using as less sources as possible etc. The construction of even a simple controller without proper theoretical investigation is a very hard and expensive task. The essential refinement of computers and corresponding software done during the last decades allows now to model, construct and test controllers implementing quite difficult tasks. This refinement has mainly done due to significant achievements in mathematical control theory. It provides abstract models of controlled systems, investigates those models, proposes solution methods and indicates ways to construct control algorithms and methods of their computer realization.

Being an important subject and having separate importance, the modelling of controlled systems is far not enough for its construction in practice. Control engineer more often needs exact or approximate control programs and more or less confidence before starting to construct something. Investigation of exact or approximate controllability of those models ensures that confidence. But, naturally, the positive answer to the question about controllability of a particular system is not yet sufficient to start the construction. The engineer has to know what will happen with his controllable system subject to a particular influence. We believe that the last aim of mathematical control theory, the construction of control algorithms and methods of their computer implementation is the most important one.

There are two possibilities of control algorithms construction; using exact or approximate approaches. The first type of approaches is efficient because unlike the second one it provides both qualitative and quantitative picture and does not require so high computational cost and is efficient for computer implementation. The complexity of the way obtaining the exact solution is mainly depend on the complexity of the mathematical model of the system and, first of all, its thorough analysis is required. By now there are several methods allowing to find the exact solution of the problem. Here we will mention only revolutionary ones: the method of dynamical programming by R. Bellman [13], A. Butkovskiy‘s method of control with compact support [22], N. Krasovskiy‘s method of moments [77], the variational approach by J.–L. Lions [83], L. Pontryagin‘s maximum principle [101], the impulsive control method by T. Yang [122].

These methods have been applied to solve numerous problems of theoretical and practical importance in almost all areas of science and manufacturing. As a result of significant progress in technology and computing, the opportunities of practical implementation and computing devices have been rapidly increased. It led to an increase in the interest among researchers and engineers to optimal systems and it turned out that present needs of both theoreticians and engineers cannot be satisfied by those methods as they initially were. So, they were required to be refined and extended in order to be applicable for solving control problems for more and more complicated systems [2, 21, 29, 78, 84, 98, 100].

That refinement brought also to development of approximation methods with very fast and easy implementable numerical schemes [12, 15, 102], due to which now we have different kind of control simulators. This branch of control theory continues developing more and more intensively and, consequently, the necessity of exact solution was pushed to the background. It is also explained by difficulties of investigation of the model. Nevertheless, occasionally there appear publications on exact solution of particular control problems [16,41,45,46,53,54,70,74], and though there are comprehensive studies of well-known control systems like wave and heat equation, Lame system, Korteweg–de Vries equation etc., there still remain fascinating problems of mentioned type rigorously unsolved yet [24, 116].

Among simple control systems which rigorous investigation requires overcoming significant difficulties are the system with variable parameters, appearing in various applied branches of science and manufacturing. First of all it is connected with difficulty of investigation of their mathematical models, formulating as constraints (ordinary and partial differential, integral and difference equations etc.) with variable coefficients: the propagation of waves in non-homogeneous media is described by the wave equation with variable coefficients, the solution of heat equation with variable coefficients evaluates the distribution of the temperature in a non-homogeneous body etc. Even though there are very accurate numerical methods for solving partial differential equations like the mentioned ones [33], explicit solution is still of interest. The explicit solution of those simple equations even for simple non-homogeneities requires to involve some special functions [1, 37, 56]. The situation gets worse if the system contains such singularities that its mathematical model does not admit classical solution and, therefore, we have to deal with generalized (or weak solution) [89, 115, 119]. For example, impulsive impact may move a point to another position in arbitrary short time leading the trajectory of the point to be discontinuous [122].

An exhaustive by that time review of control and stabilization of distributed parameter systems can be found in [42]. Let us shortly mention only several of recent results directly related to those herein. In [4] boundary L^2 control problem is considered for a non-homogeneous string and distributed L^2 control problem for a non-homogeneous ring subjected to time-dependent axial (stretching) tension. In both cases the null-controllability is aimed to achieve in finite given time by choosing appropriate controls. Using the eigenfunction expansion method, the investigation of controllability in both cases is reduced to a problem of moments. The sets of initial data for the string and ring are described which may be driven to rest by the control. The boundary C^2 controllability for a non-homogeneous string is achieved and explicit formulas for the control functions are constructed in two-part article [16]. Both cases of constrained and free initial and terminal conditions are considered. The travelling wave method is used for the solution. In [62] the necessary and sufficient conditions of Neumann null-controllability and approximate null-controllability are obtained for wave equation with coordinate dependent potential. The controllability problems are considered in the modified Sobolev spaces. The controls that solve these problems are found explicitly. It is proved that among the solutions of the Markov trigonometric moment problem there are bang-bang controls solving the approximate null-controllability problem.

The solution is even more complicated of boundary or distributed controllability analysis for systems on semi-infinite or infinite domains; that is why those problems are considered rarely. Necessary and sufficient conditions for exact and approximate L^2 null-controllability are obtained in [34] for wave equation on half-axis with coordinate-dependent potential. The Neumann boundary control function is bounded by a hard constant. Explicit expressions for controls solving the exact null-controllability problem is constructed. Using the Fourier series expansion method, the approximate null-controllability problem is reduced to the Markov power problem of moments and bang-bang resolving controls are found. In [35] necessary and sufficient conditions of Dirichlet L^∞-controllability and approximate L^∞-controllability are obtained for one-dimensional wave equation on semi-axis with coordinate dependent potential having exponential decay at infinity. These problems are considered in the Sobolev spaces. Using the transformation operator of the Sturm–Liouville problem on the positive semi-axis, the control system replicates the controllability properties of the system with constant potential. Conditions of controllability for the system are obtained from those for the system with constant potential. Article [63] deals with wave equation on semi-axis with coordinate dependent continuous potential controlled either by Dirichlet or by Neumann boundary condition chosen from L^∞. The control systems are considered in Sobolev spaces. Using the operators adjoint to the transformation operators for the Sturm–Liouville problem, necessary and sufficient conditions for null-controllability and approximate null-controllability are obtained. Explicit forms for resolving controls are obtained.

Unlike control problems for one-dimensional wave equation with variable coefficients, there is a great deal of literature devoted to (boundary and distributed) control of wave equation with constant coefficients. We refer to [9, 22, 34, 45, 46, 53, 54, 111] and the references therein. Boundary and distributed control problems for wave equation with variable coefficient in finite and semi–infinite domains are considered in Section 2.

A particular mention is given to a linear partial integro-differential equation arising in theory of waves [51, 117]. Investigation of control problems for such systems is also extremely complicated, but, with use of Butkovskiy's generalized method, we are going to derive simple formula for boundary and distributed control of a partial integro-differential equation associated with wave equation. Some optimal control problems for integro-differential equations are considered in [8, 25, 30, 61, 80]. Boundary and distributed control problems for partial integro-differential equation are considered in Section 2.

Reduction of vibration amplitudes has been the subject of study of many researchers. More recently, a quick increase in the development and application of passive energy dissipation devices, such as viscoelastic dampers, viscous fluid dampers, metallic yield dampers and friction dampers, has occurred. The objective of these devices is to absorb a portion of the input energy, due to earthquake, wind or human excitation, for instance, reducing the dynamic response of the structure. The use of passive energy dissipation devices reduces significantly the dynamic response of structures subjected to dynamic actions. However, the parameters of each damper as well as the best placement of these devices remain difficult to determine. Although some studies on optimization of tuned mass damper and viscous/viscoelastic dampers are being developed, the development of methods for optimum use of these devices are an important research issue and is still lacking.

Particularly, problems of bending vibrations control (damping) for elastic beams and plates, subjected to moving loads, are intensively studied at present owing to their practical importance [38,39,88,94,95,103,110,112]. Such problems, particularly, are the mathematical ground for modeling and engineering bridges, over which trains are moving. In common, the control (damping) of vibrations is carried out by changing the displacement of viscous or viscoelastic dampers (often the *Kelvin–Voigt* model of viscoelastic body [27] is taken into account) with fixed number and placements under the beam. The article [94] is devoted to analysis of up to date investigations in the theory of simply supported beams, subjected to moving loads. The vibrations, caused by each load separately are studied due to the fact that they can accumulate and cause resonance phenomena. In [95] is devoted to damping of resonance vibrations of a simply supported beam, subjected to moving loads, via viscous dampers connecting the beam with an auxiliary beam located under it. Numerical analysis is performed and it is proved, that it is possible to damp the beam vibrations under certain values of system parameters. In [112], under the assumptions of the linear Euler–Bernoulli theory, a control problem about damping of a simply supported finite beam vibrations, caused by influence of moving loads, is solved. A viscoelastic damper, moving with given constant speed, is attached to the beam.

Purpose of the control is the damping of vibrations of the beam at given moment. Finally, in [88] the influence of passive viscous dampers on the dynamical behavior of an orthotropic plate, subjected to moving loads, is revealed under resonance conditions. The plate is connected via viscous dampers to simply supported beams parallel to its free edges. The control is carried out by optimal choice of parameters of viscous dampers in order to damp transverse vibrations of the plate.

Especially in the last decade, for utilization of these dampers in an economic way, several researchers started to study the optimization of their parameters as well as their best positions in a structure. Recently, in problems of earthquake engineering, optimal locations and reasonable number as well of such dampers are also under intensive investigation [39, 90, 103]. In [90] the simultaneous optimization of force and placement of friction dampers is proposed and it is shown that the design of friction dampers can be done in a safe and economic way. To solve this optimization problem, the recently developed firefly algorithm is employed, which is able to deal with non-convex optimization problems, involving mixed discrete and continuous variables. For illustration purposes, two common footbridges are analysed, in which the cost function is to minimize the maximum acceleration of the structures, whereas forces and positions of friction dampers are the design variables. The results showed that the proposed method was able to determine the optimum friction forces of each damper as well as their best positions in the structures. The maximum acceleration was reduced in more than 95 per. for the Warren truss footbridge, with three friction dampers, and in more than 92 per. for the Pratt truss footbridge, with only two friction dampers. In addition, the proposed methodology is quite general and it is believed that it can be recommended as an effective tool for optimum design of friction dampers for structural response control.

The mathematical model of such systems is constructed in terms of bilinear equations, i.e. equations jointly linear in the state and control variables. They form one of the simplest nonlinear systems and therefore applicable particularly to analyse of much more complicated nonlinear systems. Such systems can be used to model a wide range of physical, chemical, biological processes that cannot be effectively modelled under the assumption of linearity.

The concept of bilinear system was introduced in the 1960s and have been applied to various areas of science and technology. A great deal of literature related to the control problems of such systems has been developed since that. Some control problems of bilinear processes were solved for plasma, quantum devices, particle accelerators, nuclear power plants and biomedicine (see [100] and the references therein). Nevertheless, there are open questions remain in this field.

Besides tremendous theoretical importance, the bilinear systems have significant practical importance: those arise in material distribution, structural and topology optimization problems. In direct statement, material distribution and topology optimization problems require to minimize the material quantity in domain occupied by a design and the volume of that domain in order to improve its desired properties. The aim of structural optimization is the determination of an optimal lower cost structure for a design preserving or even improving its feature properties. Therefore, in view of high cost of materials used in modern designs, investigation of those problems are very attractive and particular recommendation to engineer, based on solution of model problem, may be even cost saving.

For an exhaustive review of recent literature we refer to [100]. Here we mention only those investigations, which are related to ours, but not included in [100]. Monograph [6] is devoted to the exposition of new ways of formulating and solving problems of structural optimization with incomplete information. Some research results concerning the optimum shape and structural properties of bodies subjected to external loads. Particularly, optimal design with incomplete information, accounting for the interaction between the structure and its environment, properties of materials, existence of initial damages and damage accumulation are studied. It is also is devoted to overcoming mathematical difficulties caused by local functionals. Many aspects of structural optimization with incomplete information for rods, beams, plates and shells. Particularly, shape optimization of beams, plates and shells with uncertainties, crack positioning and orientations, and optimization of structural elements with uncertain material properties are studied in a systematic and careful way. Important results related to optimization of structures with a longevity constraint are obtained. Incompleteness of information arises from uncertainties in initial crack length, orientation and positioning.

Nontrivial applications in various areas of industry, such as multi-objective optimal sizing of a beam under multiple load cases, contact stress minimization for elasto-plastic body in contact with a rigid foundation, finding the shape of the header of a paper machine in order to get an appropriate distribution of a fibre suspension, multidisciplinary and multi-objective optimization of an airfoil, and shape optimization of a dividing tube, are presented in [47].

A bilinear control problem is considered in [17] for the two-dimensional Kirchhoff equation describing the bending vibrations of a thin plate clamped along one portion of its boundary and oscillating (free) along the rest portion of its boundary. The control function describes the interaction of the plate with a linear elastic one parametric base under the plate and is included in the state equation as bilinear term– as coefficient of the state function. First, the well-posedness and regularity of the problem and the existence of an optimal control is proved in appropriate Sobolev spaces, and then the optimality system is derived by means of classical variational calculus. In [18] the same control problem is considered for the same equation with free vibrating boundary conditions along the whole boundary, but the base is k-parametric, which defines the quantity of unknown controls. The well-posedness and regularity of the problem and the existence of an optimal control is proved in this case as well. In recent article [87], special features of the general statements of material optimization problems in space–time are discussed. Three open problems in this subject are proposed. In [48] a class of shape/topology optimization problems governed by the Helmholtz equation in 2D is considered. To guarantee the existence of minimizers, the functional is relaxed and a weak formulation of the problem is considered. Two numerical methods for solving such problems are proposed and theoretically justified: a direct discretization of the relaxed formulation and a level set parametrization of shapes by means of radial basis functions. Several numerical aspects are also considered. The fundamentals, classical and non-classical results and reviews on different issues of topological optimization can be found in [55].

Problems of vibration reduction of elastic structures with optimizing the dampers locations and distributions are considered in Section 3 in terms of distribution and topology optimization. In the same section a structural dynamic optimization problem is also considered.

The dissertation, particularly, is devoted to

- the extension of classes of distributed parameter systems admitting analytical exact solution. Butkovskiy‘s method of control with compact support [22] is extended in order to be applicable also to systems whose behavior is still of interest after the control influence is taken off and for systems having distributional solutions [65–71],

- the derivation of necessary and sufficient conditions for exact controllability and explicit solution of boundary and distributed control problems for various model problems,

- the explicit solution of some particular optimization problems with state constraints. Using the Bubnov–Galerkin procedure an efficient algorithm is suggested [72] for solving optimal distribution problems in dimensions one and two [60, 73, 74]. The wave field representations via vector and scalar potentials and time decomposition are applied in turn to the structure optimization of a two-dimensional system [106],

- the suggestion of particular recommendations for mechanical and civil engineers to use optimal structures for damping the bending vibrations in structures under moving loads,

- the determination of optimized structure for wave-guides propagating harmonic signals with specified parameters.

The dissertation is mainly based on research articles of the author alone and coauthored, the main results of which are published in [57–60, 65–74, 106], though not the all results are included in here. Its hard to think of a title covering the whole material and being concise. We stopped at current version as variable parameters meaning non-homogeneity and dispersive nature of controlled systems in the second chapter and distribution function and non-homogeneity of deformable bodies in the third section. Hope it will not give rise for any misunderstandings. It is organized as follows. After Introduction it comes the first preliminary chapter with list of main concepts and notations. It contains preliminaries of the theory of distributions, including the definitions, main operations and relations, which are necessary for interested readers without special mathematical background to correctly repeat the operations, transformations and manipulations done, and the detailed, step-by-step description of Butkovskiy‘s method of control with compact support [22] with proper examples, as well as of its slightly modified (generalized) version. Some applications in deformable body mechanics are described. The second and third chapters of the dissertation are devoted to the application of Butkovskiy‘s generalized method for investigating several particular interesting problems mainly arising in solid mechanics, but describing other processes in natural science as well.

The second chapter begins with introduction into Butkovskiy‘s generalized technique for a relatively general equation and boundary conditions. Then we reduce boundary and distributed control problems for finite string to an infinite system of integral linear constraints the L^p–optimal solution of which for $1 \leq p \leq \infty$ is known from Subsection 1.3.1. L^1–optimal boundary control is found for semi-infinite string providing required (non-zero) terminal conditions in the same way. Two problems of control with constant delay for the wave equation is considered when the delay exists in Dirichlet boundary conditions and in distributed controls. The system of integral constraints is derived in both cases. Finally, Butkovskiy‘s generalized method is applied for solving general boundary and distributed control problems for a particular partial integro-differential equation associated with wave equation. The system of necessary and sufficient conditions for its controllability are derived.

The third chapter contains explicit solutions of material distribution, topology and structure optimization problems formulated as initial-boundary value problems for bilinear systems in partial derivatives. Two approaches are suggested to deal with such problems. In order to reduce the model problem to a system, the Butkovskiy‘s generalized method is applicable to which, the first approach suggests to use the Bubnov–Galerkin procedure. It turns out that sometimes (when, for instance, the control function does not explicitly depends on time) it is necessary to involve Butkovskiy‘s generalized method as well. In that manner the optimal in the sense of wasted resources law of viscoelastic material distribution under an elastic finite beam subjected to moving point load is found. Among admissible controls the discrete distribution of the viscoelastic material has the minimal support (volume), corresponding to vibration absorbers (dampers). The solution is reduced to a standard problem of nonlinear programming under restrictions of both equality and inequality types with respect to switching points corresponding to the placements of the dampers under the beam. The same approach is used to optimize the topology of elastic subgrade under a simply supported rectangular elastic plate subjected to a moving normal load in order to damp its bending vibrations in required time. The same technique allows to prove that the piecewise distribution of the subgrade is optimal among all admissible distributions. Numerical analysis reveals all important dependences between optimal control function and internal, external parameters of the system. The main limitation of this approach is that linearity of boundary conditions is required.

The second approach is still valid for nonlinear boundary conditions, but applicable for solution of only dynamical problems. The main idea is in representation of the solution via wave vector and scalar potentials. It is explained on the problem of structure optimization for an infinite non-homogeneous elastic layer in order to allow propagation of periodic unidirectional waves. Assuming that the density and the Young‘s modulus of the layer material vary over its thickness according to the same law, it is proved that the piecewise non-homogeneity (layered structure) of the layer is optimal among all admissible controls. Numerical computations allows to find the thickness and the material characteristics of each component layer. The main limitation of this approach is that the solution may not always be represented via those potentials.

Then come Conclusions, and Bibliography finalizes the dissertation.

The whole dissertation and its separate parts are presented in numerous scientific conferences, seminars and colloquiums (2011–2015). Among others we appreciate the attention and comments from the audience of

- Seminar series of Chair of Mechanics, Yerevan State University, Yerevan, Armenia, 2011-2015,
- Seminar series “Wave Processes” coordinated by Prof. M. V. Belubekyan, Armenian NAS, Yerevan, Armenia, 2011-2015,
- International scientific conference dedicated to S. Banach, Lviv, Ukraine, 2012,
- International scientific conference dedicated to N. Kh. Arutyunian, Tsaghkadzor, Armenia, 2012,
- International summer-school “Geometry, Mechanics and Control”, Madrid, Spain, 2013,

- International winter-school "Applications of Mathematics in Natural Science", International Max Planck Institute, Leipzig, Germany, 2013,
- International Conference of Young Scientists and Specialists, dedicated to N. N. Bogolyubov, Joint Institute for Nuclear Research, Dubna, Russia, 2014,
- International Symposium "Youth, Science, Innovation", Kazan, Russia, 2014,
- Allrussian conference "Problems of Control", Moscow, Russia, 2014,
- Gene Golub SIAM summer school "Simulation, Optimization, and Identification in Solid Mechanics", Linz, Austria, 2014,
- Colloquium series "CeNoS" (Center for Nonlinear Sciences), Münster, Germany, 2014-2015,
- SIAM conference on "Control and its Applications", Paris, France, 2015.

The equations and their systems appearing herein covers processes also in other fields of natural science and engineering and an interested reader can find many useful tools for dealing with his particular problem. We hope, that the dissertation will make a small but appreciable contribution in areas of optimal control, controllability analysis, optimal design, distribution and topology optimization, and will be useful for engineers interested in various statements and solving techniques of various optimal control problems.

CHAPTER 1

PRELIMINARIES AND APPLICATIONS OF CONTROL WITH COMPACT SUPPORT IN DEFORMABLE BODY MECHANICS

The aim of this chapter is to outline preliminaries of the distribution theory and Butkovskiy‘s method of control with compact support, which are the basic tools for investigating the control problems considered herein. It is written as concise as possible, and therefore we do not claim the rigour. For further material and more details we refer to [20, 22, 109, 115, 118, 124].

The chapter is organized as follows. In Section 1.1 main concepts and notations, symbolic and nomenclature are brought in order to make each section or chapter of the thesis easy understandable and even readable. In Section 1.2 the fundamentals of the theory of distributions are brought. The main distributions used below are defined and their main properties are shown. Such operations as differentiation, limiting transition, convergence, Fourier integral transform are defined. Butkovskiy‘s method of control with compact support and its extension systems admitting generalized solution are described for a particular *linear* partial differential equation in Section 1.3. The solution of boundary and distributed control problems is reduced to an infinite system of integral constraints. Their L^p–optimal solution, $1 \leq p \leq \infty$, and conditions of its existence are brought treating them as a problem of moments. For further purposes, the L^1, L^2 and L^∞–optimal solutions are brought separately. Butkovskiy‘s generalized method is applied to solve boundary control problem for thermo-elastic layer modelled as a coupled system of partial differential equations. The solution is reduced to countable system of constraints.

1.1 Main Concepts and Notations

Here we bring the main notations and concepts used throughout the thesis in order to let the reader interested in a particular chapter or section herein find the definition of occurred symbols directly here.

$\mathbb{R}$ the real line

$\mathbb{R}^+$ the positive semi-axis

$\mathbb{C}$ the complex plane

$\mathbb{N}$ the set of natural numbers

$\mathcal{D}[\cdot]$ denotes state operator

Δ the Laplacian

$\mathcal{D}^q$ the weak derivative of the $q \in \mathbb{N} \cup \{0\}$–th order with respect to argument

$C_p(\mathcal{O})$ the space of piecewise continuous in $\mathcal{O}$ functions

$L^p(\mathcal{O})$, $1 \leq p \leq \infty$ the space of Lebesgue measurable in $\mathcal{O}$ functions

$\mathfrak{T}$ the space of test functions

$\mathfrak{D}$ the space of distributions

$\mathcal{U}_p = \{u \in L^p[0,T];\ \text{supp}\ u \subseteq [0,T]\}$ denotes the set of admissible controls for $1 \leq p < \infty$

$\mathcal{U}_\infty = \{u \in L^\infty[0,T];\ \text{supp}\ u \subseteq [0,T],\ |u| \leq u_0\}$

$\mathcal{U}_d$ and $\mathcal{U}_b$ denote the sets of admissible distributed and boundary controls, respectively

$\overline{\mathcal{O}}$ and $\partial\mathcal{O}$ the ordinary closure and the boundary of the domain $\mathcal{O}$

ρ density[1]

λ and μ the Lame coefficients

ν the Poisson‘s ratio

E the Young modulus

c the elastic wave propagation velocity

$D = \dfrac{2h^3E}{3(1-\nu^2)}$ the bending stiffness of elastic plate, h is its thickness

α^2 the viscosity parameter

γ the Euler constant

[1] All quantities below are supposed to have SI units of measurements.

J and S the moment of inertia and the cross sectional area of beams

Bi the Biot number,

$\mathrm{T}_m(x) = \cos(m \arccos x)$, $m \in \mathbb{N}$ the Chebyshev polynomials of the first kind [1]

$J_m(x)$ and $Y_m(x)$ the Bessel functions of the first and the second kinds [1]

$W(x)$ the Lambert function [28]

$\mathcal{F}_t[\eta] \equiv \overline{\eta} = \displaystyle\int_{-\infty}^{\infty} \eta(t) \exp[i\sigma t]\, dt$ the Fourier distributional or generalized direct transform of $\eta(t)$ with respect to t

$\mathcal{F}_t^{-1}[\overline{\eta}] \equiv \eta = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \overline{\eta}(\sigma) \exp[-i\sigma t]\, d\sigma$ the Fourier distributional inverse transform

$\mathcal{F}_s[\eta] = \displaystyle\int_0^{\infty} \eta(t) \sin(i\sigma t)\, dt$ and $\mathcal{F}_c[\eta] = \displaystyle\int_0^{\infty} \eta(t) \cos[i\sigma t]\, dt$ the Fourier sine and cosine transforms of η: $\mathcal{F}_c[\eta] + i\mathcal{F}_s[\eta] = \mathcal{F}_t[\eta]$

σ the spectral parameter of the Fourier transform

$\delta(x)$ the Dirac delta function

$\theta(x)$ the Heaviside unit step function

$\theta(x, y) = \theta(x)\theta(y)$ the Heaviside two-dimensional function

sign x the sign function

$\chi_{\mathcal{O}}(x)$ the characteristic (indicator) function of $\mathcal{O}$

supp $\eta = \overline{\{x \in \mathbb{R}^n;\ \eta(x) \neq 0\}}$ the support of the function $\eta(x)$

$v(x)$ the distribution law of the control function: $v \geq 0$, supp $v \neq \varnothing$

Re η and Im η denote the real and the imaginary parts of function η.

$\delta_k^m = \delta_m^k$ Kronecker's symbol

$\delta_{mk}^{\kappa\mu} = \delta_k^{\kappa} \cdot \delta_m^{\mu}$ Kronecker's two-index symbol

$\{1; N\}$ a short notation of $\{1, 2, ..., N\}$

The superscript T over columns or matrices denotes the transposition.

Several important theorems are brought in the following two sections in order to make the dissertation complete and independent work and not to refer the reader to a particular chapter of some book which are rare nowadays. Besides, in different books those are formulated in different terminology and with different notations, so the reader with less knowledge in mathematics (e.g. engineer) can be easily lost. That is the second reason why do we bring them in the dissertation.

1.2 Distributions and Their Fourier Transform

The necessity of introducing distributions usually arises in situations when a sufficiently "good" continuous function becomes non differentiable in some points, series or integrals diverge, and the usual Fourier transform of a sufficiently simple function, for instance, constant, does not exist. Namely, the distributions were introduced in order to "remove" such singularities. At this stage, generalized derivatives, convergence of series and integrals in a generalized sense, and the distributional Fourier transform provides invaluable service to the investigator.

In this section we will briefly outline the main ideas and concepts of the mathematical theory of distributions according to [20, 109, 115, 118, 124].

1.2.1 Test Functions and Distributions

A real valued infinitely differentiable function $\eta : \mathbb{R} \to \mathbb{R}$ with supp $\eta \subseteq [a, b]$ we shall call *test* function. The space of test functions is denoted by $\mathfrak{T}$. The convergence in $\mathfrak{T}$ is defined as follows. A given sequence $\{\eta_\iota\}_{\iota\in\mathbb{N}}$ of test functions with supp $\eta_\iota \subseteq [a, b]$, $\iota \in \mathbb{N}$, converges to $\eta \in \mathfrak{T}$, if for any $q \in \mathbb{N} \cup \{0\}$

$$\lim_{\iota\to\infty} \eta_\iota^{(q)}(x) = \eta^{(q)}(x).$$

The operations of addition and multiplication by an arbitrary infinite differentiable function are continuous with respect to that convergence.

To any ordinary function corresponds a linear continuous functional on $\mathfrak{T}$:

$$(f, \eta) = \int_{-\infty}^{\infty} f(x)\eta(x)dx, \quad \eta \in \mathfrak{T}. \tag{1.1}$$

Linear continuous functionals on $\mathfrak{T}$ are called *generalized functions* or *distributions*. The space of distributions we denote by $\mathfrak{D}$. A distribution is called *regular*, if it is representable in the form (1.1), otherwise– it is called *singular*. Distributions generated by locally measurable functions are always regular. A common example of singular distribution is the Dirac delta function, which puts in correspondence any test function η to its value at $x = 0$: $(\delta, \eta) = \eta(0), \quad \eta \in \mathfrak{T}$.

According to the definition, shifted ordinary functions lead to advance of the test function: $(f(x - x_0), \eta(x)) = (f(x), \eta(x + x_0)), \eta \in \mathfrak{T}$.

We shall say, that $(f_1,\eta) \geq (f_2,\eta)$ for any $0 \leq \eta \in \mathfrak{T}$, if $(f_1 - f_2,\eta) \geq 0$. Particularly, two distributions are equal, if $(f_1 - f_2,\eta) = 0$ almost everywhere in supp η. A sequence $\{f_\iota\}_{\iota\in\mathbb{N}} \in \mathfrak{D}$ converges to $f \in \mathfrak{D}$ if $(f_\iota,\eta) \to (f,\eta)$, $\eta \in \mathfrak{T}$. Obviously, it does not follow from relation $f_\iota \to f$ in $L^1_{loc}(\mathbb{R})$. Indeed, for example, $2\iota^3x^2(1+\iota^2x^2)^{-2} \to 0$, $x \in \mathbb{R}$, in $L^1_{loc}(\mathbb{R})$, but for any $\eta \in \mathfrak{T}$, $2\iota^3\left(x^2(1+\iota^2x^2)^{-2},\eta\right) = \pi\eta(0)$, i.e. $2\iota^3x^2\left(1+\iota^2x^2\right)^{-2} \to \pi\delta(x)$ in $\mathfrak{D}$.

Since for any $\eta \in \mathfrak{T}$, $\eta^{(q)} \in \mathfrak{T}$ as well, $q \in \mathbb{N}$, then by virtue of equality $\left(f^{(q)},\eta\right) = (-1)^q\left(f,\eta^{(q)}\right)$, all distributions are infinitely differentiable and in $\mathcal{D}$.

Further we will need to deal with derivatives of some well-known distributions. Heaviside θ function generates a regular distribution

$$(\theta,\eta) = \int_0^\infty \eta(x)dx, \quad \eta \in \mathfrak{T}.$$

According to the definition we have $(\theta',\eta) = -(\theta,\eta') = \eta(0) = (\delta,\eta)$, $\eta \in \mathfrak{T}$.

Another regular distribution is defined by the sign function. There is an obvious relation between sign and Heaviside distributions: $(1-2\theta(x),\eta) = -(\text{sign } x,\eta)$, $(1-2\theta(-x),\eta) = (\text{sign } x,\eta)$, $\eta \in \mathfrak{T}$, and therefore $(\theta(x)-\theta(-x),\eta) = (\text{sign } x,\eta)$, $(\theta(x)+\theta(-x),\eta) = (1,\eta)$, $\eta \in \mathfrak{T}$. It follows also that $2\theta(0) = 1$. According to the definition in the sense of distributions $\mathcal{D}\text{sign } x = 2\delta(x)$. Note in this regard that even though sign x is odd and δ is even, θ is neither odd, nor even.

Another regular distribution is defined by the characteristic (indicator) function

$$\left(\chi_{[0,1]}(x),\eta(x)\right) = \int_0^1 \eta(x)dx, \quad \eta \in \mathfrak{T}.$$

There are several representations of this function by means of others. For example, $\left(\chi_{[0,1]}(x),\eta(x)\right) = (\theta(x)-\theta(x-1),\eta(x))\,,\ \ \eta \in \mathfrak{T},\ x \in \mathbb{R}$ which is widely used in the preceding sections. Using the distributional derivative of the Heaviside function, we may easily obtain the distributional derivative of the characteristic function.

It will be useful to bring also the following relation

$$(\mu(x)\delta'(x),\eta(x)) = \mu(0)\left(\delta'(x),\eta(x)\right) - \mu'(0)\left(\delta(x),\eta(x)\right), \quad \eta \in \mathfrak{T},$$

which simply follows from the differentiation formula of the product $\mu(x)\delta(x)$.

While transition to non–dimensional variables and functions, the following relations are used throughout the thesis without a special reference

$$\delta\left(x-x_0\right)=\frac{1}{|x_0|}\delta\left(\frac{x}{x_0}-1\right),\quad \theta\left(x-x_0\right)=\begin{cases}\theta\left(\frac{x}{x_0}-1\right), & x_0>0,\\ 1-\theta\left(\frac{x}{x_0}-1\right), & x_0<0.\end{cases}$$

One of the most important and commonly used operations in the theory of distribution is the convolution. The relation $(f_1(x)\cdot f_2(x),\eta(x+y))\equiv(f_1*f_2,\eta)$, $\eta\in\mathfrak{T}$, in $\mathfrak{D}$ is called the convolution of distributions f_1 and f_2. If both f_1 and f_2 are regular then the convolution f_1*f_2 reads as

$$f_1*f_2=\int_{-\infty}^{\infty}f_1(x-y)f_2(y)dy=\int_{-\infty}^{\infty}f_1(y)f_2(x+y)dy=f_2*f_1.$$

Note, that so-called integral equations of convolution types are widely used in numerous applications (see Subsection 2.2).

1.2.2 The Fourier Transform of Distributions

One of the most powerful techniques to investigate the problems of mathematical physics is the technique of integral transforms. In this section we will introduce the Fourier integral transform for distributions or, simply, the distributional Fourier transform.

The Fourier integral transform of $f\in L^1_{loc}(\mathbb{R})$

$$\overline{f}(\sigma)\equiv\mathcal{F}[f]=\int_{-\infty}^{\infty}f(x)\exp\left[i\sigma x\right]dx,\quad \sigma\in\mathbb{R},$$

is bounded and, therefore, defines a regular distribution

$$(\mathcal{F}[f],\eta)=\int_{-\infty}^{\infty}\mathcal{F}[f]\eta(\sigma)d\sigma,\quad \eta\in\mathfrak{T}.$$

It is easy to see, that $(\mathcal{F}[f],\eta)=(f,\mathcal{F}[\eta])$. This relation is accepted as a definition of the Fourier transform of a distribution $f\in\mathfrak{D}$. The function f is defined via Fourier inverse transform:

$$f(x)\equiv\mathcal{F}^{-1}[\overline{f}]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}(\sigma)\exp\left[-i\sigma x\right]d\sigma.$$

A useful result for the Fourier distributional transform is the following.

Theorem 1.1 (Wiener–Paley–Schwartz, [20, 22, 109, 113, 118, 124]) *A distribution $f \in \mathfrak{D}$ is compactly supported in $[-\vartheta, \vartheta]$ if and only if $\mathcal{F}[f](\sigma + i\varsigma)$, $\varsigma \in \mathbb{R}$, satisfying*

$$\left| z^{\iota} \cdot \overline{f}(z) \right| \le \mathrm{C}_{\iota} e^{\vartheta|\varsigma|} \tag{1.2}$$

for all $\iota \in \mathbb{N} \cup \{0\}$ and corresponding real constants C_{ι}.

In other words, if we denote by $\mathfrak{E}$ the space of entire functions satisfying inequality (1.2), the Wiener–Paley–Schwartz theorem states, that the Fourier distributional transform is a bijection between $\mathfrak{D}$ and $\mathfrak{E}$. From the other hand, the distributional Fourier transform is a linear isomorphism from $\mathfrak{D}$ onto itself [118].

This theorem plays a central role in derivation of boundary and distributed controllability criteria according to Butkovskiy‘s generalized method (see Subsection 1.3.1).

In order to justify a certain phenomenon from physics viewpoint, in various applied problems of mathematical physics [43, 44, 56, 60, 67–74, 106] it is necessary to prove, that the inverse Fourier transform of the solution, obtained by means of that transformation, is real valued. For this purpose, the following result is useful.

Corollary 1.1 *It follows from the decomposition*

$$\mathcal{F}[f] = \int_{-\infty}^{\infty} [\mathrm{Re}\ f(x)\cos(\sigma x) - \mathrm{Im}\ f(x)\sin(\sigma x)]\, dx + \\ + i \int_{-\infty}^{\infty} [\mathrm{Re}\ f(x)\sin(\sigma x) + \mathrm{Im}\ f(x)\cos(\sigma x)]\, dx,$$

- *$\mathcal{F}[f]$ is real valued if and only if* $\mathrm{Re}\ f(-x) = \mathrm{Re}\ f(x)$ *and* $\mathrm{Im}\ f(-x) = -\mathrm{Im}\ f(x)$,
- *if f is real valued* ($\mathrm{Im}\ f = 0$) *then $\mathcal{F}[f](-\sigma) = \overline{\mathcal{F}[f](\sigma)}$, at this if $f(-x) = f(x)$, then $\mathcal{F}[f]$ is real valued and $\mathcal{F}[f](-\sigma) = \mathcal{F}[f](\sigma)$,*
- *if f is purely imaginary* ($\mathrm{Re}\ f = 0$) *then $\mathcal{F}[f](-\sigma) = -\overline{\mathcal{F}[f](\sigma)}$, at this if $f(-x) = -f(x)$, then $\mathcal{F}[f]$ is purely imaginary $\mathcal{F}[f](-\sigma) = -\mathcal{F}[f](\sigma)$.*

Very often the distributional Fourier sine and cosine transforms are used. According to definition, the distributional Fourier sine and cosine transforms of $f \in \mathfrak{D}$ is

$$\mathcal{F}_s[f] = \int_0^{\infty} f(x)\sin(\sigma x)dx, \quad \mathcal{F}_c[f] = \int_0^{\infty} f(x)\cos(\sigma x)dx, \quad \sigma \in \mathbb{R}^+.$$

At that, obviously, $\mathcal{F}[f] = \mathcal{F}_c[f] + i\mathcal{F}_s[f]$. The Fourier inverse sine and cosine transforms are defined in the same manner.

1.3 The Control with Compact Support

The explicit solution of control problems is always very important to considerably simplify the qualitative analysis and to design of corresponding systems. There are several methods for solving control problems explicitly for both continuous and discrete systems [13, 22, 77, 101]. Only a few of existing methods may give answer to the question about controllability of the system and, at the same time, provide a method for constructing the required controls. One of such methods is A. G. Butkovskiy‘s method of control with compact support [22]. Assuming, that the control and state functions are concentrated on some finite time-interval in which the control process is carried out, Butkovskiy‘s method provide an efficient procedure for obtaining a system of necessary and sufficient conditions for exact controllability and simultaneously restrictions on control function to be determined from. For this purpose, the well-known Wiener–Paley theorem [22] is used. From those restrictions the unknown control function may be obtained in two ways: either by constructing interpolating polynomials for the Fourier image of the unknown function or by separating the real and the imaginary parts of the system and obtain integral constraints. Actually, by means of Butkovskiy‘s method we obtain a whole class of admissible for exact controllability functions, therefore optimality conditions may be proposed.

In order to illustrate the method let us consider a one-dimensional system with distributed parameters the state of which in time is described by a second order partial differential equation[2]. Let in rectangle $\mathcal{O} = \{(x,t);\ x \in (0,1),\ t \in (0,T)\}$

$$\mathcal{D}[w] = f\left(x, t, u_d(t)\right). \tag{1.3}$$

Let also

$$\left.\begin{aligned} \alpha_0 w(0,t) + \beta_0 \frac{\partial w(x,t)}{\partial x}\right|_{x=0} &= u_b^0(t), \\ \alpha_1 w(1,t) + \beta_1 \frac{\partial w(x,t)}{\partial x}\bigg|_{x=1} &= u_b^1(t), \end{aligned} \qquad t \in [0,T]. \tag{1.4}$$

Here

$$\mathcal{D}[w] \equiv A\frac{\partial^2 w(x,t)}{\partial x^2} + B\frac{\partial w(x,t)}{\partial x} + Cw(x,t) - a\frac{\partial^2 w(x,t)}{\partial t^2} - b\frac{\partial w(x,t)}{\partial t},$$

[2]Definitely, the results remain true for higher order and higher dimensional systems of partial differential equations and their coupled systems as well: corresponding generalizations can be done easily.

the coefficients a, b, A, B and C are given and may be at most coordinate dependent, $u_d : [0,T] \to \mathbb{R}$ is a control function to be found, $f : [0,1] \times [0,T] \times \mathcal{U}_d \to \mathbb{R}$, $u_b^0 : [0,T] \to \mathbb{R}$ and $u_b^1 : [0,T] \to \mathbb{R}$ also may be considered as control functions. Choosing the constants α_0, β_0 and α_1, β_1 in appropriate way, we may obtain Dirichlet, Neumann and mixed boundary conditions.

The initial state of the system is supposed to be given:

$$w(x,0) = w_0(x), \quad \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = w_0^1(x), \quad x \in [0,1]. \tag{1.5}$$

Our aim is to find the explicit representation of distributed or boundary controls $u_d \in \mathcal{U}_d$ or $u_b^0, u_b^1 \in \mathcal{U}_b$ which ensure the given terminal state of the system

$$w(x,T) = w_T(x), \quad \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=T} = w_T^1(x), \quad x \in [0,1]. \tag{1.6}$$

In addition, a functional describing a property of the system or the resources spend on control process may be required to be minimized.

Henceforth, if the control process is carried out either by function u_d or one of the functions u_b^0, u_b^1, we will say that we have a *distributed* or *boundary* control problem, respectively, and call the corresponding control function *distributed* or *boundary* control. If the set $\mathcal{U}_b$ or $\mathcal{U}_d$ are non-empty, the system is called *controllable.* If required conditions (1.6) are satisfied exactly for some choice of controls (whether it be boundary or distributed), then (1.3)–(1.5) is called *exact controllable.* Nevertheless, in some cases it becomes impossible to find controls such that (1.6) is satisfied exactly: one can merely implement some state close to required one in some sense. Then, (1.3)–(1.5) is called *approximate controllable.* Such a problem is considered in Subsection 3.2. For simplicity of calculations, sometimes, we take $w_T(x) = w_T^1(x) \equiv 0$: the algorithm can be easily extended otherwise. Then, if the set of corresponding admissible controls is non-empty, the system is called *null-controllabile.*

It is supposed, that the transmission conditions concerning consistency of boundary conditions and initial, terminal functions (1.5), (1.6), are satisfied:

$$\alpha_0 w_0(0) + \beta_0 \mathcal{D} w_0(0) = u_b^0(0), \quad \alpha_1 w_0(1) + \beta_1 \mathcal{D} w_0(1) = u_b^1(0),$$

$$\alpha_0 w_0^1(0) + \beta_0 \mathcal{D} w_0^1(0) = \mathcal{D} u_b^0(0), \quad \alpha_1 w_0^1(1) + \beta_1 \mathcal{D} w_0^1(1) = \mathcal{D} u_b^1(0),$$

$$\alpha_0 w_T(0) + \beta_0 \mathcal{D} w_T(0) = u_b^0(T), \quad \alpha_1 w_T(1) + \beta_1 \mathcal{D} w_T(1) = u_b^1(T),$$

$$\alpha_0 w_T^1(0) + \beta_0 \mathcal{D} w_T^1(0) = \mathcal{D} u_b^0(T), \quad \alpha_1 w_T^1(1) + \beta_1 \mathcal{D} w_T^1(1) = \mathcal{D} u_b^1(T).$$

Let us now proceed to the solution of the problem. For this purpose we apply to (1.3), (1.4) the Fourier real integral transform with respect to t. After simple algebraic transformations we will finally derive:

$$\overline{\mathcal{D}}[\overline{w}] = G(x, \sigma, \overline{u}_d(\sigma)), \quad (x, \sigma) \in \mathcal{O}_1 \equiv (0, 1) \times \mathbb{R}, \tag{1.7}$$

where

$$\overline{\mathcal{D}}[\overline{w}] \equiv A \frac{d^2 \overline{w}(x, \sigma)}{dx^2} + B \frac{d\overline{w}(x, \sigma)}{dx} + \left(C + a\sigma^2 + ib\sigma\right) \overline{w}(x, \sigma),$$

$$G(x, \sigma, \overline{u}_d(\sigma)) = \overline{f}\,(x, \sigma, \overline{u}_d(\sigma)) + a\left(i\sigma w_0(x) - w_0^1(x)\right) + b w_0(x),$$

$$\left.\begin{aligned} \alpha_0 \overline{w}(0, \sigma) + \beta_0 \left.\frac{d\overline{w}(x, \sigma)}{dx}\right|_{x=0} &= \overline{u}_b^0(\sigma), \\ \alpha_1 \overline{w}(1, \sigma) + \beta_1 \left.\frac{d\overline{w}(x, \sigma)}{dx}\right|_{x=1} &= \overline{u}_b^1(\sigma), \end{aligned}\right\} \quad \sigma \in \mathbb{R}. \tag{1.8}$$

Thus, now we need to solve boundary-value problem (1.7), (1.8). Let $\overline{w}_1(x, \sigma)$ and $\overline{w}_2(x, \sigma)$ be the fundamental solutions of (1.7). Then, as it is well known [23, 123], its general solution may be represented as follows:

$$\overline{w}(x, \sigma) = h_1(\sigma)\overline{w}_1(x, \sigma) + h_2(\sigma)\overline{w}_2(x, \sigma) + \Lambda(x, \sigma, \overline{u}_d(\sigma)), \quad (x, \sigma) \in \overline{\mathcal{O}}_1, \tag{1.9}$$

where h_1 and h_2 are unknown yet,

$$\Lambda(x, \sigma, \overline{u}_d(\sigma)) = \int_0^x \frac{\overline{w}_1(x, \sigma) W_1(\xi, \sigma) + \overline{w}_2(x, \sigma) W_2(\xi, \sigma)}{W(\xi, \sigma)} G(\xi, \sigma, \overline{u}_d(\sigma)) d\xi. \tag{1.10}$$

Here $W(\xi, \sigma)$ is the main and $W_1(\xi, \sigma), W_2(\xi, \sigma)$ are the auxiliary Wronskians:

$$W(\xi, \sigma) = \begin{vmatrix} \overline{w}_1(\xi, \sigma) & \overline{w}_2(\xi, \sigma) \\ \frac{d\overline{w}_1(\xi, \sigma)}{d\xi} & \frac{d\overline{w}_2(\xi, \sigma)}{d\xi} \end{vmatrix}, \quad W_1(\xi, \sigma) = \begin{vmatrix} 0 & \overline{w}_2(\xi, \sigma) \\ 1 & \frac{d\overline{w}_2(\xi, \sigma)}{d\xi} \end{vmatrix}, \quad W_2(\xi, \sigma) = \begin{vmatrix} \overline{w}_1(\xi, \sigma) & 0 \\ \frac{d\overline{w}_1(\xi, \sigma)}{d\xi} & 1 \end{vmatrix}.$$

Substituting (1.9) into (1.8), with respect to unknowns h_1 and h_2 we will obtain a linear system of algebraic equations:

$$\begin{cases} \left(\alpha_0 \overline{w}_1(0, \sigma) + \beta_0 \left.\frac{d\overline{w}_1(x, \sigma)}{dx}\right|_{x=0}\right) h_1(\sigma) + \left(\alpha_0 \overline{w}_2(0, \sigma) + \beta_0 \left.\frac{d\overline{w}_2(x, \sigma)}{dx}\right|_{x=0}\right) h_2(\sigma) = \overline{u}_b^0(\sigma); \\ \left(\alpha_1 \overline{w}_1(1, \sigma) + \beta_1 \left.\frac{d\overline{w}_1(x, \sigma)}{dx}\right|_{x=1}\right) h_1(\sigma) + \left(\alpha_1 \overline{w}_2(1, \sigma) + \beta_1 \left.\frac{d\overline{w}_2(x, \sigma)}{dx}\right|_{x=1}\right) h_2(\sigma) = \overline{u}_b^{11}(\sigma), \end{cases}$$

in which

$$\overline{u}_b^{11}(\sigma) = \overline{u}_b^1(\sigma) - \alpha_1 \Lambda(1, \sigma, \overline{u}_d(\sigma)) - \beta_1 \frac{d\Lambda(x, \sigma, \overline{u}_d(\sigma))}{dx}\bigg|_{x=1}.$$

Let us represent its solution in the form

$$h_1(\sigma) = \frac{\Delta_1(\sigma)}{\Delta(\sigma)}, \quad h_2(\sigma) = \frac{\Delta_2(\sigma)}{\Delta(\sigma)},$$

where Δ and Δ_1, Δ_2 are the main and the auxiliary determinants of that system.

Thus, from (1.9) we have

$$\overline{w}(x, \sigma) = \frac{\Delta_1(\sigma)\overline{w}_1(x, \sigma) + \Delta_2(\sigma)\overline{w}_2(x, \sigma)}{\Delta(\sigma)} + \Lambda(x, \sigma, \overline{u}_d(\sigma)), \quad (x, \sigma) \in \overline{\mathcal{O}}_1. \tag{1.11}$$

Proceeding further, in order to apply the Wiener–Paley theorem [22] we need to extend (1.11) in $\mathbb{C}$:

$$\overline{w}(x, z) = \frac{\Delta_1(z)\overline{w}_1(x, z) + \Delta_2(z)\overline{w}_2(x, z)}{\Delta(z)} + \Lambda(x, z, \overline{u}_d(z)), \quad (x, z) \in [0, 1] \times \mathbb{C}. \tag{1.12}$$

It can be proved, that if $G(\cdot, z, \overline{u}_d(z))$ is entire, $\Lambda(\cdot, z)$ is entire as well, therefore (1.12) will be entire if and only if its first term is entire. Denote by $\{z_k\}_{k\in\mathbb{N}}$ the roots of the equation $\Delta(z) = 0$. Then, the last term will be entire if and only if the following equalities hold:

$$\Delta_1(z_k)\overline{w}_1(x, z_k) + \Delta_2(z_k)\overline{w}_2(x, z_k) = 0, \quad k \in \mathbb{N},$$

uniformly for all $x \in [0, 1]$. Since the functions $\overline{w}_1(x, \cdot)$ and $\overline{w}_2(x, \cdot)$ are linearly independent, then from the last system it follows

$$\Delta_1(z_k) = 0, \quad \Delta_2(z_k) = 0, \quad k \in \mathbb{N}, \tag{1.13}$$

It may be checked, that those equalities take place simultaneously, therefore we actually need to use only one of them.

Thus, (1.12) is entire if and only if one of (1.13) holds. Those equalities are, in fact, the constraints where the required control function must be determined from.

Substantially, the Wiener–Paley theorem gives necessary and sufficient conditions for exact controllability of considered system and a possibility to construct the set of admissible controls with compact support which ensures the controllability. As a result, we get an interpolation problem with respect to the Fourier image of the required control functions which can be attacked by efficient techniques of interpolation theory. Then we will need to apply the Fourier inverse transform to obtain the final solution. Instead of it, we may separate the real and the imaginary parts of the interpolation restrictions to obtain the corresponding infinite dimensional problem of moments.

In particular, if we have

a) Dirichlet conditions at both $x = 0$ and $x = 1$ ends, i.e. $\alpha_0 = \alpha_1 = 1$, $\beta_0 = \beta_1 = 0$, then (1.13) reads as

$$\Delta_1(z_k) = \begin{vmatrix} \overline{u}_b^0(z_k) & \overline{w}_2(0, z_k) \\ \overline{u}_b^1(z_k) - \Lambda(1, z_k, \overline{u}_d(z_k)) & \overline{w}_2(1, z_k) \end{vmatrix} = 0, \quad k \in \mathbb{N};$$

b) Neumann conditions at both $x = 0$ and $x = 1$ ends, i.e. $\alpha_0 = \alpha_1 = 0$, $\beta_0 = \beta_1 = 1$, then (1.13) reads as

$$\Delta_1(z_k) = \begin{vmatrix} \overline{u}_b^0(z_k) & \left.\frac{d\overline{w}_2(x,z_k)}{dx}\right|_{x=0} \\ \overline{u}_b^1(z_k) - \left.\frac{d\Lambda(x,z_k,\overline{u}_d(z_k))}{dx}\right|_{x=1} & \left.\frac{d\overline{w}_2(x,z_k)}{dx}\right|_{x=1} \end{vmatrix} = 0, \quad k \in \mathbb{N};$$

c) Dirichlet condition at $x = 0$ and Neumann condition at $x = 1$ end, i.e. $\alpha_0 = \beta_1 = 1$, $\alpha_1 = \beta_0 = 0$, then (1.13) reads as

$$\Delta_1(z_k) = \begin{vmatrix} \overline{u}_b^0(z_k) & \overline{w}_2(0, z_k) \\ \overline{u}_b^1(z_k) - \Lambda(1, z_k, \overline{u}_d(z_k)) & \left.\frac{d\overline{w}_2(x,z_k)}{dx}\right|_{x=1} \end{vmatrix} = 0, \quad k \in \mathbb{N};$$

d) Neumann condition at $x = 0$ end and Dirichlet condition at $x = 1$ end, i.e. $\alpha_1 = \beta_0 = 1$, $\alpha_0 = \beta_1 = 0$, then (1.13) reads as

$$\Delta_1(z_k) = \begin{vmatrix} \overline{u}_b^0(z_k) & \left.\frac{d\overline{w}_2(x,z_k)}{dx}\right|_{x=0} \\ \overline{u}_b^1(z_k) - \left.\frac{d\Lambda(x,z_k,\overline{u}_d(z_k))}{dx}\right|_{x=1} & \overline{w}_2(1, z_k) \end{vmatrix} = 0, \quad k \in \mathbb{N}.$$

For example, when it is required to find the boundary control function u_b^0 in the case d), we have

$$\overline{u}_b^0(z_k) = \frac{1}{\overline{w}_2(1, z_k)} \left.\frac{d\overline{w}_2(x, z_k)}{dx}\right|_{x=0} \left[\overline{u}_b^1(z_k) - \left.\frac{d\Lambda(x, z_k, \overline{u}_d(z_k))}{dx}\right|_{x=1} \right] \equiv \mathcal{M}_k, \quad k \in \mathbb{N},$$

as long as

$$\Delta(z) = \overline{w}_1'(0,z)\overline{w}_2(1,z) - \overline{w}_2'(0,z)\overline{w}_2(1,z) = 0, \quad z \in \mathbb{C}.$$

As it was mentioned above, we may consider this system as interpolation conditions, given in nodes z_k, construct appropriate interpolation polynomials for $\overline{u}_b^0(z)$, and then apply the Fourier inverse transform to obtain the required function u_b^0. Following [22], we also may, and are going to, separate its real and imaginary parts and obtain the countable system of integral constraints as follows:

$$\int_0^T u_b^0(t)\exp[-\varsigma_k t]\cos(\sigma_k t)dt = \mathcal{M}_{1k}, \quad \int_0^T u_b^0(t)\exp[-\varsigma_k t]\sin(\sigma_k t)dt = \mathcal{M}_{2k}, \quad k \in \mathbb{N}, \tag{1.14}$$

$$\mathcal{M}_{1k} + i\mathcal{M}_{2k} = \mathcal{M}_k.$$

The solution of (1.14) may be constructed explicitly, for instance, by the technique outlined in [77], treating those constraints as problem of moments with respect to the unknown function. This is an accepted approach in control theory of both concentrated [77] and distributed [22] parameter systems. On the other hand side (1.14) is a countable system of Fredholm integral equations of the first kind with $C^\infty[0,T]$ kernels.

Now let us consider some particular cases of (1.3) and explain the derivation procedure of corresponding problems of moments. It is supposed, that $u_d(t) = u_b^1(t) \equiv 0$ and finding of u_b^0 is required. All other cases should be analyzed in the same manner.

i) *Equations of hyperbolic type.* Partial differential equations of hyperbolic type, particularly, describe wave propagation, define electromagnetic field, and other processes in natural sciences as well. Consider the one-dimensional homogeneous wave equation ($A = a = 1$ and $B = b = C = 0$)

$$\mathcal{D}[w] \equiv \frac{\partial^2 w(x,t)}{\partial x^2} - \frac{\partial^2 w(x,t)}{\partial t^2} = 0, \quad (x,t) \in (0,1) \times (0,T).$$

In case a) of (1.4) we have $\overline{w}_1 = \exp[i\sigma x]$ and $\overline{w}_2 = \exp[-i\sigma x]$ and

$$\Delta(z) = \begin{vmatrix} 1 & 1 \\ e^{iz} & e^{-iz} \end{vmatrix} = -2i\sin z,$$

therefore the roots of the equation $\Delta(z) = 0$ will be $z_k = \pi k$, $k \in \mathbb{N}$, i.e. all the interpolation nodes are real ($\varsigma_k = 0$). Then, (1.13) provides the system of restrictions

$$\overline{u}_b^0(\pi k) = \int_0^1 \left[i w_0(\xi) - \frac{1}{\pi k} w_0^1(\xi) \right] \sin(\pi k \xi)\, d\xi, \quad k \in \mathbb{N}.$$

Separating the real and the imaginary parts we will obtain a trigonometric problem of moments

$$\left.\begin{aligned} \int_0^T u_b^0(t)\cos(\pi k t)dt &= -\frac{1}{\pi k}\int_0^1 w_0^1(\xi)\sin(\pi k\xi)d\xi, \\ \int_0^T u_b^0(t)\sin(\pi k t)dt &= \int_0^1 w_0(\xi)\sin(\pi k\xi)d\xi, \end{aligned}\right. \qquad k \in \mathbb{N}.$$

So, in order to find the control function in this case we have to solve a system of trigonometric moments problem.

ii) Equations of parabolic type. Partial differential equations of parabolic type, particularly, describe heat conduction and diffusion processes, different types of flow, and other processes in natural sciences as well. For simplicity we consider the homogeneous one-dimensional heat equation ($A = b = 1$ and $a = B = C = 0$)

$$\mathcal{D}[w] \equiv \frac{\partial^2 w(x,t)}{\partial x^2} - \frac{\partial w(x,t)}{\partial t} = 0, \quad (x,t) \in (0,1) \times (0,T).$$

In case a) of (1.4) we have $\overline{w}_1 = \exp\left[\sqrt{i\sigma}x\right]$ and $\overline{w}_2(x,\sigma) = \exp\left[-\sqrt{i\sigma}x\right]$ and

$$\Delta(z) = \begin{vmatrix} 1 & 1 \\ e^{\sqrt{iz}} & e^{-\sqrt{iz}} \end{vmatrix} = -2\sinh\sqrt{iz},$$

therefore $z_k = i(\pi k)^2$, $k \in \mathbb{N}$, i.e. all interpolation nodes are pure imaginary ($\sigma_k = 0$). Then, (1.13) provides the system of restrictions

$$\overline{u}_b^0\left(i(\pi k)^2\right) = \frac{1}{\pi k}\int_0^1 w_0(\xi)\sin(\pi k\xi)\, d\xi, \quad k \in \mathbb{N}.$$

These constraints are obviously real and can be rewritten as follows:

$$\int_0^T u_b^0(t)\exp\left[-(\pi k)^2 t\right] dt = \frac{1}{\pi k}\int_0^1 w_0(\xi)\sin(\pi k\xi)d\xi, \quad k \in \mathbb{N}.$$

In practice we need a finite number of moment equalities. To handle this lack, we will use the following theorem, giving an idea about the solution of an infinite dimensional problem of moments if the solution of any of its truncated part is known.

Theorem 1.2 ([22], p. 76) *There exists an n–dimensional vector function $u \in L^p(\mathbf{T})$, $1 \leq p \leq \infty$, $\mathbf{T} \subset \mathbb{R}^n$, resolving the infinite dimensional linear problem of moments*

$$\int_{\mathbf{T}} u(t)\mathcal{K}_k(t)dt = \mathcal{M}_k, \quad k \in \mathbb{N}, \tag{1.15}$$

for given sequence of functions $\{\mathcal{K}_k\}_{k\in\mathbb{N}} \in L^q(\mathbf{T})$, $p^{-1}+q^{-1}=1$, and constants $\mathcal{M}_k$, if and only if the inequality

$$\left|\sum_{k=1}^{m} \mathcal{M}_k l_k\right| \leq l \left[\int_{\mathbf{T}} \sum_{\kappa=1}^{n} \left|\sum_{k=1}^{m} l_k \mathcal{K}_k^{\kappa}(t)\right|^q dt\right]^{\frac{1}{q}}, \quad ||u||_{L^p(\mathbf{T})} \leq l < \infty,$$

holds for any finite set of constants $\{l_\kappa\}_{\kappa\in\{1;m\}} \in \mathbb{R}$.

In other words, an infinite dimensional linear problem of moments is resolvable if and only if the solution of its truncated part for any $m \in \mathbb{N}$ exists. Questions concerning the convergence of the solution of the truncated part to that of the infinite system are studied in traditional manners [22].

In [77] the L^1, L^2 and L^∞–optimal solutions of finite dimensional linear problem of moments are constructed rigorously. In [22] the general solution of linear problem of moments minimizing L^p–norm, $1 \leq p \leq \infty$, is brought.

Theorem 1.3 ([22], p. 72) *The general solution $u \in L^p(\mathbf{T})$, $1 < p \leq \infty$, of (1.15) for some finite m exists, is unique and reads as*

$$u^o(t) = \lambda_m^q \left|\sum_{k=1}^{m} l_k^o \mathcal{K}_k(t)\right|^{q-1} \operatorname{sign} \sum_{k=1}^{m} l_k^o \mathcal{K}_k(t), \quad t \in \mathbf{T}, \tag{1.16}$$

where λ_m and $\{l_k^o\}_{k\in\{1;m\}}$, are determined as a solution of the following equivalent problems of conditional extrema:

i) find

$$\lambda_m = \sum_{k=1}^{m} l_k^o \mathcal{M}_k = \max_{\{l_k\}_{k\in\{1;m\}}} \sum_{k=1}^{m} l_k \mathcal{M}_k,$$

under the condition

$$\left[\int_{\mathbf{T}} \left[\sum_{k=1}^{m} l_k \mathcal{K}_k(t)\right|^q dt\right]^{\frac{1}{q}} = 1.$$

ii) find

$$\frac{1}{\lambda_m} = \min_{\{l_k\}_{k\in\{1;m\}}} \left[\int_{\mathbf{T}} \left[\sum_{k=1}^{m} l_k \mathcal{K}_k(t)\right|^q dt\right]^{\frac{1}{q}},$$

under the condition

$$\sum_{k=1}^{m} l_k \mathcal{M}_k = 1.$$

The minimal value of the norm $||u||_{L^p(\mathbf{T})}$ is equal to $||u||_{L^p(\mathbf{T})} = \lambda_m$.

In the case when $p = 1$ ($q = \infty$) the resolving admissible controls are determined as a weak limit of (1.16) as follows

$$u^o(t) = \sum_{j=1}^{J} u_j^o \delta\left(t - t_j^o\right) \operatorname{sign} \sum_{k=1}^{m} l_k^o \mathcal{K}_k\left(t_j^o\right), \quad t \in \mathbf{T}, \tag{1.17}$$

where $t_j^o \in \mathbf{T}$ are determined from

$$\operatorname*{vrai\,max}_{t\in\mathbf{T}} \left|\sum_{k=1}^{m} l_k^o \mathcal{K}_k(t)\right| = \min_{\{l_k\}_{k\in\{1;m\}}} \operatorname*{vrai\,max}_{t\in\mathbf{T}} \left|\sum_{k=1}^{m} l_k \mathcal{K}_k(t)\right|,$$

and u_j^o– from (1.15), with substituted (1.17) into it.

Remark 1.1 *It is true, that $\delta \notin L^1$ in usual sense and therefore the meaning of (1.17) must be clarified. Furthermore, dealing with L^1–optimal controls we will take into account only its physical treatment: $\delta(t - t_0)$ describes unit impulse acting at t_0.*

We will pass the proof of both Theorems 1.2 and 1.3, as they are brought in details in [22].

Definition 1.1 *If the corresponding set $\mathcal{U}_p \neq \varnothing$, $1 \leq p \leq \infty$, the system is called L^p–controllable. The corresponding resolving controls are called L^p– controls. If in addition, the solution is a minimizer of the L^p norm of the control function, then it is called L^p– optimal control.*

We finalize the subsection with this very useful and attractive result.

Theorem 1.4 ([22], p. 101) *Assume that the solution u_j of the following problem of moments is known:*

$$\int_{\mathbf{T}} u_j(t)\mathcal{K}_k(t)dt = \delta_j^k, \quad j, k \in \mathbb{N}.$$

Then, the solution of (1.15) reads as

$$u(t) = \sum_{j\in\mathbb{N}} \mathcal{M}_j u_j(t), \quad t \in \mathbf{T}.$$

1.3.1 L^1, L^2 and L^∞-Optimal Controls

In particular, let us consider the L^1, L^2 and L^∞–optimal solutions of (1.14) for finite N:

$$\int_0^T u(t)\exp\left[-\varsigma_k t\right]\cos(\sigma_k t)dt = \mathcal{M}_{1k}, \quad \int_0^T u(t)\exp\left[-\varsigma_k t\right]\sin(\sigma_k t)dt = \mathcal{M}_{2k}, \tag{1.18}$$

$$k \in \{1; N\},$$

i.e. those minimizing the functionals

$$\kappa_1[u] = ||u||_{L^1[0,T]} \equiv \int_0^T |u(t)|dt, \quad u \in \mathcal{U}_1, \tag{1.19a}$$

$$\kappa_2[u] = ||u||^2_{L^2[0,T]} \equiv \int_0^T u^2(t)dt, \quad u \in \mathcal{U}_2, \tag{1.19b}$$

$$\kappa_\infty[u] = ||u||_{L^\infty[0,T]} \equiv \max_{t\in[0,T]} |u(t)|, \quad u \in \mathcal{U}_\infty, \tag{1.19c}$$

respectively, characterizing the linear momentum, the energy and the intensity of controls.

Remark 1.2 *Naturally, we should put the subscript N to the unknown u in (1.18) to differ it from the solution of the infinite system, but we omit it implying that dependence obvious.*

The L^1–optimal solution of (1.18) has the form (1.17) [67–74]. The intensities u_j^o are constrained by

$$\text{sign } u_j^o = \text{sign } h^o\left(t_j^o\right), \quad j \in \{1; J\},$$

and are determined from the system

$$\sum_{j=1}^J u_j^o \exp\left[-\varsigma_k t_j^o\right]\cos\left(\sigma_k t_j^o\right) = \mathcal{M}_{1k}, \quad \sum_{j=1}^J u_j^o \exp\left[-\varsigma_k t_j^o\right]\sin\left(\sigma_k t_j^o\right) = \mathcal{M}_{2k}, \quad k \in \{1; N\}.$$

The moments t_j^o are determined from the equality

$$\kappa_\infty[h^o] = \left[\sum_{j=1}^J |u_j^o|\right]^{-1}.$$

The number J of impacts must be determined from the condition $\left\{t_j^o\right\}_{j=1}^J \subset (0,T)$. Unfortunately, it is non-unique [22, 67–74, 77].

Here

$$h^o(t) = \sum_{k=1}^{N} \exp\left[-\varsigma_k t\right]\left[l_{1k}^o \cos\left(\sigma_k t\right) + l_{2k}^o \sin\left(\sigma_k t\right)\right], \quad t \in [0,T],$$

and the optimal coefficients l_{1k}^o, l_{2k}^o are determined from the problem of conditional extremum:

$$h^o\left(t_j^o\right) \xrightarrow[l_{1k},l_{2k}]{} \min, \quad \text{when} \ \sum_{k=1}^{N}\left[l_{1k}\mathcal{M}_{1k} + l_{2k}\mathcal{M}_{2k}\right] = 1.$$

At this, the solution $u^o(t)$ of the truncated system (1.18) exists if and only if $\kappa_\infty[h^o] \neq 0$ [77]. Then, for solvability of (1.14) according to Theorem 1.2 we have

Theorem 1.5 *The L^1–optimal solution of (1.14) exists if and only if*

$$\sum_{j=1}^{J} |u_j^o| \neq 0$$

for all $N \in \mathbb{N}$. Then, (1.17) converges in L^1 to the solution with $N \to \infty$ uniformly.

According to Theorem 1.3 the L^2–optimal solution is [68–71]

$$u^o(t) = \sum_{k=1}^{N} \exp\left[-\varsigma_k t\right]\left[l_{1k}^o \cos(\sigma_k t) + l_{2k}^o \sin(\sigma_k t)\right], \ t \in [0,T], \tag{1.20}$$

where the coefficients l_{pk}^o, $p \in \{1;2\}$, are determined from the system of linear algebraic equations

$$\mathbf{J}\mathbf{L}^o = \mathbf{M},$$

where $\mathbf{L}^o = (l_{11}^o \ldots l_{1n}^o \ l_{21}^o \ldots l_{2N}^o)^{\mathrm{T}}$, $\mathbf{M} = (\mathcal{M}_{11} \ldots \mathcal{M}_{1N} \ \mathcal{M}_{21} \ldots \mathcal{M}_{2N})^{\mathrm{T}}$,

$$\mathbf{J} = \begin{pmatrix} J_{11}^+ & J_{12}^+ & \ldots & J_{1N}^+ & J_{11} & J_{12} & \ldots & J_{1N} \\ J_{21}^+ & J_{22}^+ & \ldots & J_{2N}^+ & J_{21} & J_{22} & \ldots & J_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ J_{N1}^+ & J_{N1}^+ & \ldots & J_{NN}^+ & J_{N1} & J_{N2} & \ldots & J_{NN} \\ J_{11} & J_{21} & \ldots & J_{1N} & J_{11}^- & J_{12}^- & \ldots & J_{1N}^- \\ J_{12} & J_{22} & \ldots & J_{2N} & J_{21}^- & J_{22}^- & \ldots & J_{2N}^- \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ J_{1N} & J_{2N} & \ldots & J_{NN} & J_{N1}^- & J_{N2}^- & \ldots & J_{NN}^- \end{pmatrix}, \tag{1.21}$$

$$J_{jk}^{\pm} = \int_0^T \exp\left[-\left(\varsigma_j + \varsigma_k\right)t\right] \begin{pmatrix} \cos\left(\sigma_j t\right)\cos(\sigma_k t) \\ \sin\left(\sigma_j t\right)\sin(\sigma_k t) \end{pmatrix} dt,$$

$$J_{jk} = \int_0^T \exp\left[-\left(\varsigma_j + \varsigma_k\right)t\right]\cos\left(\sigma_j t\right)\sin(\sigma_k t)dt.$$

The following theorem gives necessary and sufficient conditions for exact controllability in this case.

Theorem 1.6 *L^2–optimal solution of (1.14) exists if and only if*

$$\kappa_2[h^o] = \sum_{k=1}^{N} (l_{1k}^o)^2 J_{kk}^{+} + 2\sum_{j=1}^{N-1}\sum_{k=j+1}^{N} l_{1j}^o \left(l_{1k}^o J_{jk}^{+} + l_{2k}^o J_{jk}\right) + \\ + \sum_{k=1}^{N} (l_{2k}^o)^2 J_{kk}^{-} + 2\sum_{j=1}^{N-1}\sum_{k=j+1}^{N} l_{2j}^o \left(l_{2k}^o J_{jk} + l_{2k}^o J_{jk}^{-}\right) \tag{1.22}$$

is positive for all $N \in \mathbb{N}$. Then, (1.20) converges in L^2 to the solution with $N \to \infty$ uniformly.

Here the selfadjointness of $L^2[0,T]$ is taken into account.

If all roots z_k, $k \in \mathbb{N}$, are real, i.e. $\varsigma_k = 0$, $k \in \mathbb{N}$, instead of formulas (1.20)–(1.22) the following should be used: the optimal control (1.20) reads as

$$u^o(t) = \sum_{k=1}^{N} \left[l_{1k}^o \cos(\sigma_k t) + l_{2k}^o \sin(\sigma_k t)\right], \ t \in [0,T],$$

the system (1.21) is separated into two independent systems with respect to the coefficients l_{pk}^o, $p \in \{1;2\}$, correspondingly:

$$\mathbf{J}^{\pm}\mathbf{L}_p^o = \mathbf{M}_p, \ p \in \{1;2\},$$

$$\mathbf{L}_p^o = (l_{p1}^o \dots l_{pN}^o)^T, \ \mathbf{M}_p = (\mathcal{M}_{p1} \dots \mathcal{M}_{pN})^T, \ \mathbf{J}^{\pm} = \{J_{jk}^{\pm}\}_{j,k=1}^{N},$$

$$J_{jk}^{\pm} = \int_0^T \begin{pmatrix} \cos(\sigma_j t)\cos(\sigma_k t) \\ \sin(\sigma_j t)\sin(\sigma_k t) \end{pmatrix} dt,$$

and the expression (1.22) reads as

$$\kappa_2[h^o] = \sum_{k=1}^{N} (l_{1k}^o)^2 J_{kk}^{+} + 2\sum_{j=1}^{N-1} l_{1j}^o \sum_{k=j+1}^{N} l_{1k}^o J_{jk}^{+} + \sum_{k=1}^{N} (l_{2k}^o)^2 J_{kk}^{-} + 2\sum_{j=1}^{N-1} l_{2j}^o \sum_{k=j+1}^{N} l_{2k}^o J_{jk}^{-}.$$

This corresponds to the case of the harmonic control [31].

L^∞–optimal solution of (1.18) is

$$u^o(t) = u_0 \cdot \operatorname{sign} h^o(t), \quad t \in [0,T]. \tag{1.23}$$

The solvability of (1.14) in this case is provided by the following theorem.

Theorem 1.7 *L^∞–optimal solution of (1.14) exists if and only if*

$$\kappa_1[h^o] \equiv \frac{1}{u_0} \le \sum_{k=1}^{N} \frac{1-\exp\left[-\varsigma_k T\right]}{\varsigma_k} \sqrt{\left(l^o_{1k}\right)^2 + \left(l^o_{2k}\right)^2}$$

for all $N \in \mathbb{N}$. Then, (1.23) converges in L^2 to the solution with $N \to \infty$ uniformly.

The case when all z_k, $k \in \mathbb{N}$, are real ($\varsigma_k \to 0$) is obtained using l'Hospital's rule.

In a particular problem of optimal control, Theorems 1.5, 1.6 and 1.7, essentially, help to construct the set of initial, terminal data, boundary functions, right-hand side and other parameters of the state equation and control time T in terms of which the system is exact controllable.

1.3.2 The Generalized Control with Compact Support

It follows from physical reasons that supp $w(x,\cdot) \subseteq [0,1]$, but in the general case it is not necessarily identically zero outside $[0,T]$. In such cases the Fourier transform cannot be replaced by a definite integral with bounds 0 and T as it was done before. It turns out, that from the viewpoint of the theory of distributions it is possible to handle this lack. For that purpose we suggest to introduce a new generalized state function which will be compactly supported in the rectangle $[0,1]\times[0,T]$ and coincide with the state function therein [57–60, 65–74].

Suppose we have to solve a control problem with fix end-points for abstract differential equation

$$\mathcal{D}[w] = f(\boldsymbol{x}, t, u_d(t)), \quad (\boldsymbol{x}, t) \in \mathcal{O} \times [0,T], \quad \text{in } \mathfrak{D}, \tag{1.24}$$

where, for instance, $\mathcal{O} \subset \mathbb{R}^3$, subjected to boundary conditions

$$\mathcal{B}[w] = u_b(t), \quad (\boldsymbol{x}, t) \in \partial\mathcal{O} \times [0,T]. \tag{1.25}$$

Here we have $f \in \mathfrak{D}$, therefore, the state function $w \in \mathfrak{D}$ as well (see [89, 119]). $\mathcal{D}[\cdot]$ is the state operator, $\mathcal{B}[\cdot]$ is the operator of the boundary conditions: they both are supposed to have at most *stationary* or coordinate dependent coefficients.

The initial state of the system is supposed to be known:

$$\mathcal{D}_t^m w(\boldsymbol{x}, t)\Big|_{t=0} = w_0^m(\boldsymbol{x}), \quad m \in \{0; n-1\}, \ \boldsymbol{x} \in \overline{\mathcal{O}}, \tag{1.26}$$

n is the order of the highest time-derivative of the state function in (1.24).

The aim of the control problem is the ensuring of terminal condition

$$\mathcal{D}_t^m w(\boldsymbol{x}, t)\big|_{t=T} = w_T^m(\boldsymbol{x}), \quad m \in \{0; n-1\}, \ \boldsymbol{x} \in \overline{\mathcal{O}}, \tag{1.27}$$

by means of an appropriate choice of control function. Functions (1.26) and (1.27) are chosen from appropriate Sobolev spaces (see [42] for details). All data are supposed to be real-valued.

As above, the control may be implemented either by means of boundary function $u_b \in \mathcal{U}_b$ or distributed control $u_d \in \mathcal{U}_d$.

In order to obtain a state function compactly supported in $\mathcal{O} \times [0, T]$ we introduce the operator

$$\mathcal{A}_{[0,T]}[\eta] \equiv \eta_1(t) = \begin{cases} \eta(t), & t \in [0, T], \\ 0, & t \notin [0, T], \end{cases} \qquad t \in \mathbb{R},$$

which puts arbitrary function η in correspondence with a function η_1 compactly supported in $[0, T]$ where it coincides with the main function η.

This operator may be represented in terms of the characteristic function

$$\mathcal{A}_{[0,T]}[\eta] = \chi_{[0,T]}(t)\eta(t) \equiv \eta_1(t), \quad t \in \mathbb{R}.$$

Since further we will need to deal with distributions and their derivatives, it will be advisable to have other explicit forms of this operator. For that purpose, we may use the representation of the indicator function by means of Heaviside‘s function, given in Section 1.2:

$$\mathcal{A}_{[0,T]}[\eta] = [\theta(t) - \theta(t-T)]\eta(t) \equiv \eta_1(t), \quad t \in \mathbb{R}. \tag{1.28}$$

It is important to reveal, that the introduced operator $\mathcal{A}_{[0,T]}[\cdot]$ is a linear continuous mapping from the space of ordinary functions η into the Sobolev space of functions, which are concentrated on $[0, T]$ with generalized derivatives of arbitrary order.

In order to solve the problem we apply the operator $\mathcal{A}_{[0,T]}[\cdot]$ to governing system (1.24), (1.25). Taking into account the Leibniz rule of differentiation, we have

$$\mathcal{D}^n \eta_1(t) = \sum_{m=0}^{n} \mathrm{C}_m^n \mathcal{D}^{n-m} \left[\theta(t) - \theta(t-T)\right] \mathcal{D}^m \eta(t), \quad t \in \mathbb{R}.$$

Since $\mathcal{D}\theta(t) = \delta(t)$ in $\mathfrak{D}$, then

$$\mathcal{D}^n\eta_1(t) = [\theta(t) - \theta(t-T)]\,\mathcal{D}^n\eta(t) + \sum_{m=0}^{n-1} \mathrm{C}_m^n \mathcal{D}^{n-m-1}\left[\delta(t) - \delta(t-T)\right]\mathcal{D}^m\eta(t) =$$

$$= \mathcal{A}_{[0,T]}\left[\mathcal{D}^n\eta\right] + \sum_{m=0}^{n-1} \mathrm{C}_m^n \mathcal{D}^{n-m-1}\left[\delta(t) - \delta(t-T)\right]\mathcal{D}^m\eta(t), \quad t \in \mathbb{R}.$$

Using the filtering property of the Dirac's delta, the second term at the right hand side is well-defined by means of initial and terminal data (1.26), (1.27):

$$\sum_{m=0}^{n-1} \mathrm{C}_m^n \mathcal{D}^{n-m-1}\left[\delta(t) - \delta(t-T)\right]\mathcal{D}^m w(x,t) =$$

$$= \sum_{m=0}^{n-1} \mathrm{C}_m^n \left[w_0^m(\boldsymbol{x})\mathcal{D}^{n-m-1}\delta(t) - w_T^m(\boldsymbol{x})\mathcal{D}^{n-m-1}\delta(t-T)\right], \quad t \in \mathbb{R}.$$

For instance,

$$\frac{\partial w_1}{\partial t} = [\theta(t) - \theta(t-T)]\,\frac{\partial w}{\partial t} + w_0^1(\boldsymbol{x})\delta(t) + w_T^1(\boldsymbol{x})\delta(t-T), \quad t \in \mathbb{R},$$

$$\frac{\partial^2 w_1}{\partial t^2} = [\theta(t) - \theta(t-T)]\,\frac{\partial^2 w}{\partial t^2} + w_0^1(\boldsymbol{x})\delta'(t) - w_T^1(\boldsymbol{x})\delta'(t-T) + w_0^2(\boldsymbol{x})\delta(t) - w_T^2(\boldsymbol{x})\delta(t-T), \quad t \in \mathbb{R}.$$

Thus, from (1.24), (1.25) we will have

$$\mathcal{D}[w_1] = f_1(\boldsymbol{x}, t, u_{d1}(t)) - W(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \mathcal{O} \times \mathbb{R}, \tag{1.29}$$

$$\mathcal{B}[w_1] = u_{b1}(t), \quad (\boldsymbol{x}, t) \in \partial\mathcal{O} \times \mathbb{R}, \tag{1.30}$$

where $W(\boldsymbol{x}, t)$ contains the initial and the terminal data of the system derived in the mentioned manner. $w_1(\boldsymbol{x}, t) = \mathcal{A}_{[0,T]}[w]$, defined for all t will be called generalized state function. Obviously, the generalized state function is concentrated on $[0, T]$, i.e. supp $w_1(\cdot, t) = [0, T]$, where the control process is carried out and it coincides with the state function w therein.

Now, we may apply the distributional Fourier transform to (1.29) and (1.30). As a result we will obtain a Cauchy problem with respect to Fourier image of the generalized state function:

$$\overline{\mathcal{D}}[\overline{w}_1] = \overline{f}_1(\boldsymbol{x}, \sigma, \overline{u}_{d1}(\sigma)) - \overline{W}(\boldsymbol{x}, \sigma), \quad (\boldsymbol{x}, \sigma) \in \mathcal{O} \times \mathbb{R}, \tag{1.31}$$

$$\overline{\mathcal{B}}[\overline{w}_1] = \overline{u}_{b1}(\sigma), \quad (\boldsymbol{x}, \sigma) \in \partial\mathcal{O} \times \mathbb{R}, \tag{1.32}$$

and we will be able to proceed as is suggested above and obtain equalities like (1.14). On this way we have to use Wiener–Paley–Schwartz theorem 1.2 for distributions.

As you can see, the main distinguishing feature of Butkovskiy's method of control with compact support [22] and the method of control with compact support used in this thesis is that in [22] the compactness of the support of the control and the state functions is *assumed*, whereas here it is *ensured* by means of introduction of a generalized state function.

Coming back to (1.3) and (1.4) according to Butkovskiy's generalized method we have

$$\mathcal{D}[w_1] = G(x,t,u_{d1}(t)), \quad (x,t) \in (0,1) \times \mathbb{R}, \tag{1.33}$$

$$\left.\begin{aligned} \alpha_0 w_1(0,t) + \beta_0 \frac{\partial w_1(x,t)}{\partial x}\bigg|_{x=0} &= u_{b1}^0(t), \\ \alpha_1 w_1(1,t) + \beta_1 \frac{\partial w_1(x,t)}{\partial x}\bigg|_{x=1} &= u_{b1}^1(t), \end{aligned}\right. \quad t \in \mathbb{R}, \tag{1.34}$$

where

$$G(x,t,u_{d1}(t)) = f_1(x,t,u_{d1}(t)) - a w_0(x)\delta'(t) - [a w_0^1(x) + b w_0(x)]\delta(t).$$

Now we may apply the distributional Fourier transform to (1.3), (1.4) without troubling about values of the state function $w(x,t)$ outside the interval $[0,T]$ and proceed as it is recommended above.

1.3.2.1 Applications in Deformable Body Mechanics

In this subsection we illustrate the use of Butkovskiy's method for solving boundary control problems for a coupled systems of partial differential equations arise in thermo-elasticity. As it was mentioned above, Butkovskiy's method is a powerful tool for dealing with control problems for coupled systems of partial differential equations. Such systems arise in problems of connected fields (thermo-elasticity, magneto-electro-elasticity, diffusion, etc.), that is why illustration of this approach for such a system has practical importance. A relevant problem of distributed control is considered in [93].

Let an isotropic homogeneous elastic infinite layer $\mathcal{O}_* = \{(x_*, y_*, z_*);\ \ x_* \in [0,h],\ y_* \in \mathbb{R},\ z_* \in \mathbb{R}\}$ is subjected to uniformly distributed normal (compressing) boundary stresses of intensity σ_0. The layer is in convective heat transfer with external medium according to Newton's law. This leads to coupling of stresses σ_{x*} normal to sections in x_* direction of the layer and temperature Θ_* of the layer. Introducing dimensionless variables and functions

$$x = \frac{x_*}{h}, \quad t = \frac{2t_1 - T_1}{T_1}\pi, \quad t_1 = \frac{\alpha_\Theta}{h^2}t_*, \quad \sigma_x = \frac{1-2\nu}{E}\sigma_{x*}, \quad \Theta = \frac{1-\nu}{1+\nu}\beta_\Theta \Theta_*, \quad \alpha^2 = \frac{\alpha_\Theta^2}{h^2 c^2},$$

the state of the layer can be described by [57]

$$\frac{\partial^2\sigma_x}{\partial x^2}=\alpha^2\left[\frac{\partial^2\sigma_x}{\partial t^2}+\frac{\partial^2\Theta}{\partial t^2}\right],\quad \frac{\partial^2\Theta}{\partial x^2}=(1+\varepsilon)\frac{\partial\Theta}{\partial t}+\varepsilon\frac{\partial\sigma_x}{\partial t},\quad x\in[0,1],\ t\in[-\pi,\pi]. \tag{1.35}$$

Here α_Θ is the thermal diffusivity, β_Θ is the coefficient of linear thermal expansion, α characterizes the ratio of elastic wave and temperature propagation times, ε is the temperature and deformation coupling factor. The boundary conditions take the form

$$\sigma_x=-\sigma_0 u_{b1}(t),\quad \frac{\partial\Theta}{\partial x}\mp\mathrm{Bi}\,(\Theta-\Theta_0 u_{b2}(t))=0,\quad x=0;1,\ t\in[-\pi,\pi], \tag{1.36}$$

Here $0<\sigma_0,\Theta_0=const$. The "initial" conditions are supposed to be

$$\sigma_x=0,\quad \frac{\partial\sigma_x}{\partial t}=0,\quad \Theta=0,\quad t=-\pi,\ x\in[0,1].$$

Our aim is the (implicit or explicit) representation of the set of admissible controls $\mathcal{U}_\infty$ providing "terminal" required conditions

$$\sigma_x=-\sigma_0,\quad \frac{\partial\sigma_x}{\partial t}=\sigma_\pi,\quad \Theta=\Theta_\pi(x),\quad t=\pi,\ x\in[0,1].$$

Here $\sigma_\pi=const$.

Applying the operator $\mathcal{A}_{[-\pi,\pi]}$ to (1.35) and (1.36) we will obtain

$$\frac{\partial^2\sigma_{x1}}{\partial x^2}=\alpha^2\left[\frac{\partial^2\sigma_{x1}}{\partial t^2}+\frac{\partial^2\Theta_1}{\partial t^2}\right]+W_1(x,t),\quad \frac{\partial^2\Theta_1}{\partial x^2}=(1+\varepsilon)\frac{\partial\Theta_1}{\partial t}+\varepsilon\frac{\partial\sigma_{x1}}{\partial t}+W_2(x,t), \tag{1.37}$$

$$W_1(x,t)=-\alpha^2\left[(\sigma_0-\Theta_\pi(x))\,\delta'(t-\pi)-\frac{\sigma_\pi+\Theta_\pi''(x)}{1+\varepsilon}\delta(t-\pi)\right],$$

$$W_2(x,t)=[(1+\varepsilon)\Theta_\pi(x)-\varepsilon\sigma_0]\,\delta(t-\pi),\quad x\in[0,1],\ t\in\mathbb{R},$$

$$\sigma_{x1}=-\sigma_0 u_{b11}(t),\quad \frac{\partial\Theta_1}{\partial x}\mp\mathrm{Bi}\,(\Theta_1-\Theta_0 u_{b21}(t))=0,\quad x=0;1,\ t\in\mathbb{R}. \tag{1.38}$$

Applying the procedure described in the previous subsection we will arrive at

$$\begin{gathered} g_1(\sigma)h_1(\sigma)+g_1(\sigma)h_2(\sigma)+g_2(\sigma)h_3(\sigma)+g_2(\sigma)h_4(\sigma)=-\sigma_0\overline{u}_{b11}(\sigma)+\frac{1}{(-i\sigma)\varepsilon}\overline{W}_2(0,\sigma),\\ g_1(\sigma)\exp\left[\lambda_1(\sigma)\right]h_1(\sigma)+g_1(\sigma)\exp\left[-\lambda_1(\sigma)\right]h_2(\sigma)+g_2(\sigma)\exp\left[\lambda_2(\sigma)\right]h_3(\sigma)+\\ +g_2(\sigma)\exp\left[-\lambda_2(\sigma)\right]h_4(\sigma)=-\sigma_0\overline{u}_{b11}(\sigma)-\Lambda_2(\sigma),\\ (\lambda_1(\sigma)-\mathrm{Bi})\,h_1(\sigma)+(-\lambda_1(\sigma)-\mathrm{Bi})\,h_2(\sigma)+(\lambda_2(\sigma)-\mathrm{Bi})\,h_3(\sigma)+\\ +(-\lambda_2(\sigma)-\mathrm{Bi})\,h_4(\sigma)=-\mathrm{Bi}\cdot\Theta_0\overline{u}_{b21}(\sigma), \end{gathered} \tag{1.39}$$

$$(\lambda_1(\sigma)+\mathrm{Bi})\exp[\lambda_1(\sigma)]\,h_1(\sigma)+(-\lambda_1(\sigma)+\mathrm{Bi})\exp[-\lambda_1(\sigma)]\,h_2(\sigma)+$$

$$+(\lambda_2(\sigma)+\mathrm{Bi})\exp[\lambda_2(\sigma)]\,h_3(\sigma)+(-\lambda_2(\sigma)+\mathrm{Bi})\exp[-\lambda_2(\sigma)]\,h_4(\sigma)=$$

$$=\mathrm{Bi}\cdot\Theta_0\overline{u}_{b21}(\sigma)+\Lambda_1(\sigma),$$

with

$$\lambda_p=\sqrt{\frac{1}{2}\left[(-i\sigma)\left[(-i\sigma)\alpha^2+1+\varepsilon\right]\pm\sqrt{(-i\sigma)^2\left[(-i\sigma)^2\alpha^4+(1+\varepsilon)^2-2(-i\sigma)(1+\varepsilon)\alpha^2\right]}\right]},$$

$$g_p(\sigma)=\frac{\lambda_p^2(\sigma)-(-i\sigma)(1+\varepsilon)}{(-i\sigma)\varepsilon},\ p\in\{1;2\},$$

$$\Lambda_1(\sigma)=\left.\frac{d\Lambda}{dx}\right|_{x=1}+\mathrm{Bi}\cdot\Lambda(1,\sigma),\ \Lambda_2(\sigma)=\frac{1}{(-i\sigma)\varepsilon}\left.\left[\frac{d\Lambda}{dx}-(-i\sigma)(1+\varepsilon)\Lambda(x,\sigma)-\overline{W}_2(x,\sigma)\right]\right|_{x=1},$$

$$\Lambda(x,\sigma)=\int_0^x\frac{\overline{W}(\xi,\sigma)}{W(\xi,\sigma)}\sum_{j=1}^4 W_j(\xi,\sigma)w_j(x,\sigma)d\xi,$$

$w_1(x,\sigma)=\exp[\lambda_1 x]$, $w_2(x,\sigma)=\exp[-\lambda_1 x]$, $w_3(x,\sigma)=\exp[\lambda_2 x]$, $w_4(x,\sigma)=\exp[-\lambda_2 x]$ are the fundamental solutions of transformed (1.37) by Fourier, $W(x,\sigma)$ is the main and $W_j(x,\sigma)$, $j\in\{1;4\}$, are the auxiliary Wronskians of (1.37).

Further, equating to zero the main determinant of (1.39) expanded in the whole $\mathbb{C}$:

$$\left(\mathrm{Bi}^2(g_1(z)-g_2(z))^2-g_1^2(z)\lambda_2^2(z)-g_2^2(z)\lambda_1^2(z)\right)\sinh[\lambda_1(z)]\sinh[\lambda_2(z)]-$$

$$-2g_1(z)\lambda_1(z)g_2(z)\lambda_2(z)(1-\cosh[\lambda_1(z)]\cosh[\lambda_2(z)])=0,\quad z\in\mathbb{C},\tag{1.40}$$

we have to find such $\overline{u}_{b11}(z_k)$ and $\overline{u}_{b12}(z_k)$[3] that two auxiliary determinants (since we seek for two boundary controls) of (1.39)

$$\Delta_1(z_k)=0,\quad \Delta_2(z_k)=0$$

for all $k\in\mathbb{N}$. It will serve us as a system to derive constraints on control functions like (1.14). Further, optimal in the sense of (1.19c) control may be found in the similar way as above [57].

[3] z_k, $k\in\mathbb{N}$, are the roots of (1.40).

CHAPTER 2

BOUNDARY AND DISTRIBUTED CONTROL OF VIBRATIONS IN NON-HOMOGENEOUS STRUCTURES

The aim of this chapter is to show the efficiency of Butkovskiy‘s generalized method, outlined in Subsection 1.3.2, in dealing with particular equations in partial derivatives and underline its main features. In this order we have chosen such equations which widely appear in many different areas of applications, and their well-posedness was already proved.

Below we solve boundary and distributed control problems for differential and integro-differential equations in partial derivatives arising in numerous problems of wave propagation in non-homogeneous and dispersive media, respectively. Using Butkovskiy‘s generalized method the solution is reduced to infinite dimensional linear problem of moments. Here we collect only the controllability criteria allowing explicit representation for required controls. Those controls are determined up to a function from wide class, so optimality of the control process also can be considered in terms of results of Section 1. Nevertheless, we do not stop on that issue in this chapter.

The chapter is organized as follows: in Section 2.1 the method is applied to solve boundary and distributed exact controllability problems for one-dimensional wave equation with variable coefficients in finite and semi-infinite domains. Particular mention is given to the case where both boundary and distributed controls contain constant delay. Section 2.2 is devoted to analysis of an integro-differential equation in partial derivatives for boundary and distributed exact controllability and explicit representation of corresponding controls.

The material of this chapter is based on articles [65–71].

2.1 Boundary and Distributed Control of Vibrations of Non-Homogeneous Elastic Systems

Now we begin to apply Butkovskiy's generalized method to analyse the exact controllability of one-dimensional wave equation with variable coefficients and explain the procedure of derivation of resolving systems which can be used in obtaining explicit solution. The state equation is considered in finite and in semi-infinite domains in order to show the efficiency of the method in both cases. The exact controllability is the main aim of the investigation. We also concern with the case where the control function contains constant delay.

The propagation of one-dimensional waves in non-homogeneous media is of special theoretical and practical interest; first because the are described by almost-periodic function (in contrast to the case of homogeneous media when periodic functions are enough), second they appear in many real-life processes (in signal transfer [31], finance [36], damage identification [114], etc.). The process is described by one-dimensional wave equation with variable coefficients, characterizing the non-homogeneity of the medium. It considerably complicates the analytical investigation.

In this section we are going to gather the main results of articles [65–70]. In those investigations the boundary and distributed exact controllability is considered for one-dimensional wave equation with variable coefficients (all quantities and variables are supposed to be dimensionless)

$$\frac{\partial}{\partial x}\left[N(x)\frac{\partial w(x,t)}{\partial x}\right]-\rho(x)\frac{\partial^2 w(x,t)}{\partial t^2}=f(x,t)+u_d(t)v(x),\quad (x,t)\in\mathcal{O}\times(0,T),\tag{2.1}$$

subjected to some linear boundary conditions

$$\mathcal{B}[w]=u_b(t),\quad (x,t)\in\partial\mathcal{O}\times[0,T].\tag{2.2}$$

The domain $\mathcal{O}\subset\mathbb{R}$ in some cases is considered to be finite, in some cases– semi-infinite.

The initial state of the string is supposed to be known:

$$w(x,0)=w_0(x),\ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0}=w_0^1(x),\quad x\in\overline{\mathcal{O}}.\tag{2.3}$$

Equation (2.1), particularly, describes the propagation of one-dimensional waves in non–homogeneous media; for instance the longitudinal vibrations of linear-elastic, non-homogeneous rod, at that w is its displacement, N is the elastic (Young's) modulus and ρ– its density. In particular, if $N \equiv const$ and only $\rho = \rho(x)$, (2.1) describes the transverse vibrations of mechanical string, at that N is its tensile force. That is why hereinafter in this chapter we do not specify the type of vibrating system and merely write vibrating system. Relying on physical considerations, we assume that $0 \leq N \in C_p^{(1)}(\mathcal{O})$, $0 < \rho \in C_p(\mathcal{O})$.

Our aim is to find explicit representation for admissible controls $u \in \mathcal{U}$ and derive controllability conditions for initial-boundary value problem (2.1)–(2.3) providing the terminal conditions

$$w(x,T) = w_T(x), \ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=T} = w_T^1(x), \quad x \in \overline{\mathcal{O}}. \tag{2.4}$$

If we use the notation u without specifying whether is it boundary or distributed, we imply that the reasoning is the same for both.

2.1.1 Elastic System Subjected to General Linear Boundary Conditions: Finite Domain

We will deal with (2.1) in $\mathcal{O} = (0,1)$ subject to boundary conditions

$$\begin{aligned} \alpha_0 w(0,t) + \beta_0 \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} &= u_b^0(t), \\ \alpha_1 w(1,t) + \beta_1 \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=1} &= u_b^1(t), \end{aligned} \qquad t \in [0,T]. \tag{2.5}$$

Here $\alpha_0 + \beta_0 > 0$, $\alpha_1 + \beta_1 > 0$ are constants: choosing them in a proper way, we may obtain Dirichlet, Neumann or mixed boundary conditions.

Our aim is the providing terminal steady state for the vibrations system in finite fixed time $T > 0$ choosing appropriate boundary or distributed controls. Suppose, that all the transmission conditions concerning (2.3), (2.5) and null terminal conditions (2.4) are satisfied:

$$\alpha_0 w_0(0) + \beta_0 \mathcal{D} w_0(0) = u_b^0(0), \quad \alpha_1 w_0(1) + \beta_1 \mathcal{D} w_0(1) = u_b^1(0),$$

$$\alpha_0 w_0^1(0) + \beta_0 \mathcal{D} w_0^1(0) = \mathcal{D} u_b^0(0), \quad \alpha_1 w_0^1(1) + \beta_1 \mathcal{D} w_0^1(1) = \mathcal{D} u_b^1(0),$$

$$u_b^0(T) = u_b^1(T) = \mathcal{D} u_b^0(T) = \mathcal{D} u_b^1(T) = 0.$$

Applying the operator $\mathcal{A}_{[0,T]}[\cdot]$ to (2.1), (2.5), we will arrive at

$$\frac{\partial}{\partial x}\left[N(x)\frac{\partial w_1(x,t)}{\partial x}\right]-\rho(x)\frac{\partial^2 w_1(x,t)}{\partial t^2}=f_1(x,t)+u_d(t)v(x)-W(x,t), \tag{2.6}$$

$$W(x,t)=\rho(x)\left[w_0(x)\delta'(t)+w_0^1(x)\delta(t)\right], \quad (x,t)\in(0,1)\times\mathbb{R},$$

$$\left.\begin{aligned}\alpha_0 w_1(0,t)+\beta_0\left.\frac{\partial w_1(x,t)}{\partial x}\right|_{x=0}&=u_{b1}^0(t),\\ \alpha_1 w_1(1,t)+\beta_1\left.\frac{\partial w_1(x,t)}{\partial x}\right|_{x=1}&=u_{b1}^1(t),\end{aligned}\right. \quad t\in\mathbb{R}. \tag{2.7}$$

It is quite obvious, that in order to provide null terminal conditions, the external perturbations f must vanish before T. Let supp $f_1(\cdot,t)\subseteq[0,\vartheta]$ with $0<\vartheta<T$. Then, in view of the obvious relation

$$\mathcal{A}_{[0,T]}\left[\mathcal{A}_{[0,\vartheta]}[\eta]\right]=\mathcal{A}_{[0,\vartheta]}[\eta], \quad t\in\mathbb{R},$$

we have $f_1(x,t)\equiv\mathcal{A}_{[0,T]}[f]=f(x,t)$ for all $x\in[0,1]$.

Applying distributional Fourier transform with respect to t to (2.6), (2.7) we will obtain

$$\begin{aligned}\frac{d}{dx}\left[N(x)\frac{d\overline{w}_1(x,\sigma)}{dx}\right]&+\sigma^2\rho(x)\overline{w}_1(x,\sigma)=\\ &=\overline{f}_1(x,\sigma)+\overline{u}_{d1}(\sigma)v(x)-\rho(x)\left[w_0^1(x)-i\sigma w_0(x)\right],\end{aligned} \tag{2.8}$$

$$(x,\sigma)\in(0,1)\times\mathbb{R},$$

$$\left.\begin{aligned}\alpha_0 \overline{w}_1(0,\sigma)+\beta_0\left.\frac{d\overline{w}_1(x,\sigma)}{dx}\right|_{x=0}&=\overline{u}_{b1}^0(\sigma),\\ \alpha_1 \overline{w}_1(1,\sigma)+\beta_1\left.\frac{d\overline{w}_1(x,\sigma)}{\partial x}\right|_{x=1}&=\overline{u}_{b1}^1(\sigma),\end{aligned}\right. \quad \sigma\in\mathbb{R}. \tag{2.9}$$

Thus, we arrive at boundary-value problem (2.8), (2.9), therefore we may proceed as before. Its general solution may be represented in the form

$$\overline{w}_1(x,\sigma)=h_1(\sigma)\exp\left[i\lambda(x,\sigma)\right]+h_2(\sigma)\exp\left[-i\lambda(x,\sigma)\right]+\Lambda(x,\sigma,\overline{u}_{d1}(\sigma)), \tag{2.10}$$

$$(x,\sigma)\in[0,1]\times\mathbb{R},$$

where $\lambda=\lambda(x,\sigma)$ is determined from the Riccati differential equation

$$\frac{\partial\nu}{\partial x}+\frac{1}{N(x)}\nu^2+\sigma^2\rho(x)=0, \quad (x,\sigma)\in[0,1]\times\mathbb{R}, \tag{2.11}$$

through the relation

$$i\lambda(x,\sigma)=\int_0^x\frac{\nu(\xi,\sigma)}{N(\xi)}d\xi, \tag{2.12}$$

and [23, 123]

$$\Lambda(x,\sigma,\overline{u}_{d1}(\sigma)) = \int_0^x \left[\overline{f}_1(\xi,\sigma) + \overline{u}_{d1}(\sigma)v(\xi) - \overline{W}(\xi,\sigma)\right] \mathcal{K}(x,\xi,\sigma)d\xi, \tag{2.13}$$

$$\mathcal{K}(x,\xi,\sigma) = \frac{\sin\left[\lambda(x,\sigma) - \lambda(\xi,\sigma)\right]}{\lambda'(\xi,\sigma)}.$$

The unknown coefficients $h_1(\sigma)$ and $h_2(\sigma)$ are determined from (2.9) as follows:

$$h_p(\sigma) = \frac{\Delta_p(\sigma)}{\Delta(\sigma)}, \quad p \in \{1;2\}, \tag{2.14}$$

where (see Section 1.3)

$$\Delta(\sigma) = \begin{vmatrix} [\alpha_0 + i\beta_0\lambda'(0,\sigma)]\exp[i\lambda(0,\sigma)] & [\alpha_0 - i\beta_0\lambda'(0,\sigma)]\exp[-i\lambda(0,\sigma)] \\ [\alpha_1 + i\beta_1\lambda'(1,\sigma)]\exp[i\lambda(1,\sigma)] & [\alpha_1 - i\beta_1\lambda'(1,\sigma)]\exp[-i\lambda(1,\sigma)] \end{vmatrix}, \tag{2.15}$$

$$\Delta_1(\sigma) = \begin{vmatrix} \overline{u}_{b1}^0(\sigma) & [\alpha_0 - i\beta_0\lambda'(0,\sigma)]\exp[-i\lambda(0,\sigma)] \\ \overline{u}_{b1}^{11}(\sigma) & [\alpha_1 - i\beta_1\lambda'(1,\sigma)]\exp[-i\lambda(1,\sigma)] \end{vmatrix}, \tag{2.16}$$

$$\Delta_2(\sigma) = \begin{vmatrix} [\alpha_0 + i\beta_0\lambda'(0,\sigma)]\exp[i\lambda(0,\sigma)] & \overline{u}_{b1}^0(\sigma) \\ [\alpha_1 + i\beta_1\lambda'(1,\sigma)]\exp[i\lambda(1,\sigma)] & \overline{u}_{b1}^{11}(\sigma) \end{vmatrix}, \tag{2.17}$$

$$\overline{u}_{b1}^{11}(\sigma) = \overline{u}_{b1}^1(\sigma) - \alpha_1\Lambda\left(1,\sigma,\overline{u}_{d1}(\sigma)\right) - \beta_1 \left.\frac{d\Lambda'\left(x,\sigma,\overline{u}_{d1}(\sigma)\right)}{dx}\right|_{x=1}.$$

Now, in terms of obtained results we may conclude.

Theorem 2.1 *Function u is admissible for (2.1)–(2.4) if and only if it is determined from the countable system of equalities*

$$\Delta_1(z) = 0, \tag{2.18}$$

as long as

$$\Delta(z) = 0, \quad z \in \mathbb{C}. \tag{2.19}$$

The proof of the theorem merely repeats the steps described in Section 1.3.

Particularly, if we seek boundary controls $u_{b1}^0 \in \mathcal{U}_2$, then the corresponding constants $\mathcal{M}_k = \mathcal{M}_{1k} + i\mathcal{M}_{2k}$ will have the form

$$\mathcal{M}_k = \overline{u}_{b1}^{11}(z_k)\frac{\alpha_0 - i\beta_0\lambda'(0,z_k)}{\alpha_1 - i\beta_1\lambda'(1,z_k)}\exp\left[i(\lambda(1,z_k) - \lambda(0,z_k))\right].$$

Seeking boundary controls $u_{b1}^1 \in \mathcal{U}_2$, we will arrive at

$$\mathcal{M}_k = \overline{u}_{b1}^0(z_k)\frac{\alpha_1 - i\beta_1\lambda'(1,z_k)}{\alpha_0 - i\beta_0\lambda'(0,z_k)}\exp\left[-i(\lambda(1,z_k) - \lambda(0,z_k))\right] +$$
$$+ \alpha_1\Lambda(1,z_k,\overline{u}_{d1}(z_k)) + \beta_1\Lambda'(1,z_k,\overline{u}_{d1}(z_k)).$$

Finally, seeking distributed control $u_{d1} \in \mathcal{U}_1$, we will obtain

$$\mathcal{M}_k = \left[\alpha_1\mathcal{R}\left[\overline{f}_1 - \overline{W}\right](z_k)\Big|_{x=1} + \beta_1\frac{\partial\mathcal{R}[\overline{f}_1 - \overline{W}](z_k)}{\partial x}\Bigg|_{x=1} - \overline{u}_{b1}^0(z_k)\frac{\alpha_1 - i\beta_1\lambda'(1,z_k)}{\alpha_0 - i\beta_0\lambda'(0,z_k)}\times\right.$$
$$\left.\times\exp\left[-i(\lambda(1,z_k) - \lambda(0,z_k))\right] + \overline{u}_{b1}^1(z_k)\right]\left[\alpha_1\mathcal{R}[v](z_k)\Big|_{x=1} + \beta_1\frac{\partial\mathcal{R}[v](z_k)}{\partial x}\Bigg|_{x=1}\right]^{-1},$$
$$\mathcal{R}[\eta](z_k) = \int_0^x \eta(\xi)\mathcal{K}(x,\xi,z_k)d\xi.$$

If one needs the controlled motion of the system, he has to apply Fourier inverse distributional transform to (2.10), taking into account (2.11)–(2.17):

$$w(x,t) = \mathcal{F}_t^{-1}[\overline{w}] =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[h_1(\sigma)\exp\left[i\lambda(x,\sigma)\right] + h_2(\sigma)\exp\left[-i\lambda(x,\sigma)\right] + \Lambda(x,\sigma)\right]\exp\left(-i\sigma t\right)d\sigma,$$
$$(x,t) \in [0,1]\times[0,T].$$

It is easy to check that

$$\mathrm{Re}\,\overline{w}(x,-\sigma) = \mathrm{Re}\,\overline{w}(x,\sigma), \quad \mathrm{Im}\,\overline{w}(x,-\sigma) = -\mathrm{Im}\,\overline{w}(x,\sigma),$$

therefore, according to Corollary 1.1, its Fourier inverse transform is a real valued function:

$$w(x,t) = \frac{1}{\pi}\int_0^{\infty}\left[\mathrm{Re}\,\overline{w}(x,\sigma)\cos(\sigma t) + \mathrm{Im}\,\overline{w}(x,\sigma)\sin(\sigma t)\right]d\sigma =$$
$$= \mathcal{F}_c^{-1}\left[\mathrm{Re}\,\overline{w}\right](t) + \mathcal{F}_s^{-1}\left[\mathrm{Im}\,\overline{w}\right](t),\ (x,t) \in [0,1]\times[0,T].$$

As the extension of the integrand in the whole complex plane has no singularities, the integral is well defined.

Let us illustrate the procedure of derivation of (1.14) in this case. Let

$$f(x,t) = P\delta(t - t_0)\delta(x - x_0), \quad (x,t) \in [0,1]\times[0,T],$$

which corresponds to constant load P acting at $t_0 \in [0,\vartheta]$ and concentrated at isolated point $x_0 \in (0,1)$. Then

$$\Lambda(x,\sigma,\overline{u}_{d1}(\sigma)) = P\exp\left[i\sigma t_0\right]\mathcal{K}(x,x_0,\sigma) + \overline{u}_{d1}(\sigma)\mathcal{R}[v](\sigma) - \mathcal{R}\left[\rho w_0^1\right](\sigma) + i\sigma\mathcal{R}[w_0](\sigma).$$

Suppose, that the displacements of the system are fixed at both ends, i.e. $u_b^0(t) = u_b^1(t) = 0$, $\alpha_0 = \alpha_1 = 1$ and $\beta_0 = \beta_1 = 0$, and we are required to determine admissible distributed controls $u_d \in \mathcal{U}_1$.

In order to proceed, we need to solve special Riccati equation (2.11) and calculate λ from (2.12). It is well-known in general it is not integrated by quadrature, nevertheless this can be done in some particular cases [123]. Let, for simplicity

$$N(x) = N_0 \exp\left[\gamma_1 x\right], \quad \rho(x) = \rho_0 \exp\left[-\gamma_2 x\right],$$

with natural restriction from above N_0, $\rho_0 > 0$, γ_1, γ_2 are real parameters with $\gamma_1 \cdot \gamma_2 > 0$.

Then, (2.11) admits exact solution [123]:

$$\nu(x,\sigma) = \frac{\gamma_1}{N_0} \frac{\eta'_\zeta(\zeta,\sigma)}{\eta(\zeta,\sigma)}, \quad \zeta(x) = \exp\left[-\gamma_1 x\right],$$

$$\eta(\zeta,\sigma) = \zeta^{1/2}\left[C_1 J_{\frac{1}{2q}}\left(\frac{|\sigma|}{|\gamma_1| q}\sqrt{\rho_0 N_0}\zeta^q\right) + C_2 Y_{\frac{1}{2q}}\left(\frac{|\sigma|}{|\gamma_1| q}\sqrt{\rho_0 N_0}\zeta^q\right)\right], \quad q = 1 + \frac{\gamma_2 - \gamma_1}{2}:$$

It provides an oscillating solution of (2.8) for some values of the parameters γ_p, $p \in \{1;2\}$. Indeed, in order to have an oscillating solution, the coefficients of (2.8) must satisfy the inequality [123]

$$2\sigma^2 \frac{\rho}{N} - \left[\frac{N'}{N}\right]' - \frac{1}{2}\left[\frac{N'}{N}\right]^2 > 0, \quad (x,\sigma) \in [0,1] \times \mathbb{R}.$$

In this particular case we would have

$$4\sigma^2 \frac{\rho_0}{N_0} \exp\left[-(\gamma_1 + \gamma_2)x\right] - \gamma_1^2 > 0,$$

which provides a whole range of parameters γ_p, $p \in \{1;2\}$.

Restricting consideration to the case $\gamma_1 = \gamma_2$ (softening factor), i.e. $q = 1$, from (2.12) we have

$$i\lambda(x,\sigma) = -\frac{1}{N_0^2} \ln\left|\frac{\eta\left(\zeta(x),\sigma\right)}{\eta\left(\zeta(0),\sigma\right)}\right|,$$

and, in particular, if $C_1 = 1$, $C_2 = 0$–

$$\eta\left(\zeta(x),\sigma\right) = \zeta^{1/2} J_{\frac{1}{2}}\left(\frac{|\sigma|}{|\gamma_1|}\sqrt{\rho_0 N_0}\zeta\right) = \sqrt{\frac{2|\gamma_1|}{\pi|\sigma|\sqrt{\rho_0 N_0}}} \sin\left(\frac{|\sigma|}{|\gamma_1|}\sqrt{\rho_0 N_0}\zeta\right).$$

In this case

$$\Delta(\sigma) = \left[\frac{\sin\left(\frac{|\sigma|}{|\gamma_1|}\sqrt{\rho_0 N_0}\exp\left[-\gamma_1\right]\right)}{\sin\left(\frac{|\sigma|}{|\gamma_1|}\sqrt{\rho_0 N_0}\right)}\right]^{\frac{1}{N_0^2}} - \left[\frac{\sin\left(\frac{|\sigma|}{|\gamma_1|}\sqrt{\rho_0 N_0}\exp\left[-\gamma_1\right]\right)}{\sin\left(\frac{|\sigma|}{|\gamma_1|}\sqrt{\rho_0 N_0}\right)}\right]^{-\frac{1}{N_0^2}},$$

$$\Delta_1(\sigma) = \Lambda(1, \sigma, \overline{u}_{d1}(\sigma)).$$

The general solution of corresponding equation (2.19) for $\alpha \in \mathbb{R}$ seems to be unreachable. Though, for $\gamma_1 = 1$ ($\alpha = 1$) we obtain

$$z_k^{\pm} = \sigma_k^{\pm} = \frac{2\pi k}{1 \pm \exp[-1]}, \quad k \in \mathbb{N},$$

i.e. they all are real: $\varsigma_k = 0$. We omit the case $k = 0$, since then we would have an identity.

In the same way when $\alpha = 0.5, 1.5, 2$ etc. we obtain

$$z_k^{\pm} = \frac{2\pi k}{1 \pm \exp[-1]},$$

correspondingly. Both z_k^+ and z_k^- satisfy asymptotic expansion of eigenvalues of differential operator (2.8), which according to [22] is

$$z_k \sim 2\pi k \left[\int_0^1 \frac{1}{N(x)} dx\right]^{-1}, \quad k \to \infty.$$

For k large enough $z_k^{\pm} \to \pi k$, which is well-known [22]. Thus, the system (1.14) must be considered for $\sigma_k = z_k^{\pm}$ and $\varsigma_k = 0$.

Substituting this roots into extension (2.18) of determinant $\Delta_1(\sigma)$, and equating it to zero, we will have

$$\Lambda\left(1, \frac{2\pi k}{1 \pm \exp[-\gamma_1]}, \overline{u}_{1d}\left(\frac{2\pi k}{1 \pm \exp[-\gamma_1]}\right)\right) = 0, \quad k \in \mathbb{N}.$$

After simple algebraic transformations with respect to Fourier transform of unknown control function we will obtain

$$\overline{u}_{d1}\left(\frac{2\pi k}{1 \pm \exp[-\gamma_1]}\right) = \mathcal{M}_k, \quad k \in \mathbb{N}, \tag{2.20}$$

where

$$\mathcal{M}_k = \frac{P \exp[i\sigma_k t_0] \mathcal{K}(1, x_0, \sigma_k) - [\mathcal{R}[\rho w_0^1](z_k) - i\sigma\mathcal{R}[w_0](z_k)]\big|_{x=1}}{\mathcal{R}[v](z_k)\big|_{x=1}}.$$

In particular case, where $v(x) = \delta(x - x_1)$, $x_1 \in (0, 1)$, which corresponds to concentrated control, we have

$$\mathcal{R}[v](z_k)\big|_{x=1} = \mathcal{K}(1, x_1, z_k).$$

When control "reaches" one of the ends, i.e. $x_1 \to 0+$ or $x_1 \to 1-$,

$$\lim_{x_1 \to 0+} \mathcal{K}(1, x_1, z_k) = \lim_{x_1 \to 1-} \mathcal{K}(1, x_1, z_k) = 0, \quad k \in \mathbb{N},$$

i.e. admissible controls cannot be determined from (2.20), which should be expected [68–70]. On the other hand side, when $x_0 \to 0+$ or $x_0 \to 1-$, the coefficients $\mathcal{M}_k$ become independent from external perturbations, which is also quite expected.

Separating the real and the imaginary parts of (2.20), we will obtain trigonometric infinite-dimensional moments problem for admissible controls:

$$\int_0^T u_d(t)\cos(\sigma_k t)dt = \mathcal{M}_{1k}, \quad \int_0^T u_d(t)\sin(\sigma_k t)dt = \mathcal{M}_{2k}, \quad u \in \mathcal{U}, \quad k \in \mathbb{N}.$$

Then, in order to find the L^p–optimal control, $1 \le p \le \infty$, we will be able to apply the procedure described in Subsection 1.3.1.

2.1.2 Boundary Controllability of Non-Homogeneous Elastic System: Semi-Infinite Domain

Now we will deal with (2.1) in $\mathcal{O} = \mathbb{R}^+$ subject to Dirichlet boundary condition

$$w(0,t) = u_b(t), \quad t \in [0,T], \tag{2.21}$$

and the condition at infinity

$$\lim_{x\to+\infty} w(x,t) = m(t), \quad t \in [0,T]. \tag{2.22}$$

The last condition is related with practical importance of one-dimensional waves propagating in non-homogeneous media and having the given behaviour far from the wave source. We additionally suppose that $m \in L^2[0,T]$.

In this case the control is implemented by the boundary input $u \in \mathcal{U}_1$ and we are required to find L^1–optimal among those admissible controls. The existence of controls providing the null-controllability for the system seems to be non-realistic, therefore we require the providing non-zero conditions at $t = T$. The transmission conditions between (2.21), (2.22) and (2.3), (2.4) are supposed to be satisfied:

$$u(0) = w_0(0), \quad m(0) = \lim_{x\to+\infty} w_0(x), \quad m(T) = \lim_{x\to+\infty} w_T(x),$$

$$\mathcal{D}u(0) = w_0^1(0), \quad \mathcal{D}u(T) = w_T^1(0), \quad \mathcal{D}m(0) = \lim_{x\to+\infty} w_0^1(x), \quad \mathcal{D}m(T) = \lim_{x\to+\infty} w_T^1(x).$$

Application of the operator $\mathcal{A}_{[0,T]}[\cdot]$ to governing system will lead us to

$$\frac{\partial}{\partial x}\left[N(x)\frac{\partial w_1(x,t)}{\partial x}\right]-\rho(x)\frac{\partial^2 w_1(x,t)}{\partial t^2}=f_1(x,t)-W(x,t),\quad (x,t)\in\mathbb{R}^+\times\mathbb{R}, \tag{2.23}$$

$$W(x,t)=\rho(x)[w_0(x)\delta'(t)+w_0^1(x)\delta(t)-w_T(x)\delta'(t-T)-w_T^1(x)\delta(t-T)],$$

$$w_1(0,t)=u_{b1}(t),\quad \lim_{x\to+\infty}w_1(x,t)=m_1(t),\quad t\in\mathbb{R}. \tag{2.24}$$

Note, that a wave equation, similar to (2.23) but with constant coefficients is obtained in [44] in a contact problem about longitudinal vibrations of two semi-infinite elastic homogeneous stringers attached to an elastic half-plane caused by harmonic perturbations applied on ends of the stringers.

Repeating the same procedure as above in this chapter we will arrive at:

Theorem 2.2 *Admissible control $u_1\in\mathcal{U}_1$ resolves the boundary control problem (2.23), (2.24) if and only if it satisfies the countable system of equalities (1.14) with*

$$\mathcal{M}_k=\mathcal{M}_{1k}+i\mathcal{M}_{2k}=\overline{m}_1(z_k)+(-1)^k\Lambda(0,z_k),$$

$$\Lambda(x,\sigma)=\int_x^\infty\left[\overline{f}_1(\xi,\sigma)-\overline{W}(\xi,\sigma)\right]\mathcal{K}(x,\xi,\sigma)d\xi,$$

as long as

$$\lambda(0,z_k)=\pi k,\quad k\in\mathbb{N}, \tag{2.25}$$

in terms of [1]

$$i\lambda(x,\sigma)=\int_x^\infty\frac{\nu(\xi,\sigma)}{N(\xi)}d\xi,\qquad (x,\sigma)\in\mathbb{R}^+\times\mathbb{R}. \tag{2.26}$$

Here $\nu(\xi,\sigma)$ is still the solution of the Riccati equation (2.11) in $\mathbb{R}^+$.

Let in the particular case of the non-homogeneity considered in the last subsection $\gamma_1=\gamma_2$, $C_1=0$, $C_2=1$. Then, from (2.26) when $\gamma_1=\gamma_2=1$ we have

$$i\lambda(x,\sigma)=\ln\left|\cos\left[|\sigma|\exp[-x]\right]\right|,\quad (x,\sigma)\in\mathbb{R}^+\times\mathbb{R},$$

and from (2.25)–

$$z_k=\sigma_k=2\pi k,\quad k\in\mathbb{N},$$

[1] Arising improper integral must be understood in the sense of its Cauchy principal value.

i.e. all roots of equation (2.25) are real ($\varsigma_k = 0$), which leads in system (1.14) to trigonometric problem of moments.

Let for simplicity, that in the above mentioned case of the non-homogeneity

$$f(x,t) \equiv 0,\ m(t) \equiv 0,\ w_0(x) = x^2 \exp\left[-0.25x\right],\ w_0^1(x) = x \exp\left[-0.25x^2\right],$$

and we are aimed to ensure the steady state for the system at

$$T = 2\int_0^\infty \frac{dx}{N(x)} = \frac{2}{N_0}.$$

As it is known [22], it is the minimal time needed to damp vibrating non-homogeneous system. It is an expected result: the time required to damp the vibrations decreases with increase of the softening factor of the string. Then

$$\mathcal{M}_{1k} = \frac{1}{2\pi k}\int_0^\infty \xi \exp\left[-\gamma_1\xi - 0.25\xi^2\right] \mathcal{K}(0,\xi,2\pi k)d\xi,$$

$$\mathcal{M}_{2k} = -\int_0^\infty \xi^2 \exp\left[-(\gamma_1 + 0.25)\xi\right] \mathcal{K}(0,\xi,2\pi k)d\xi,$$

$$\mathcal{K}(0,\xi,2\pi k) = \frac{N_0}{2}\exp\left[\gamma_1\xi\right]\cot\left[2\pi k\exp\left[-\gamma_1\xi\right]\right]\frac{1-\cos^{\frac{2}{N_0^2}}\left[2\pi k\exp\left[-\gamma_1\xi\right]\right]}{\left|\cos^{\frac{1}{N_0^2}}\left[2\pi k\exp\left[-\gamma_1\xi\right]\right]\right|}.$$

Particularly, when $\gamma_1 = 1$

$$\mathcal{K}(0,\xi,\pi k) = \frac{N_0}{2}\exp\left[\xi\right]\sin\left[2\pi k\exp\left[-\xi\right]\right]\operatorname{sign}\left[\cos\left[2\pi k\exp\left[-\xi\right]\right]\right].$$

Nevertheless, even in this simple case integrals above seem to be explicitly non-calculable. Numerical calculations are done instead, and discrete graphics of the coefficients $\mathcal{M}_{1k}$ and $\mathcal{M}_{2k}$ are plotted in Figures 2.1, 2.2. It is easy to see that those coefficients decrease with k very fast.

Suppose now that

$$w_0(x) = H\delta(x-\xi_0),\ w_0^1(x) = v_0\delta(x-\xi_0),\quad 0 < H, v_0 = const,$$

which corresponds to a system the $x = \xi_0$ point of which is initially shifted to small height H at and left with constant velocity v_0. Then

$$\mathcal{M}_{1k} = \frac{v_0}{2\pi k}\exp\left[-\xi_0\right]\mathcal{K}(0,\xi_0,\pi k),\quad \mathcal{M}_{2k} = -\frac{H}{2}\exp\left[-\xi_0\right]\mathcal{K}(0,\xi_0,\pi k).$$

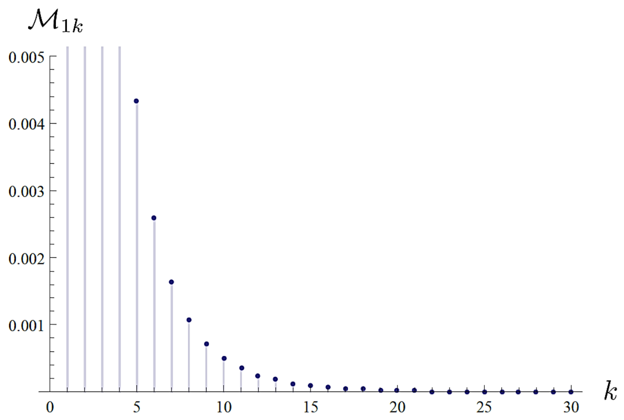

Figure 2.1: Discrete plot of $\mathcal{M}_{1k}$ for $\gamma_1 = 1$.

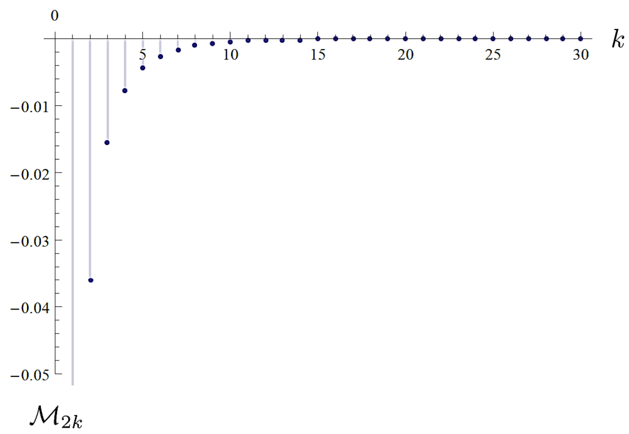

Figure 2.2: Discrete plot of $\mathcal{M}_{2k}$ for $\gamma_1 = 1$.

The L^1–optimal solution in this particular case by virtue of (1.17) takes the form

$$u^o(t) = \sum_{j=1}^{J} u_j^o \delta(t - t_j^o), \quad t \in [0, 2] .$$

under the following system of restrictions:

$$\left.\begin{aligned}
\sum_{j=1}^{J} u_j^o \cos(\pi k t_j^o) &= \frac{v_0}{2(\pi k)^2} \sin[\pi k \exp[-\xi_0]] \mathrm{sign}\left[\cos\left[\pi k \exp\left[-\xi_0\right]\right]\right], \\
\sum_{j=1}^{J} u_j^o \sin(\pi k t_j^o) &= -\frac{H}{2\pi k} \sin[\pi k \exp[-\xi_0]] \mathrm{sign}\left[\cos[\pi k \exp[-\xi_0]]\right],
\end{aligned}\right. \quad k \in \{1; N\},$$

$$\text{sign } u_j^o = \text{sign } h^o(t_j^o), \quad j \in \{1; J\},$$

where

$$h^o(t) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{v_0^2 + (\pi k H)^2} \left[v_0 \cos(\pi k t) + \pi k H \sin(\pi k t)\right], \quad t \in [0, 2] .$$

Solution of this system may be obtained by numerical techniques of non-linear programming [15].

Results of calculations done for $\xi_0 = 1$ and different values of v_0 and H are presented in Table 2.1. Though the parameter ξ_0 is fixed, the behaviour of optimal control against it may give a rise to a new investigation. For example, when $v_0 = 1$ and $H = 0.1$ the resolving control u^o takes the form

$$u^o(t) = 9.05\delta(t - 0.2) - 1.32\delta(t - 1.) - 1.28\delta(t - 1.79) - 6.45\delta(t - 3.37), \quad t \in [0, 2] \,.$$

It is plotted in Figure 2.3.

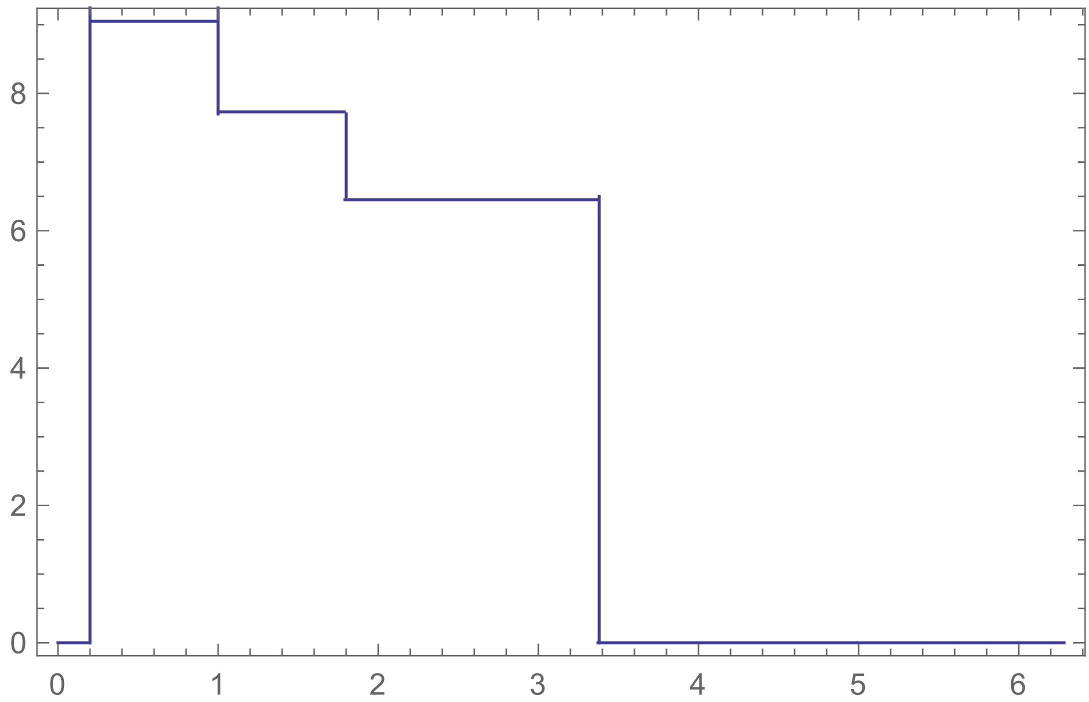

Figure 2.3: The optimal bang-bang control u against time.

2.1.3 Boundary and Distributed Control of Finite Elastic String with Constant Delay

Using the fact that the Fourier integral transform provides a powerful technique to deal with systems containing delay (after-effect or prehistory), we have considered also boundary and distributed control problems in this case. Let (2.1) be subject to Dirichlet boundary conditions

$$w(0, t) \equiv 0, \; w(1, t) = u_b(t - \tau), \; t \in [0, T], \tag{2.27}$$

v_0	H	u_j^o	t_j^o
1	0.01	$u_1^o = 8.75$, $u_2^o = u_3^o = -1.25$, $u_4^o = -6.25$	$t_1^o = 0.19$, $t_2^o = 1.$, $t_3^o = 1.8$, $t_4^o = 3.37$
1	0.1	$u_1^o = 9.05$, $u_2^o = -1.32$, $u_3^o = -1.28$, $u_4^o = -6.45$	$t_1^o = 0.2$, $t_2^o = 1.$, $t_3^o = 1.79$, $t_4^o = 3.37$
10	1	$u_1^o = -1.53$, $u_2^o = 1.21$, $u_3^o = 1.38$, $u_4^o = -1.4$	$t_1^o = 0.16$, $t_2^o = 0.94$, $t_3^o = 2.13$, $t_4^o = 3.43$
20	1	$u_1^o = 1.49$, $u_2^o = -5.92$, $u_3^o = -2.06$, $u_4^o = 5.54$	$t_1^o = 0.23$, $t_2^o = 1.$, $t_3^o = 1.765$, $t_4^o = 3.03$

Table 2.1: Control impacts and moments for $N = 30, J = 4$.

when $u_d \equiv 0$, and

$$u_d(t) = u_d(t - \tau) \tag{2.28}$$

when mixed boundary conditions are given:

$$w(0,t) = 0, \quad \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=1} = u_b(t), \quad t \in [0,T]. \tag{2.29}$$

In both cases $0 < 2\tau < T$ is a constant delay. The control process is carried out by the boundary input u_b and distributed impact u_d, respectively. It means that we must put $\alpha_0 = \alpha_1 = 1$, $\beta_0 = \beta_1 = 0$, $u_b^0 \equiv 0$ in (2.5) in the first case and $\alpha_0 = \beta_1 = 1$, $\beta_0 = \alpha_1 = 0$, $u_b^0(t) \equiv 0$ in the second case. Then, repeating the procedure described in Subsection 2.1.1 we will obtain

$$\Delta(\sigma) = -2i\sin[\lambda(1,\sigma) - \lambda(0,\sigma)], \quad \Delta_1(\sigma) = [\Lambda(1,\sigma,0) - \exp[i\sigma\tau]\,\overline{u}_{b1}(\sigma)]\exp[-i\lambda(0,\sigma)],$$

in the first case, and

$$\Delta(\sigma) = -2\lambda'(1,\sigma)\cos[\lambda(1,\sigma) - \lambda(0,\sigma)], \quad \Delta_1(\sigma) = [\Lambda'(1,\sigma,\overline{u}_{d1}(\sigma)) - \overline{u}_{b1}(\sigma)]\exp[-i\lambda(0,\sigma)],$$

in the second case.

The initial data (2.3) are supposed to be the same. Our aim is still to find restrictions on admissible control functions $u \in \mathcal{U}$ providing conditions (2.4) and represent them explicitly.

Solution of boundary control problem gives the following theorem.

Theorem 2.3 *Function $u_b \in \mathcal{U}_2$ resolves the boundary control problem for (2.1), (2.27) if and only if its Fourier generalized transform satisfies countable system of equalities*

$$\overline{u}_{b1}(z_k) = \Lambda(1, z_k, 0) \exp\left[-iz_k\tau\right], \; k \in \mathbb{N}, \tag{2.30}$$

where points z_k are determined from characteristic equations

$$\lambda(1, z_k) - \lambda(0, z_k) = \pi k, \quad k \in \mathbb{N}.$$

Solution of distributed control problem gives the following theorem.

Theorem 2.4 *Function $u_d \in \mathcal{U}_1$ resolves the distributed control problem for (2.1), (2.28), (2.29) if and only if its Fourier generalized transform satisfies countable system of equalities*

$$\overline{u}_d(z_k) = \left[\frac{\overline{u}_b^1(z_k)}{\lambda'(1, z_k)} + \left.\frac{\partial \mathcal{R}\left[\overline{W}\right](z_k)}{\partial x}\right|_{x=1}\right] \frac{\exp\left[-iz_k\tau\right]}{\left.\frac{\partial \mathcal{R}[v](z_k)}{\partial x}\right|_{x=1}}, \; k \in \mathbb{N}, \tag{2.31}$$

where points z_k are determined from characteristic equations

$$\lambda(1, z_k) - \lambda(0, z_k) = \frac{2k+1}{2}\pi, \quad k \in \mathbb{N}.$$

Separating the real and the imaginary parts in (2.30) and (2.31) we will obtain infinite system of problem of moments like (1.14). The explicit solution may be obtained by technique described in Subsection 2.1.2.

2.1.4 Some Other Types of Non-Homogeneities Providing Explicit Solution

There is a wide class of coefficients $N(x)$ and $\rho(x)$ providing explicit solution. As a matter of fact, Riccati equation (2.11) admits analytical solution for power, trigonometric, inverse trigonometric, hyperbolic and logarithmic coefficients [123].

Let[2]

$$N(x) = x^{-\gamma}, \quad \rho(x) = x^{-\gamma-2}, \; \gamma \in \mathbb{R}.$$

[2]Naturally, in all cases the parameter γ is taken in sense that $N \geq 0$ and $\rho > 0$ for all x in considered domain.

Then, the general solution of (2.11) will be

$$|\sigma| \ln x = \int \frac{d\eta}{\eta^2 + \beta\eta + 1} + C,$$

where C is a constant,

$$\eta = \frac{1}{|\sigma|} x^{\gamma+1} y, \quad \beta = \frac{\gamma+1}{|\sigma|}.$$

Interesting examples of non-homogeneity allowing explicit solution as well is

$$N(x) = a + b\sin(\gamma x), \quad \rho(x) = \rho_0 N^{-1}(x),$$

for $a \pm b > 0$, $\rho_0 > 0$, corresponding to so-called weak non-homogeneity. Then the particular solution of (2.11) is $\nu_0 = -\sigma\rho_0$, therefore its general solution will be

$$\nu(x, \sigma) = -\sigma\rho_0 + \zeta(x) \exp\left[-2\sigma\rho_0 \int_0^x \frac{d\xi}{N(\xi)}\right],$$

in which

$$\beta(x) = \left[c - \int_0^x \exp\left[-2\sigma\rho_0 \int_0^\xi \frac{d\chi}{N(\chi)}\right] d\xi\right]^{-1}.$$

The general solution of (2.11) has the same form also for $N(x) = a + b\cos(\gamma x)$, $N(x) = a + b\sinh(\gamma x)$ $N(x) = a + b\cosh(\gamma x)$, $N(x) = a + b\tanh(\gamma x)$, $N(x) = a + b\coth(\gamma x)$ with corresponding expression for $\rho(x) = \rho_0 N^{-1}(x)$.

Suppose now that

$$N(x) = \frac{N_0}{\sin(\gamma x)}, \quad \rho(x) = \rho_0 \sin(2\gamma x), \quad \gamma > 0.$$

Then

$$\nu(x, \sigma) = -\frac{\eta'_\zeta(\zeta, \sigma)}{\eta(\zeta, \sigma)}, \quad \zeta(x) = \cos(\gamma x),$$

in which

$$\eta(\zeta, \sigma) = \zeta^{\frac{1}{2}} \left[C_1 J_{\frac{1}{3}}\left(\frac{2\sqrt{2}}{3}\left|\frac{\sigma}{\lambda}\right| \sqrt{N_0\rho_0}\zeta^{\frac{3}{2}}\right) + C_2 Y_{\frac{1}{3}}\left(\frac{2\sqrt{2}}{3}\left|\frac{\sigma}{\lambda}\right| \sqrt{N_0\rho_0}\zeta^{\frac{3}{2}}\right)\right].$$

The general solution of (2.11) has the same form also for

$$N(x) = \frac{N_0}{\cos(\gamma x)}, \quad \rho(x) = \rho_0 \sin(2\gamma x),$$

with $\zeta(x) = \sin(\gamma x)$.

Let

$$N(x) = N_0 \frac{x}{\arcsin x}, \quad \rho(x) = \rho_0 \frac{\arcsin x}{x},$$

with $N_0, \rho_0 > 0$. Then the general solution of (2.11) reads as

$$\nu(x, \sigma) = \beta \tan\left(C + \beta\zeta[x]\right),$$

in which

$$\beta = |\sigma|\sqrt{\frac{\rho_0}{N_0}}, \; \zeta\left[\frac{\arcsin x}{x}\right] = \int \frac{\arcsin x}{x} dx,$$

and C is unknown constant. The same form with $\zeta\left[\arccos x\right]$, $\zeta\left[\arctan x\right]$ and $\zeta\left[\mathrm{arccot}\ x\right]$ has the solution of (2.11) for

$$N(x) = N_0 \frac{x}{\arccos x}, \quad \rho(x) = \rho_0 \frac{\arccos x}{x},$$

$$N(x) = N_0 \frac{x}{\arctan x}, \quad \rho(x) = \rho_0 \frac{\arctan x}{x},$$

and

$$N(x) = N_0 \frac{x}{\mathrm{arccot}\ x}, \quad \rho(x) = \rho_0 \frac{\mathrm{arccot}\ x}{x},$$

respectively.

Riccati equation (2.11) can be integrated by quadrature also in the case of some arbitrary function ζ describing either N and ρ:

$$\nu'_x + N_0\zeta(x)\nu^2 + \rho_0\sigma^2\zeta(x) = 0.$$

Naturally, in order to keep the physical treatment of the problem we have to consider only the case $\zeta > 0$. Then its particular solution will, obviously, be

$$\nu_0 = i|\sigma|\sqrt{\frac{\rho_0}{N_0}}.$$

Therefore, its general solution can be written as follows

$$\nu(x, \sigma) = \nu_0 + \beta(x) \exp\left[2\nu_0 N_0 \int \zeta(x) dx\right],$$

in which

$$\beta(x) = \left[C - N_0 \int \zeta(x) \exp\left[2\nu_0 N_0 \int \zeta(\xi) d\xi\right] dx\right]^{-1}.$$

2.2 Optimal Wave Processing and Transmission in Dispersive Media

Many real processes very often are roughly modelled by ordinary or partial differential equations. However, local character of differential equations does not allow us to take into account such real-world phenomena, as, for instance, processes with memory (prehistory). Introduction of integral terms into differential equation, thereby transforming it into that of integro-differential type, is of help in some cases. As a common one-dimensional example of such equations in finite interval the following one may serve:

$$\mathcal{D}_1[w] + \int_a^b \mathcal{K}(x, s, t)\mathcal{D}_2[w]ds = F(x, t), \tag{2.32}$$

where $w = w(x, t)$ is the unknown function, $\mathcal{K} : [a, b] \times [a, b] \times \mathrm{T} \to \mathbb{R}$ is a given function, called kernel of the equation, T is finite or infinite interval, $\mathcal{D}_1[\cdot]$ and $\mathcal{D}_2[\cdot]$ are differential operators, acting on unknown function, and $F : [a, b] \times \mathrm{T} \to \mathbb{R}$ is the given right hand side, satisfying certain constraints. If operators $\mathcal{D}_1[\cdot]$ and $\mathcal{D}_2[\cdot]$ are linear, contains ordinary or partial derivatives, then equation (2.32) is called linear, ordinary or partial, respectively. Furthermore, equation (2.32) is called symmetric, if its kernel is symmetric: $\mathcal{K}(x, s, \cdot) = \mathcal{K}(s, x, \cdot)$, and difference or convolution type, if $\mathcal{K}(x, s, \cdot) = \mathcal{K}(x - s, \cdot)$.

The monograph [3] is devoted to thorough investigation and classification of integro-differential operators. Main areas of integro-differential equations application are described, and numerous problems mathematically formulated by those equations, particularly in theories of elasticity and viscoelasticity, continuum mechanics, contact interactions mechanics, growing body mechanics and fracture mechanics are considered ibid. In [105] the description of some financial processes by partial integro-differential equations are investigated.

Accounting of such irreversible processes of wave energy transfer to medium particle as dispersion and dissipation, is accompanied with introduction in ordinary wave equation of some additional linear terms acting on unknown function, the form of which depends only on physical mechanism of interaction between wave and medium. In general, the description of wave propagation in a homogeneous isotropic medium with dissipation or dispersion properties occupied domain $\mathcal{O} \subset \mathbb{R}^3$ in three-dimensional formulation is mathematically equivalent to solution of the following wave equation:

$$\Delta w = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} + \mathcal{L}[w], \quad (x, y, z) \in \mathcal{O}, \; t > 0, \tag{2.33}$$

where $w = w(x, y, z, t)$ is the state function, and $\mathcal{L}[w]$ is a linear operator. Particularly, in sound wave propagation problem in a dissipative medium [117]

$$\mathcal{L}[w] \equiv \alpha \frac{\partial \Delta w}{\partial t},$$

where α characterizes the dissipative nature of the medium.

In propagation problem of electromagnetic wave in a dispersive medium [117]

$$\mathcal{L}[w] \equiv \beta \frac{\partial^2}{\partial t^2} \int_0^\infty \mathcal{K}(\tau) w(x, y, z, t - \tau) d\tau, \tag{2.34}$$

where $\mathcal{K}$, is a bounded function, characterizing the medium dielectric susceptibility, β characterizes the dispersive nature of the medium. Typical examples of electromagnetic waves include radio waves, TV signals, radar beams, and light rays. Besides, it arises in modelling of many other phenomena in deformable body mechanics (electro-elasticity, growth mechanics, statistical mechanics, etc.), that is why the investigation of control problems for equations with such terms has also practical importance.

So, we begin our investigation with non-homogeneous, one-dimensional analogue of equation (2.33) in finite time interval (for simplicity, symmetric: it can always be done by linear transformation of variable t) when (2.34) is taken into account:

$$\frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial t^2} + \epsilon \frac{\partial^2}{\partial t^2} \int_{-\vartheta}^{\vartheta} \mathcal{K}(\tau) w(x, t - \tau) d\tau + u_d(t) v(x), \tag{2.35}$$

$$(x, t) \in \mathcal{O} := (0, 1) \times (-\vartheta, \vartheta),$$

subjected to boundary conditions

$$w(0, t) = u_b^0(t), \; w(1, t) = u_b^1(t), \; t \in [-\vartheta, \vartheta], \tag{2.36}$$

(all quantities and variables are supposed to be dimensionless). ϵ is a positive constant.

From mathematics point of view, control process in boundary-value problem (2.35), (2.36) may be carried out either by boundary function u_b^0 (input signal), or by function u_d (signal filter located in the medium).

The state function is given at $t = -\vartheta$:

$$w(x, -\vartheta) = w_{-\vartheta}(x), \ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=-\vartheta} = w^1_{-\vartheta}(x), \ x \in [0,1]. \tag{2.37}$$

In the framework of accepted physical interpretation, (2.35), (2.36), in particular, describes the transfer of one-dimensional electromagnetic signal (impulse) on a finite distance in dispersive medium, at that u_b^0 and u_b^1 are input and output signals, respectively. It covers also other phenomena in electro- and magneto-elasticity. Unlike traditional statement of optimal control problems for distributed parameter systems [22], demanding to ensure given state at $t = \vartheta$

$$w(x, \vartheta) = w_{\vartheta}(x), \ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=\vartheta} = w^1_{\vartheta}(x), \ x \in [0,1], \tag{2.38}$$

we may demand to ensure one boundary condition by appropriate choice of the other one when functions (2.37), (2.38) are given.

Our main purpose is to construct an efficient for numerical reasons algorithm of the following two problems solution.

The boundary control problem requires determination of admissible control functions $u_b^0 \in \mathcal{U}_2$ ensuring given output signal u_b^1 when data (2.37) and (2.38) are given and derivation of necessary and sufficient conditions for exact controllability.

The distributed control problem requires determination of admissible control functions $u_d \in \mathcal{U}_1$ ensuring given output signal u_b^1 when input signal u_b^0 and states (2.37), (2.38) are given, as well as find necessary and sufficient conditions for exact controllability.

Applying the operator $\mathcal{A}_{[-\vartheta,\vartheta]}[\cdot]$ to (2.35), (2.36), in terms of $\mathfrak{D}$ we shall derive

$$\frac{\partial^2 w_1(x,t)}{\partial x^2} = \frac{\partial^2 w_1(x,t)}{\partial t^2} + \epsilon \frac{\partial^2}{\partial t^2}[\mathcal{K}_1 * w_1] + u_{d1}(t) v(x) + W(x,t), \ (x,t) \in (0,1) \times \mathbb{R}, \tag{2.39}$$

$$w_1(0,t) = u^0_{b1}(t), \ w_1(1,t) = u^1_{b1}(t), \ t \in \mathbb{R}. \tag{2.40}$$

$$W(x,t) = W_-(x,t) - W_+(x,t),$$

$$W_\pm(x,t) = [\delta(t \pm \vartheta) + \epsilon \mathcal{K}_1(t \pm \vartheta)] w^1_{\pm\vartheta}(x) + \frac{d}{dt}[\delta(t \pm \vartheta) - \epsilon \mathcal{K}_1(t \pm \vartheta)] w_{\pm\vartheta}(x).$$

It should be noted, that the function $W(x,t)$ containing delay and advance of argument t

is identically zero when $t \pm \vartheta \notin [-\vartheta, \vartheta]$, at that

$$W(x,t) = \begin{cases} -W_+(x,t), \quad t \in [-\vartheta, 0); \\ W_0(x), \quad t = 0; \\ W_-(x,t), \quad t \in (0, \vartheta], \end{cases} \qquad x \in [0,1],$$

where

$$W_0(x) = \epsilon \left[\mathcal{K}_1(-\vartheta) w^1_{-\vartheta}(x) - \mathcal{K}'_1(-\vartheta) w_{-\vartheta}(x) - \mathcal{K}_1(\vartheta) w^1_{\vartheta}(x) + \mathcal{K}'_1(\vartheta) w_{\vartheta}(x) \right].$$

In transformed form (2.39) often occurs also in optics, mechanics and theory of probability [40, 51, 117].

2.2.1 Control via Input Signal and Signal Filter

Applying now Fourier real generalized integral transform with respect to variable t to (2.39), (2.40), after some algebraic transformations we will obtain:

$$\frac{d^2 \overline{w}_1(x,\sigma)}{dx^2} + \sigma^2 [1 + \epsilon \overline{\mathcal{K}}_1(\sigma)] \overline{w}_1(x,\sigma) = \overline{u}_{d1}(\sigma) v(x) + \overline{W}(x,\sigma), \quad (x,\sigma) \in (0,1) \times \mathbb{R}, \tag{2.41}$$

$$\overline{w}_1(0,\sigma) = \overline{u}^0_{b1}(\sigma), \ \overline{w}_1(1,\sigma) = \overline{u}^1_{b1}(\sigma). \tag{2.42}$$

According to Butkovskiy‘s generalized method we have the following theorem.

Theorem 2.5 $u^0_{b1} \in \mathcal{U}_2$ *resolves the first problem if and only if*

$$\overline{u}^0_{b1}(z_k) = (-1)^k \left[\overline{u}^1_{b1}(z_k) - \Lambda\left(1, z_k, \overline{u}_{d1}(z_k)\right) \right], \quad k \in \mathbb{N}, \tag{2.43}$$

as long as

$$\chi(z_k) = i\pi k, \quad k \in \mathbb{N}. \tag{2.44}$$

It was obtained taking into account that the general solution of (2.41), (2.42) reads as

$$\overline{w}_1(x,\sigma) = h_1(\sigma) \exp\left[\chi(\sigma) x\right] + h_2(\sigma) \exp\left[-\chi(\sigma) x\right] + \Lambda(x, \sigma, \overline{u}_{d1}(\sigma)), \ (x,\sigma) \in [0,1] \times \mathbb{R}, \tag{2.45}$$

with

$$h_p(\sigma) = \frac{\left[\overline{u}^1_{b1}(\sigma) - \Lambda\left(1, \sigma, \overline{u}_{d1}(\sigma)\right)\right] \exp\left[\pm \chi(\sigma)\right] - \overline{u}^0_{b1}(\sigma)}{\exp\left[\pm 2\chi(\sigma)\right] - 1}, \quad p \in \{1; 2\},$$

$$\Lambda(x,\sigma,\overline{u}_{d1}(\sigma)) = \frac{1}{\chi(\sigma)}\int_0^x \left[\overline{u}_{d1}(\sigma)v(\xi) + \overline{W}(\xi,\sigma)\right]\sinh[\chi(\sigma)(x-\xi)]d\xi,$$
$$\chi(\sigma) = \sqrt{(-i\sigma)^2\left[1+\epsilon\overline{\mathcal{K}}_1(\sigma)\right]} = i|\sigma|\sqrt{1+\epsilon\overline{\mathcal{K}}_1(\sigma)}.$$

From Theorem 2.5 immediately follows

Corollary 2.1 *If the kernel of (2.39) satisfies conditions*

a) $\mathcal{F}_t[\mathcal{K}_1]$ *is a real-valued function*[3],

b) $1+\epsilon\overline{\mathcal{K}}_1(\sigma) > 0,\ \ \sigma\in\mathbb{R}$,

then $u_{b1}^0 \in \mathcal{U}_2$ *resolves the first problem if and only if (cf. (2.43), (2.44))*

$$\overline{u}_1^0(z_k) = (-1)^k\left[\overline{u}_{b1}^1(z_k) - \Lambda\left(1, z_k, \overline{u}_{d1}(z_k)\right)\right], \quad k\in\mathbb{N}, \tag{2.46}$$

as long as

$$\chi_0(z_k) = \pi k, \quad k\in\mathbb{N}. \tag{2.47}$$

To proof the Corollary 2.1, instead of (2.45) the following must be used:

$$\overline{w}_1(x,\sigma) = \overline{u}_{b1}^0(\sigma)\cos\left[\chi_0(\sigma)x\right] + \frac{\overline{u}_{b1}^1(\sigma) - \overline{u}_{b1}^0(\sigma)\cos\left[\chi_0(\sigma)\right] - \Lambda\left(1,\sigma,\overline{u}_{d1}(\sigma)\right)}{\sin\left[\chi_0(\sigma)\right]}\sin\left[\chi_0(\sigma)x\right] +$$
$$+\,\Lambda(x,\sigma,\overline{u}_{d1}(\sigma)), \quad (x,\sigma)\in[0,1]\times\mathbb{R},$$

with

$$\Lambda(x,\sigma,\overline{u}_{d1}(\sigma)) = \frac{1}{\chi_0(\sigma)}\int_0^x \left[\overline{u}_{d1}(\sigma)v(\xi) + \overline{W}(\xi,\sigma)\right]\sin[\chi_0(\sigma)(x-\xi)]d\xi,$$
$$\chi_0(\sigma) = \sqrt{\sigma^2\left[1+\epsilon\overline{\mathcal{K}}_1(\sigma)\right]} = |\sigma|\sqrt{1+\epsilon\overline{\mathcal{K}}_1(\sigma)}.$$

According to Corollary 1.1, condition a) of Corollary 2.1 holds, particularly, when $\mathcal{K}_1(t)$, $t\in\mathbb{R}$, is an even function: $\mathcal{K}_1(-t) = \mathcal{K}_1(t)$. Let us add that a condition similar to condition b) of Corollary 2.1 is obtained also in [40] when introducing the solvability of Riemann problem in theory of analytical functions. the well-posedness of (2.39), (2.40), when $\mathcal{K}_1 \in L^1[-\vartheta,\vartheta]$ function is even and u_{d1}, $W \in L^1[-\vartheta,\vartheta]$ is proved in [120]. It should be noted also, that conditions a) and b) of Corollary 2.1 have simple physical treatment: the quantity $\varepsilon(\sigma) = 1+\epsilon\overline{\mathcal{K}}_1(\sigma)$ is complex dielectric permittivity of isotropic medium with dispersion property [51, 117], and those conditions are equivalent to assumption, that quantity $\varepsilon(\sigma)$, $\sigma\in\mathbb{R}$, is

[3]See Corollary 1.1.

real-valued, because if so it is positive for all known materials. On the other hand, with increase of propagating signal frequency to values similar to eigenfrequency of medium, the difference between dielectric and conducting abilities of medium decreases, and it turns out, that existence of imaginary part in dielectric permittivity expression from macroscopic point of view is indistinguishable from conducting ability; they both lead to heat evaluation. Thus, both conditions of Corollary 2.1 are practically realizable.

To obtain controlled electromagnetic wave field, we have to apply Fourier inverse generalized transform to (2.45). Then, on the basis of Corollary 1.1 from (2.45) we have

$$\begin{aligned} w(x,t) &= \mathcal{F}_t^{-1}[\overline{w}_1] = \mathcal{F}_c^{-1}[\mathrm{Re}\,\overline{w}_1] - \mathcal{F}_s^{-1}[\mathrm{Im}\,\overline{w}_1] = \\ &= \frac{1}{\pi}\int_0^\infty [\mathrm{Re}\,\overline{w}_1(x,\sigma)\cos(\sigma t) - \mathrm{Im}\,\overline{w}_1(x,\sigma)\sin(\sigma t)]\,d\sigma, \end{aligned} \quad (x,t)\in[0,1]\times[-\vartheta,\vartheta].$$

Separating the real and the imaginary parts of equalities (2.43), we will obtain the following countable system of equalities:

$$\int_{-\vartheta}^{\vartheta} u_b(t)\exp[-\varsigma_k t]\cos(\sigma_k t)dt = \mathcal{M}_{1k},\ \int_{-\vartheta}^{\vartheta} u_b(t)\exp[-\varsigma_k t]\sin(\sigma_k t)dt = \mathcal{M}_{2k},\quad k\in\mathbb{N},\quad (2.48)$$

$$\mathcal{M}_{1k} + i\mathcal{M}_{2k} \equiv \mathcal{M}_k = (-1)^k\left[\overline{u}_{b1}^1(\sigma_k + i\varsigma_k) - \Lambda\left(1,\sigma_k + i\varsigma_k, \overline{u}_{d1}(\sigma_k + i\varsigma_k)\right)\right].$$

Let us consider two examples of optimal controls determination. First, determine L^2–optimal solution of system (2.48) for boundary control problem, when the control process is considered in time-interval $t\in[-\pi,\pi]$, $u_d\equiv 0$, the second boundary condition, initial and terminal data read as follows

$$u_b^1(t) = \cos(2t),\ w_{-\pi}(x) = -\cos(\pi x),\ w_{-\pi}^1(x) = \sin(\pi x),\ w_\pi(x) = \cos(2\pi x),\ w_\pi^1(x) = \sin(2\pi x),$$

respectively, and the kernel of equation (2.39)– $\mathcal{K}_1(t) = \mathcal{A}_{[-\pi,\pi]}[\exp[-a|t|]]$, $t\in\mathbb{R}$, a is a positive constant (Debye dispersion model). Note, that the transmission conditions, concerning chosen data are satisfied.

It obviously satisfies condition a) of Corollary 2.1. Furthermore, as $\overline{\mathcal{K}}_1(\sigma) = a(a^2+\sigma^2)^{-1}$, therefore $1+\epsilon\overline{\mathcal{K}}_1(\sigma) > 0$ for all $\sigma\in\mathbb{R}$ and positive constants ϵ and a. Then, from (2.47) we will obtain

$$\sigma_k = \left[\sqrt{\gamma_k^2 + a^2(\pi k)^2} + \gamma_k\right]^{\frac{1}{2}},\ \varsigma_k = \left[\sqrt{\gamma_k^2 + a^2(\pi k)^2} - \gamma_k\right]^{\frac{1}{2}},$$

$$2\gamma_k = (\pi k)^2 - a(a+2\epsilon), \quad k \in \mathbb{N}.$$

It is easy to see, that for large k roots σ_k do not depend on parameter a, whereas $\lim_{k\to\infty} \varsigma_k = a$. But as ς_k are controls damping factor, then one may conclude, that as faster the kernel $\mathcal{K}_1(t)$ decreases, as faster controls damp.

Add, that $\sigma_k = \mathrm{O}(k)$ when $k \to \infty$, i.e. for k large enough they became equidistant. Moreover, as a result of calculations was observed, that $\mathcal{M}_{pk} = \mathrm{O}\left(k^{-3}\right), \; p \in \{1; 2\}$, when $k \to \infty$, which justifies truncation of infinite system (2.48). At that for large k they do not depend on factor α in range $[0.01, 10]$. The boundary optimal control function is plotted in Figures 2.4, 2.5, when $N = 80$ and $a = 0.25;\ 0.5;\ 1;\ 2$. From this graphs it is obvious, that when parameter a increases, absolute value of controls also increases.

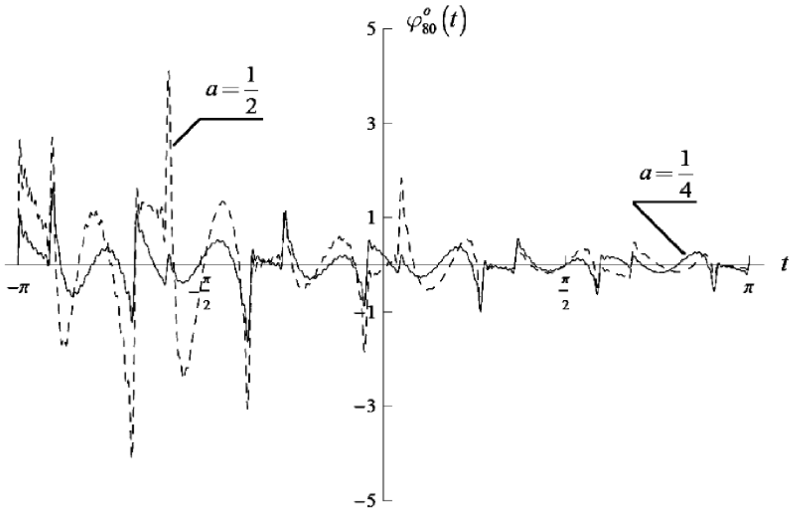

Figure 2.4: Optimal control function for $N = 80$ when $a = 0.25$ and $a = 0.5$.

Let us proceed now to solution of *the second problem.* Instead of resolving system (2.43) in this case we will obtain

$$\overline{u}_{d1}(z_k) = \frac{\overline{u}^1_{b1}(z_k) - (-1)^k \overline{u}^0_{b1}(z_k) - \mathcal{R}\left[\overline{W}\right](z_k)}{\mathcal{R}\left[v\right](z_k)}, \quad k \in \mathbb{N}, \tag{2.49}$$

for the same roots z_k, where

$$\mathcal{R}\left[\eta\right](z_k) = \frac{1}{i\pi k} \int_0^1 \eta(\xi) \sinh\left[i\pi k(1-\xi)\right] d\xi.$$

A system of equalities of (2.48) type with respect to unknown control function can also be obtained in this case with the same kernels, but with other right hand sides.

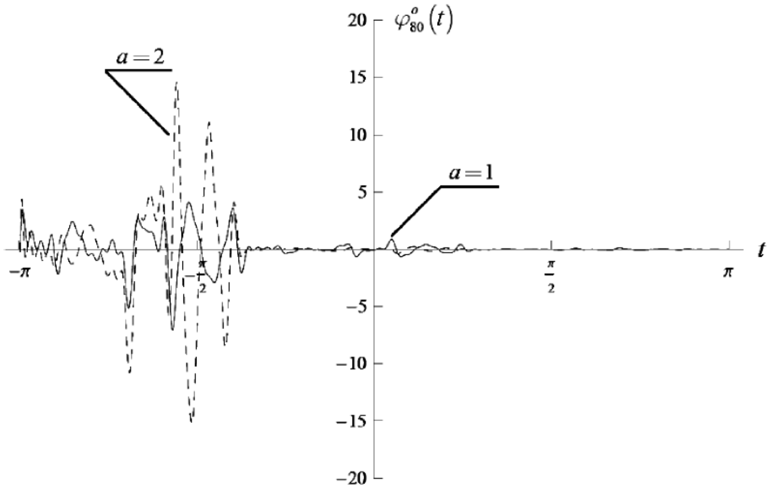

Figure 2.5: Optimal control function for $N = 80$ when $a = 1$ and $a = 2$.

From an applications point of view, electromagnetic signals controlled by a point source are especially important. This case corresponds to the substitution in resolving conditions (2.49) $v(x) = \delta(x - x_0)$ with $x_0 \in (0, 1)$. Then the relation

$$\mathcal{R}\left[v\right](z_k) = \frac{1}{i\pi k} \sinh\left[i\pi k(1 - x_0)\right],\ k \in \mathbb{N},$$

should be taken into account. Note also, that when $x_0 \to 0$ or $x_0 \to 1$, i.e. when the source "reaches" any of the medium boundaries, $\mathcal{R}\left[v\right](z_k) \to 0$, while the numerator of (2.49) remains bounded, therefore $\overline{u}_{d1}(z_k)$ can not be determined from (2.49).

Let us add that under the conditions of Corollary 2.1 in this case we will obtain:

$$\overline{u}_{d1}(z_k) = \frac{\overline{u}^1_{b1}(z_k) + (-1)^{k+1}\overline{u}^0_{b1}(z_k) - \mathcal{R}\left[\overline{W}\right](z_k)}{\mathcal{R}\left[v\right](z_k)}, \quad k \in \mathbb{N},$$

for the same roots z_k, where

$$\mathcal{R}\left[\eta\right](z_k) = \frac{1}{\pi k}\int_0^1 \eta(\xi)\sin\left[\pi k(1 - \xi)\right]d\xi.$$

Thus, admissible controls ought to be determined from infinite system (2.48). Its L^p–optimal solutions, $1 \le p \le \infty$, may be constructed according to Theorem 1.3. Corresponding existence results also remain true (see Theorem 1.2).

As second illustrative example, let us consider the L^2–optimal solution of system (2.48) for distributed control problem, when

$$u^0_b(t) = \cos t,\ u^1_b(t) = \sin(2t),\ w_{-\pi}(x) = (x - 1)\cos(\pi x),\ w^1_{-\pi}(x) = -2x\cos(\pi x),$$

$$w_\pi(x) = (x-1)\cos(2\pi x),\ w_\pi^1(x) = 2x\cos(2\pi x),\quad (x,t) \in [0,1] \times [-\pi,\pi],$$

and $v(x) = \delta(x - x_0)$, $\mathcal{K}_1(t) = -\mathcal{A}_{[-\pi,\pi]}\left[b|t|^{-1}\right]$, $t \in \mathbb{R}$, b is a positive constant. Note, that the transmission conditions, concerning chosen data are satisfied. The kernel $\mathcal{K}_1$ in this case also satisfies condition a) of Corollary 2.1. Moreover, $\overline{\mathcal{K}}_1(\sigma) = 2b(\gamma + \ln|\sigma|)$, where γ is Euler's constant, and, therefore choosing parameter b in a certain manner one may satisfy condition b) of Corollary 2.1 as well. Substituting results in (2.47), we will obtain

$$z_k = \frac{\beta_k}{\sqrt{W\left(\beta_k^2 \cdot \exp\left[\frac{1}{b\epsilon} + 2\gamma\right]\right)}},\quad \beta_k^2 = \frac{(\pi k)^2}{b\epsilon},\quad k \in \mathbb{N}.$$

Taking into account properties of that function we may prove, that all roots z_k, $k \in \mathbb{N}$, in this case are real, i.e. $\varsigma_k = 0$. Moreover, $z_k = \mathrm{O}(k)$ and $\mathcal{M}_{pk} = \mathrm{O}\left(k^{-3}\right),\ p \in \{1;2\}$, when $k \to \infty$ as well. At that for large k they do not depend on factor ϵ in range $[0.01, 10]$. The optimal control function is plotted in Figures 2.6–2.7 when $N = 80$ and $x_0 = 0.685$.

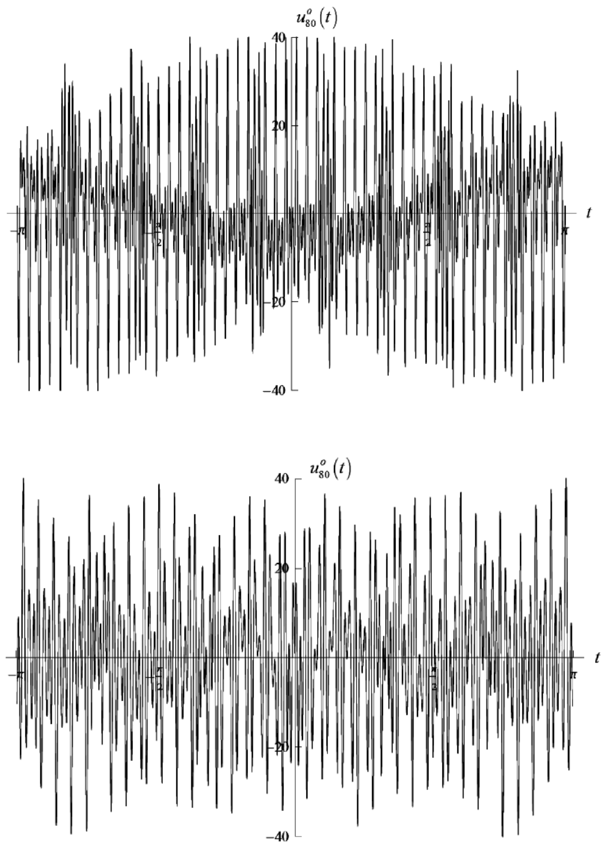

Figure 2.6: Optimal control function for $b = 0.001$ (the upper) and $b = 0.01$ (the lower).

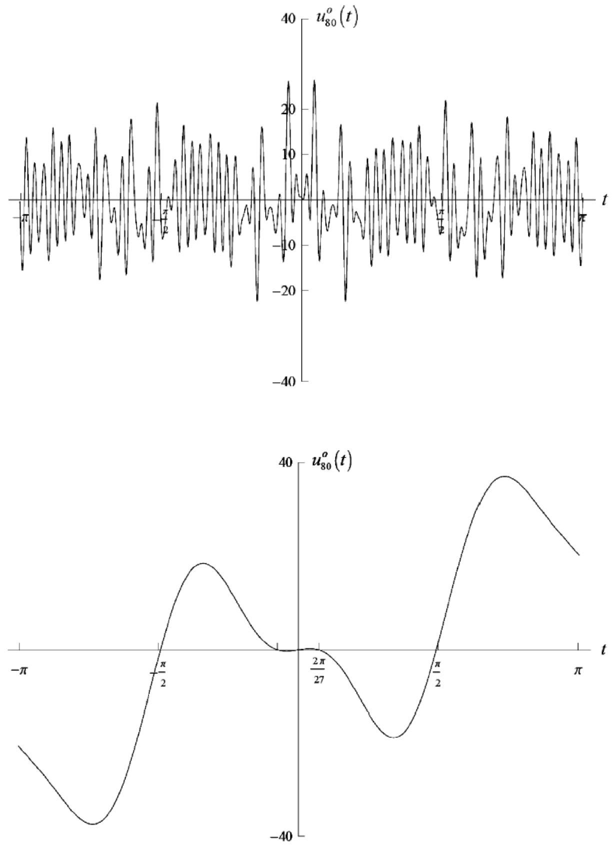

Figure 2.7: Optimal control function for $b = 0.1$ (the upper) and $b = 20$ (the lower).

It was observed, that when parameter b increases, the absolute value and frequency of control impacts decrease, and for large values of that parameter the coefficients of sines (l^o_{1k}) in (1.20) dominate those of cosine (l^o_{2k}). It was also observed, that the absolute value of the control function does not signally depend on the point x_0, except for singular cases $x_0 \to 0+$ and $x_0 \to 1-$ mentioned above.

Note that all parameters of the system (2.36)–(2.38) were chosen in such manners that the criterion (1.22) is always positive, so the optimal solution exists.

On the basis of obtained explicit formulas for optimal boundary control a simulation is implemented using the features of COMSOL Multiphysics 5.0 package. The medium is modelled as a layer of constant thickness $h =$ infinite in two other directions. The dispersion model is set to $\mathcal{K}(t) = \exp[-at]$ (Debye model considered for the boundary control problem) with $a = 10^6$ Hz: so the dispersion is nonlinear. Initially the electromagnetic wave along the layer thickness has the form $w(x, 0) = \sin(\pi x)$. The optimal resolving control is implemented from its boundary in order to have $w(x, T) = \sin(2\pi x)$ at moment $T = 10^{-}6$ s. The results of simulation is plotted in Figure 2.8.

For comparison the same boundary input was given to similar layer with constant parameters (dielectric permittivity), i.e. linear dispersion. The results of simulation is plotted in Figure 2.9.

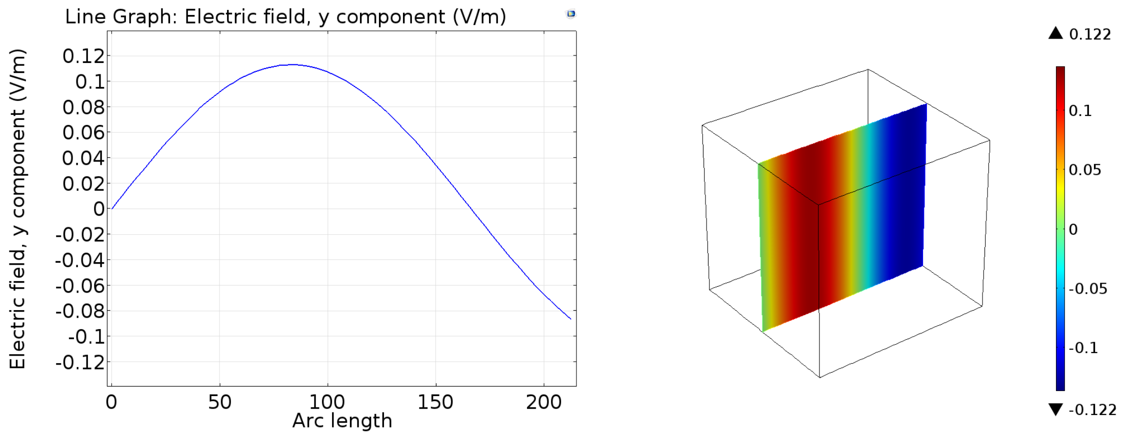

Figure 2.8: The result of simulation: nonlinear (Debye) dispersion. The electric field is plotted against the thickness of the layer on one cross section (from the left) and in dimensions three (from the right).

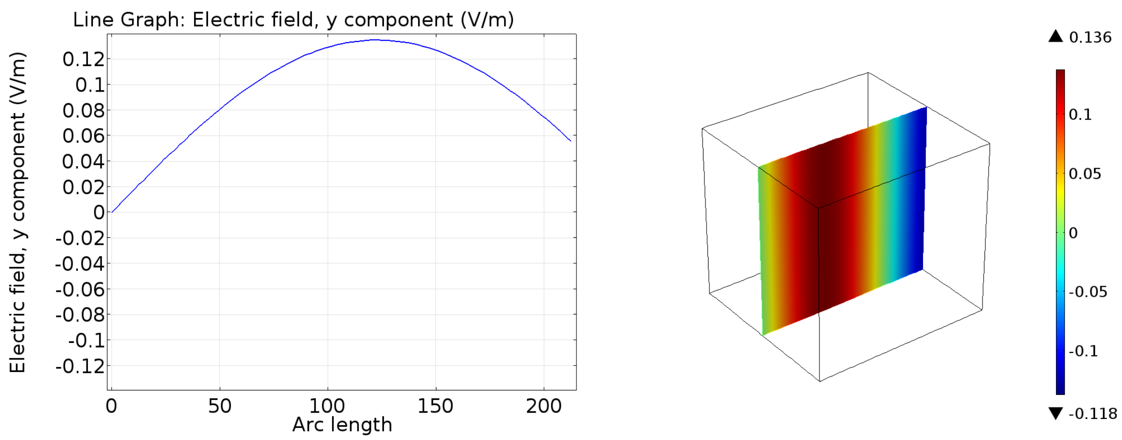

Figure 2.9: The result of simulation: linear dispersion. The electric field is plotted against the thickness of the layer on one cross section (from the left) and in dimensions three (from the right).

CHAPTER 3

MATERIAL DISTRIBUTION, TOPOLOGY AND STRUCTURAL OPTIMIZATION FOR DEFORMABLE STRUCTURES

Optimal design problems are traditionally considered in order to minimize or maximize some parameters (weight, volume, load capacity and etc.) of a design with given structure [6, 7, 26, 107]. In monograph [26] a wide range of construction optimization problems of three main kinds: optimization of size, form and structure, are investigated. Recently, so–called topology optimization problems have begun to be investigated in order to minimize a specific functional describing the material distribution in the given domain, retaining or even improving desired properties of the structure. The solution of topology optimization problems, unlike those of structural optimization, where necessary conditions of optimality have to be solved, are generally reduced to problem of nonlinear programming [15] (see, for example, [5, 14, 19, 32, 47–50, 52, 59, 60, 72, 73, 76, 82, 87, 96, 106, 121]). Nevertheless, the explicit analytical form solution in such problems is connected with significant difficulties, and the numerical solution requires high computational costs.

Such problems have not only practical importance, but stand out with complexity of investigation, because mathematically they are formulated as nonlinear control systems, therefore they have also theoretical importance.

In the general statement the topology optimization problems require minimization of given criterion

$$\kappa[u] \underset{u}{\to} \min, \quad u \in \mathcal{U},$$

usually describing the distribution of the material in design under certain geometrical and characteristic constraints.

If, for example, the control problem is considered for a deformable design, equation of its motion, e.g.

$$\mathcal{D}_u[w] = f(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \mathcal{O} \times (0, T), \tag{3.1}$$

(or static equilibrium) may be considered as characteristic (differential) constraints, whereby $\boldsymbol{x} \in \mathcal{O} \subset \mathbb{R}^3$ will be the geometrical constraint. Besides, the solution of (3.1) satisfies given conditions on boundary:

$$\mathcal{B}[w] = w_\partial(t), \quad (\boldsymbol{x}, t) \in \partial\mathcal{O} \times [0, T]. \tag{3.2}$$

In dynamical problems initial conditions of form

$$\mathcal{I}[w] = 0, \quad \boldsymbol{x} \in \overline{\mathcal{O}},$$

are given. The main purpose of the control in dynamical problem may be, for instance, ensuring the given terminal conditions (see [22] and Chapter 2):

$$\mathcal{T}[w] = 0, \quad \boldsymbol{x} \in \overline{\mathcal{O}}.$$

The nonlinear differential operator $\mathcal{D}_u[\cdot]$ is defined in $\mathcal{O} \times (0, T)$, $\mathcal{B}[\cdot]$ is a linear or nonlinear operator, representing the boundary conditions, $f : \mathcal{O} \times (0, T) \to \mathbb{R}$ is a given function satisfying certain conditions.

Examples of operators $\mathcal{D}_u[\cdot]$ and $\mathcal{B}[\cdot]$ may be found, for instance, in monographs [6, 7, 14, 47, 50, 52, 100, 107] and in articles [10, 17–19, 32, 48, 49, 59, 60, 64, 72–76, 79, 81, 82, 85–87, 96, 99, 104, 106, 108].

In order to solve the bilinear control problems when $\mathcal{B}[\cdot]$ is *linear* we suggest to use the Bubnov–Galerkin procedure [91]. Suppose, that we have already constructed a system of (linearly independent) approximating (also called basis or Galerkin) functions $\{\varphi_n(\boldsymbol{x})\}_{n=0}^N$, the first one of which, φ_0, satisfies non–homogeneous, and the rest– to homogeneous boundary conditions (3.2). Then the residue, obtained as a result of substitution of approximating solution

$$w_N(\boldsymbol{x}, t) = \varepsilon_0(t)\varphi_0(\boldsymbol{x}) + \sum_{n=1}^{N} \varepsilon_n(t)\varphi_n(\boldsymbol{x}), \quad (\boldsymbol{x}, t) \in \overline{\mathcal{O}} \times [0, T], \tag{3.3}$$

into (3.1), will be

$$\mathcal{R}_N(\boldsymbol{x},t) = \mathcal{D}_u[w_N] - f(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \overline{\mathcal{O}} \times [0,T]. \tag{3.4}$$

Above the coefficients ε_n, $n \in \{0, N\}$, must be determined, and t is considered as a parameter. According to the Bubnov–Galerkin procedure, the required coefficients ε_n, $n \in \{0, N\}$, are determined from orthogonality conditions of basis functions $\{\varphi_n(\boldsymbol{x})\}_{n=0}^{N}$ to residual (3.4):

$$\int_{\overline{\mathcal{O}}} \mathcal{R}_N(\boldsymbol{x},t)\varphi_n(\boldsymbol{x})d\boldsymbol{x} = 0, \quad t \in [0,T], \quad n \in \{0; N\}. \tag{3.5}$$

If for some $N_0 \in \mathbb{N}$ the residual (3.4) is identically zero: $\mathcal{R}_{N_0}(\boldsymbol{x},t) \equiv 0$, then corresponding function $w_{N_0}(\boldsymbol{x},t)$ (3.3) will be the exact solution of boundary–value problem (3.1), (3.2). Otherwise, increasing number N of approximating functions $\{\varphi_n(\boldsymbol{x})\}_{n=0}^{N}$ we may increase the accuracy of approximation (3.3).

Since the state operator $\mathcal{D}_u[\cdot]$ is bilinear, (3.5) provides linear equations (differential, integral, integro-differential, functional, algebraic, etc.). After determining the unknown coefficients $\varepsilon_n(t)$ from the system of linear equations (3.5) and substituting them into (3.3), and taking into account, that at T given terminal conditions must be satisfied, we will derive the system of nonlinear restrictions

$$\mathcal{T}[w] = \mathcal{T}\left[\varepsilon_0\varphi_0(\boldsymbol{x}) + \sum_{n=1}^{N}\varepsilon_n\varphi_n(\boldsymbol{x})\right], \quad \boldsymbol{x} \in \overline{\mathcal{O}}. \tag{3.6}$$

which the unknown function u [1] has to be determined from.

Remark 3.1 *i) The complexity of the technique application in a particular problem depends on the complexity of operator $\mathcal{D}_u[\cdot]$ and, especially, its component with respect to t: it defines the type (algebraic, differential, integral etc.) of equation (3.5). Its solution from appropriate space can be approximated, for example, by expansion*

$$\varepsilon_n(t) = \sum_{k=1}^{\infty}\varepsilon_k \mathrm{T}_k(t).$$

ii) By all means, the procedure can also be applied in the case when $\mathcal{D}_u[\cdot]$ is nonlinear in both control and state functions, which seems to be very complicated.

[1]Recall Remark 1.2.

iii) Since (3.6) contains the unknown function only with inclusion in some functional while the rest parameters of the equality are known, after computation (in most cases numerical) we will be able to obtain the values of that functional, say, $J[u] = \mathcal{M}$. *This will be the system of necessary and sufficient conditions, which the control function must be determined from.*

To be clear let us explain the derivation of controllability conditions (3.6) on a certain example. Let, for simplicity, (3.1) has the form

$$\mathcal{D}_u[w] - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = 0, \quad (\boldsymbol{x}, t) \in \mathcal{O} \times (0, T).$$

Then initial and terminal conditions may have the form

$$w(\boldsymbol{x}, 0) = w_0(\boldsymbol{x}), \quad \left.\frac{\partial w}{\partial t}\right|_{t=0} = w_0^1(\boldsymbol{x}), \quad \boldsymbol{x} \in \overline{\mathcal{O}},$$
$$w(\boldsymbol{x}, T) = w_T(\boldsymbol{x}), \quad \left.\frac{\partial w}{\partial t}\right|_{t=T} = w_T^1(\boldsymbol{x}), \quad \boldsymbol{x} \in \overline{\mathcal{O}},$$

respectively. Therefore (3.6) will take the form

$$w_T(\boldsymbol{x}) = \varepsilon_0(T)\varphi_0(\boldsymbol{x}) + \sum_{n=1}^{N} \varepsilon_n(T)\varphi_n(\boldsymbol{x}), \; w_T^1(\boldsymbol{x}) = \dot{\varepsilon}_0(T)\varphi_0(\boldsymbol{x}) + \sum_{n=1}^{N} \dot{\varepsilon}_n(T)\varphi_n(\boldsymbol{x}),$$

in which $\dot{\varepsilon}$ denotes its time derivative. Expanding terminal functions into basis functions and using the linearly independence of those, we will arrive at required restrictions.

The case when the control function does not explicitly depend on time t is covered in the first two subsections of this chapter, which is based on the articles [60, 72–74, 106] and contains solutions to several problems of optimization. First, in Section 3.1, optimal distribution of viscoelastic material with given characteristics under elastics beams with given geometry and characteristics is suggested, when the beam is subjected to constant normal load uniformly passing through its upper bound. In Section 3.2 a two dimensional distribution optimization problem is considered for elastic material under an elastic rectangular plate subjected to uniformly moving normal load. The problem of optimization in both cases is the choice of a distribution providing null terminal state in given time after the load is detached of the corresponding structure. The last Section 3.3 is devoted to choice of optimal structure for an infinite layer allowing the propagation of periodic waves with given phase speed in the direction perpendicular to the structure non–homogeneity.

3.1 Optimization of Viscoelastic Material Distribution under Finite Elastic Beam Subjected to Moving Load

To show how the procedure described above should be applied for solving particular problems, we consider a finite, elastic beam undergo a constant concentrated load P_*, moving over the beam uniformly with constant speed v_* (Figure 3.1). Our aim is to damp the beam vibrations in required time after the load detaches from the beam. For that purpose we have only limited quantity of viscoelastic material at our disposal. The material may be distributed under the beam in several manners: continuously, partially, point-wise etc. As it supposed to be, we are interested in wasting as less material as possible: it is quite obvious that the point-wise distribution, corresponding to vibration absorbers (dampers), has minimal volume.

The presence of viscoelastic base will lead to significant change of the beam bending energy: when it increases, the frequency of the vibrations increases as well, and as soon as the energy of the beam turns to zero, vibrations damp (it is quite obvious, that it may happen only after the moving load detaches from the beam). Quantitatively that change will, naturally, depend on the manner of distribution of the base. Thence, we may achieve our aim by varying all possible distribution of the base under the beam. Therefore, the achieving of the beam vibrations damping with as little quantity of the material as possible will be, by all means, very important.

Let us now construct the mathematical model of the problem described above. Let the beam is homogeneous and is made of linear–elastic, isotropic material. Suppose also, that the beam is simply supported. With respect to viscoelastic material we accept the Kelvin–Voigt model [27] as it is usually done in such problems [39, 103, 112]. This section is based one the results of [72, 73].

Then the bending vibrations of the beam will be described by the following bilinear partial differential equation [112]

$$EJ\frac{\partial^4 w_*(x_*,t_*)}{\partial x_*^4} + u(x_*)\left[\alpha_*^2\frac{\partial w_*(x_*,t_*)}{\partial t_*} + \beta_*^2 w_*(x_*,t_*)\right] + \rho S\frac{\partial^2 w_*(x_*,t_*)}{\partial t_*^2} = f_*(x_*,t_*), \quad (3.7)$$

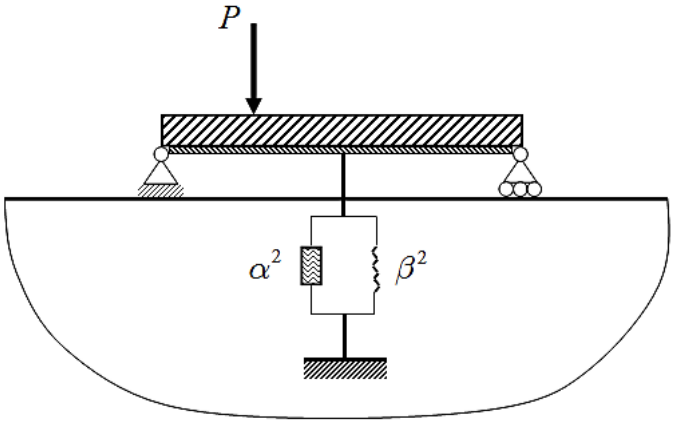

Figure 3.1: Illustration of the beam on viscoelastic base.

$$(x_*, t_*) \in (-l, l) \times (0, T),$$

subjected to the boundary conditions of simply support

$$w_*(\pm l, t_*) = \left. \frac{\partial^2 w_*(x_*, t_*)}{\partial x_*^2} \right|_{x_*=\pm l} = 0, \quad t_* \in [0, T]. \tag{3.8}$$

Here $u(x)$ is a dimensionless function, describing the distribution of the base under the beam. The right hand side of (3.7) describes influence of the moving load.

The initial state of the beam is given:

$$w_*(x_*, 0) = w_{0*}(x_*), \quad \left. \frac{\partial w_*(x_*, t_*)}{\partial t_*} \right|_{t_*=0} = w_{0*}^1(x_*), \quad x_* \in [-l, l]. \tag{3.9}$$

The moving load detaches from the beam at given (fixed) moment $\tau_* = 2l/v_*$. Then, the explicit form of that influence may be represented as follows:

$$f_*(x_*, t_*) = P_* \mathcal{A}_{[0,\tau_*]} \left[\delta(x_* + l - v_* t_*) \right], \quad (x_*, t_*) \in (-l, l) \times (0, T). \tag{3.10}$$

In order to be allowed to control over discontinuous (piece-wise, point-wise, step-wise, etc.) functions, the set of admissible controls is taken to be

$$\mathcal{U}_1 = \left\{ u \in \mathfrak{D}; \ \int_{-l}^{x_*} u(\xi) d\xi \in L^1[-l, l]; \ \text{supp}\ u \subset [-l, l], \ 0 \le u \le 1 \right\}$$

Our main aim is to choose an admissible control $u^o \in \mathcal{U}_1$, providing the terminal state

$$w_*(x_*, T) = 0, \quad \left. \frac{\partial w_*(x_*, t_*)}{\partial t_*} \right|_{t_*=T} = 0, \quad x_* \in [-l, l], \tag{3.11}$$

as well as minimizing the functional

$$\kappa_1[u] = \int_{-l}^{l} u(x_*)dx_*, \quad u \in \mathcal{U}_1. \tag{3.12}$$

Up to a constant multiplier, the functional $\kappa_1[u]$ describes the mass of viscoelastic material spent for implementing the distribution u. The natural restriction on control time is $\tau_* < T \leq T_0$, in which T_0 is the time when the bending vibrations of the beam vanish in the case $u \equiv 1$.

We suppose finally, that the transmission conditions between boundary and initial and terminal functions are satisfied:

$$w_{0*}(-l) = w''_{0*}(-l) = w_{0*}(l) = w''_{0*}(l) = 0, \quad w^1_{0*}(-l) = w^{1''}_{0*}(-l) = w^1_{0*}(l) = w^{1''}_{0*}(l) = 0.$$

Before proceeding further, let us introduce the dimensionless variables and functions:

$$x = \frac{x_*}{l}, \quad t = \frac{2t_* - T}{T}\vartheta, \quad w = \frac{w_*}{l}, \quad \alpha^2 = \frac{\alpha_*^2 l^4}{EJ}\frac{2\vartheta}{T}, \quad \beta^2 = \frac{\beta_*^2 l^4}{EJ}, \quad \gamma^2 = \frac{\rho S l^4}{EJ}\frac{4\vartheta^2}{T^2},$$

$$P = \frac{P_* l^3}{EJ}, \quad v = \frac{v_*}{l}\frac{T}{2\vartheta}, \quad \tau = \tau_*\frac{2\vartheta}{T}.$$

Then, the system (3.7)–(3.12) can be written in the following form:

$$\frac{\partial^4 w(x,t)}{\partial x^4} + u(x)\left[\alpha^2\frac{\partial w(x,t)}{\partial t} + \beta^2 w(x,t)\right] + \gamma^2\frac{\partial^2 w(x,t)}{\partial t^2} = f(x,t), \tag{3.13}$$

$$(x,t) \in (-1,1) \times (-\vartheta, \vartheta),$$

$$f(x,t) = P\mathcal{A}_{[-\vartheta,\vartheta-\tau_*]}\left[\delta(x + 1 - v(t+\vartheta))\right], \tag{3.14}$$

$$w(\pm 1, t) = \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=\pm 1} = 0, \quad t \in (-\vartheta, \vartheta), \tag{3.15}$$

$$w(x, -\vartheta) = w_0(x), \quad \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=-\vartheta} = w_0^1(x), \quad x \in (-1,1), \tag{3.16}$$

$$w(x, \vartheta) = 0, \quad \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=\vartheta} = 0, \quad x \in (-1,1), \tag{3.17}$$

$$\kappa_1[u] = \int_{-1}^{1} u(x)dx. \tag{3.18}$$

Now, proceeding to the solution of the problem, we use Butkovskiy‘s generalized method. Applying the operator $\mathcal{A}_{[-\vartheta,\vartheta]}[\cdot]$ to (3.13)–(3.15), we will obtain

$$\frac{\partial^4 w_1(x,t)}{\partial x^4} + u(x)\left[\alpha^2\frac{\partial w_1(x,t)}{\partial t} + \beta^2 w_1(x,t)\right] + \gamma^2\frac{\partial^2 w_1(x,t)}{\partial t^2} = G(x,t), \tag{3.19}$$

$$(x,t) \in (-1,1) \times \mathbb{R},$$

$$G(x,t) = f_1(x,t) + \alpha^2 u(x) w_0(x) \delta(t+\vartheta) + \gamma^2 \left[w_0(x) \delta'(t+\vartheta) + w_0^1(x) \delta(t+\vartheta) \right], \qquad (3.20)$$

$$w_1(\pm 1, t) = \left. \frac{\partial^2 w_1(x,t)}{\partial x^2} \right|_{x=\pm 1} = 0, \quad t \in \mathbb{R}. \qquad (3.21)$$

Thus, we have a boundary–value problem in $\mathfrak{D}$ over $[-1,1] \times \mathbb{R}$. Therefore, we may involve Fourier generalized integral transform with respect to t. Applying the Fourier generalized distributional to (3.19)–(3.21) we will obtain

$$\frac{d^4 \overline{w}_1(x,\sigma)}{dx^4} + \left[\left(\beta^2 - i\sigma\alpha^2 \right) u(x) - \sigma^2 \gamma^2 \right] \overline{w}_1(x,\sigma) = \overline{G}(x,\sigma), \quad (x,\sigma) \in (-1,1) \times \mathbb{R}, \qquad (3.22)$$

$$\overline{G}(x,\sigma) = \overline{f}_1(x,\sigma) + \left[\gamma^2 \left[w_0^1(x) - i\sigma w_0(x) \right] + \alpha^2 u(x) w_0(x) \right] \exp\left[-i\sigma\vartheta \right], \qquad (3.23)$$

$$\overline{w}_1(\pm 1, \sigma) = \left. \frac{d^2 \overline{w}_1(x,\sigma)}{dx^2} \right|_{x=\pm 1} = 0, \quad \sigma \in \mathbb{R}. \qquad (3.24)$$

To find the Fourier image of $f_1(x,t)$ we take into account the obvious relation

$$f_1(x,t) \equiv \mathcal{A}_{[-\vartheta,\vartheta]}[f] = f(x,t), \quad (x,t) \in (-1,1) \times \mathbb{R}.$$

Then

$$\overline{f}_1(x,\sigma) = \frac{P}{v} \left[\theta(x+1) - \theta(x-1) \right] \exp\left[i\sigma \left(\frac{x+1}{v} - \vartheta \right) \right].$$

Here we have taken into account that $v\tau = 2$.

It is characteristic for (3.22), that the control function is included as in its coefficients, as well as in its right hand side. However, in particular case when $\alpha_*^2 = 0$, and therefore $\alpha^2 = 0$, which corresponds to pure elastic material [60], the control function is included only in coefficients of (3.22). In the case when $\beta_*^2 = 0$, and therefore $\beta^2 = 0$, which corresponds to pure viscous material [72], introducing the auxiliary function

$$\overline{w}(x,\sigma) = i\sigma \exp\left[i\sigma\vartheta \right] \overline{w}_1(x,\sigma) + w_0(x),$$

we may reduce the control function from the right hand side of (3.22) and obtain

$$\frac{d^4 \overline{w}(x,\sigma)}{dx^4} - \left[\sigma^2 \gamma^2 + i\sigma\alpha^2 u(x) \right] \overline{w}(x,\sigma) = \overline{G}_0(x,\sigma), \quad (x,\sigma) \in [-1,1] \times \mathbb{R},$$

$$\overline{G}_0(x,\sigma) = i\sigma \left[\overline{f}_1(x,\sigma) + \gamma^2 w_0^1(x) \right] + w_0^{IV}(x).$$

Now we are able to involve the Bubnov–Galerkin procedure as described above. In our case, the system $\{\sin(\pi n x)\}_{n \in \mathbb{N}}$ is orthonormal in $[-1,1]$ and satisfies the boundary conditions

(3.24), therefore can be taken as family of approximate functions. The approximating solution of (3.22)–(3.24) therefore will take the form

$$\overline{w}_{1N}(x,\sigma) = \sum_{n=1}^{N} \overline{\varepsilon}_n(\sigma)\sin(\pi n x), \quad (x,\sigma) \in [-1,1]\times\mathbb{R}. \tag{3.25}$$

Then, the residual (3.4) we will be

$$\mathcal{R}_N(x,\sigma) = \sum_{n=1}^{N} \left[(\pi n)^4 + \left(u(x)\left(\beta^2 - i\sigma\alpha^2\right) + \gamma^2\right)\right]\overline{\varepsilon}_n(\sigma)\sin(\pi n x) - \overline{G}(x,\sigma).$$

Taking into account this relation, from (3.5) we will derive the following system of linear algebraic equations with respect to unknown coefficients $\overline{\varepsilon}_n$

$$\sum_{n=1}^{N} \Lambda_{km}(\sigma)\overline{\varepsilon}_n(\sigma) = \Omega_\nu(\sigma), \quad \nu \in \{1;N\}, \tag{3.26}$$

where

$$\Lambda_{n\nu}(\sigma) = \left[(\pi n)^4 - \sigma^2\gamma^2\right]\delta_n^\nu + \left[\beta^2 - i\sigma\alpha^2\right]J_{n\nu}[u],$$
$$\Omega_\nu(\sigma) = \int_{-1}^{1} \overline{G}(x,\sigma)\sin(\pi\nu x)dx,$$
$$J_{n\nu}[u] = J_{\nu n}[u] = \int_{-1}^{1} u(x)\sin(\pi n x)\sin(\pi\nu x)dx.$$

It is obvious, that $J_{nn}[u] \geq 0$. From the other hand side $|J_{n\nu}[u]| \leq \kappa[u], \quad n,\nu \in \{1;N\}$.

To apply Butkovskiy's generalized technique, let us represent the solution of (3.26) as follows:

$$\overline{\varepsilon}_n(\sigma) = \frac{\Delta_n(\sigma)}{\Delta(\sigma)}, \quad \sigma \in \mathbb{R}, \tag{3.27}$$

and substitute it (3.25):

$$\overline{w}_{1N}(x,\sigma) = \sum_{n=1}^{N} \frac{\Delta_n(\sigma)}{\Delta(\sigma)}\sin(\pi n x). \tag{3.28}$$

Since $w_1(x,t) \equiv \mathcal{A}_{[-\vartheta,\vartheta]}[w]$ by the definition is compactly supported in $[-\vartheta,\vartheta]$, then, according to Wiener–Paley–Schwartz theorem, the extension $\overline{w}_1(x,\sigma+i\varsigma)$ is entire. In the same way as we did in Section 1.3, from (3.28) we will have

$$\Delta_1(z_k) = \begin{vmatrix} \Omega_1(z_k) & \Lambda_{12}(z_k) & \dots & \Lambda_{1N}(z_k) \\ \Omega_2(z_k) & \Lambda_{22}(z_k) & \dots & \Lambda_{2N}(z_k) \\ \dots & \dots & \dots & \dots \\ \Omega_N(z_k) & \Lambda_{n2}(z_k) & \dots & \Lambda_{NN}(z_k) \end{vmatrix} = 0, \quad k \in \{1;2N\}, \tag{3.29}$$

as long as

$$\Delta(z_k) = 0. \tag{3.30}$$

Remark 3.2 *It is easy to see from the expression of* $\Lambda_{n\nu}(\sigma)$*, the determinant* $\Delta(\sigma)$ *is a polynomial of degree* $2N$*, therefore (3.30) admits* $2N$ *complex roots. But since* $\mathrm{Re}\ \Delta(-\sigma - i\varsigma) = \mathrm{Re}\ \Delta(\sigma + i\varsigma)$*, and* $\mathrm{Im}\ \Delta(-\sigma - i\varsigma) = \mathrm{Im}\ \Delta(\sigma + i\varsigma)$*, which means that* $z_k = \sigma_k + i\varsigma_k$ *and* $z_k = -\sigma_k + i\varsigma_k$ *satisfy (3.30) simultaneously, we conclude that (3.29) contains only* N *independent conditions. At this, that number cannot be reduced further.*

Solving (3.30) and substituting z_k into (3.29), with respect to $J_{n\nu}[u]$ we will obtain the system of restrictions

$$J_{n\nu}[u] = \mathcal{M}_{n\nu}, \quad n, \nu \in \{1; N\},$$

where the constants $\mathcal{M}_{n\nu}$ depend on all data of (3.29), and therefore (3.13)–(3.18), except unknown u.

Thus, the solution of the optimal control problem (3.13)–(3.18) is reduced to minimization of the functional (3.18) under integral constraints of equality type. Since the kernels of $J_{n\nu}[u]$ are bounded, we can solve the reduced problem by means of problem of moments. According to Subsection 1.3.1[2]:

$$u^o(x) = \sum_{j=1}^{J} \delta(x - x_j^o), \quad x \in (-1, 1). \tag{3.31}$$

The solution obtained corresponds to discrete distribution of viscoelastic material under the beam. As it was mentioned above, it is the most common model of vibration absorbers (dampers) [39, 88, 94, 95, 103, 112]: the switching points $-1 < x_j^o < x_{j+1}^o < 1$ corresponds to the placements of the dampers under the beam and are determined from the system [67–74]

$$\sum_{j=1}^{J} \sin\left(\pi n x_j^o\right) \sin\left(\pi \nu x_j^o\right) = \mathcal{M}_{n\nu}, \quad n, \nu \in \{1; n\}. \tag{3.32}$$

It should be noted, that (3.31) is not unique in the sense that the number of dampers J satisfying $\left\{x_j^o\right\}_{j=1}^{J} \in (-1, 1)$ may be reduced. This may put a start to study of very important problem about finding the reasonable number of the dampers ensuring the same result. The maximum number of dampers under use may be defined as it is done in [90].

[2] We recall Remarks 1.1, 1.2 once again.

After determining the optimal resolving controls, we may aim to find also the deflection of the beam. Applying to (3.25) with substituted u^o the Fourier inverse transform, we will obtain

$$w_{1N}(x,t) = \sum_{n=1}^{N} \varepsilon_n(t)\sin(\pi n x), \quad (x,t) \in [-1,1] \times \mathbb{R}. \tag{3.33}$$

Remark 3.3 *As the real parts of expressions $\Lambda_{n\nu}(\sigma)$ and $\Omega_\nu(\sigma)$ are even, and the imaginary parts are odd:*

$$\mathrm{Re}\ \Lambda_{n\nu}(-\sigma) = \mathrm{Re}\ \Lambda_{n\nu}(\sigma), \quad \mathrm{Re}\ \Omega_\nu(-\sigma) = \mathrm{Re}\ \Omega_\nu(\sigma),$$

$$\mathrm{Im}\ \Lambda_{n\nu}(-\sigma) = -\mathrm{Im}\ \Lambda_{n\nu}(\sigma), \quad \mathrm{Im}\ \Omega_\nu(-\sigma) = -\mathrm{Im}\ \Omega_\nu(\sigma),$$

it is clear that

$$\mathrm{Re}\ \overline{\varepsilon}_n(-\sigma) = \mathrm{Re}\ \overline{\varepsilon}_n(\sigma), \quad \mathrm{Im}\ \overline{\varepsilon}_n(-\sigma) = -\mathrm{Im}\ \overline{\varepsilon}_n(\sigma).$$

According to Corollary 1.1 it is necessary and sufficient for $\varepsilon_n(t) = \mathcal{F}_t^{-1}[\overline{\varepsilon}_n]$, and therefore for $w_{1N}(x,t)$, to be real valued. Then,

$$\varepsilon_n(t) = \frac{1}{\pi}\int_0^\infty [\mathrm{Re}\ \overline{\varepsilon}_n(\sigma)\cos(\sigma t) + \mathrm{Im}\ \overline{\varepsilon}_n(\sigma)\sin(\sigma t)]\, d\sigma = \mathcal{F}_c^{-1}[\mathrm{Re}\ \overline{\varepsilon}_n] + \mathcal{F}_s^{-1}[\mathrm{Im}\ \overline{\varepsilon}_n],$$

$$\mathrm{Re}\ \overline{\varepsilon}_n(\sigma) = \frac{\mathrm{Re}\ \Delta_n(\sigma)\mathrm{Re}\ \Delta(\sigma) + \mathrm{Im}\ \Delta_m(\sigma)\mathrm{Im}\ \Delta(\sigma)}{(\mathrm{Re}\ \Delta(\sigma))^2 + (\mathrm{Im}\ \Delta(\sigma))^2},$$

$$\mathrm{Im}\ \overline{\varepsilon}_n(\sigma) = -\frac{\mathrm{Re}\ \Delta_n(\sigma)\mathrm{Im}\ \Delta(\sigma) - \mathrm{Im}\ \Delta_n(\sigma)\mathrm{Re}\ \Delta(\sigma)}{(\mathrm{Re}\ \Delta(\sigma))^2 + (\mathrm{Im}\ \Delta(\sigma))^2}.$$

We finalize with numerical simulation based on the obtained results. Let $N = 3$, $w_0(x) = \sin(\pi x)$, $w_0^1(x) \equiv 0$, $x \in [-1,1]$. Then we get $\Delta(\sigma) = \Gamma_1\Gamma_2\Gamma_3 + \mathrm{G}_4\mathrm{E}^4 + \mathrm{G}_5\mathrm{E}^3 - \mathrm{G}_6\mathrm{E}^2$,

$$\Delta_1(\sigma) = \Omega_1\Gamma_2\Gamma_3 + \mathrm{E}^2 J_{23}[u]\left[-\Omega_1 J_{23}[u] + \Omega_2 J_{13}[u] + \Omega_3 J_{12}[u]\right] -$$
$$- \mathrm{E}\left[\Omega_2\Gamma_3 J_{12}[u] + \Omega_3\Gamma_2 J_{13}[u]\right],$$

$$\Delta_2(\sigma) = \Omega_2\Gamma_1\Gamma_3 + \mathrm{E}^2 J_{13}[u]\left[\Omega_1 J_{23}[u] - \Omega_2 J_{13}[u] + \Omega_3 J_{12}[u]\right] -$$
$$- \mathrm{E}\left[\Omega_1\Gamma_3 J_{12}[u] + \Omega_3\Gamma_1 J_{23}[u]\right],$$

$$\Delta_3(\sigma) = \Omega_3\Gamma_1\Gamma_2 + \mathrm{E}^2 J_{12}[u]\left[\Omega_1 J_{23}[u] + \Omega_2 J_{13}[u] - \Omega_3 J_{12}[u]\right] -$$
$$- \mathrm{E}\left[\Omega_1\Gamma_2 J_{13}[u] + \Omega_2\Gamma_1 J_{23}[u]\right],$$

where

$$\Omega_\nu = (-1)^\nu \lambda_\nu P \cdot \frac{1 - e^{2i\frac{\sigma}{v}}}{\lambda_\nu^2 - \sigma^2} - i\sigma\gamma^2\delta_1^\nu + \alpha^2 J_{1\nu}[u], \quad \lambda_\nu = \pi\nu v,$$

$$\Gamma_\nu = \mathrm{E}J_{\nu\nu}[u] + \mathrm{G}_\nu, \quad \mathrm{E} = \beta^2 - i\sigma\alpha^2, \quad \mathrm{G}_\nu = (\pi\nu)^4 - \gamma^2\sigma^2, \quad \nu \in \{1;3\},$$

$$\mathrm{G}_4 = J_{13}[u]\left[J_{12}[u]J_{23}[u] + J_{13}[u]J_{22}[u]\right],$$

$$\mathrm{G}_5 = J_{23}[u]\left[J_{11}[u]J_{23}[u] - J_{12}[u]J_{13}[u]\right] - J_{12}^2[u]J_{33}[u],$$

$$\mathrm{G}_6 = \mathrm{G}_1 J_{23}^2[u] + \mathrm{G}_2 J_{13}^2[u] + \mathrm{G}_3 J_{12}^2[u].$$

It should be taken into account as well the obvious relation $J_{11}[u] + J_{22}[u] = 2J_{13}[u]$.

On the basis of (3.32) the switching points are calculated for different values of parameters $P, \upsilon, \tau, \alpha^2, \beta^2$. Some of them are brought in Table 3.1.

P	α^2	β^2	υ	x_j^o
100	0.1	100	100	$x_1^o = -0.87$, $x_2^o = -0.26$, $x_3^o = 0.$, $x_4^o = 0.72$
200	1	150	50	$x_1^o = -0.77$, $x_2^o = -0.06$, $x_3^o = 0.$, $x_4^o = 0.69$
200	5	200	40	$x_1^o = -0.56$, $x_2^o = -0.26$, $x_3^o = 0.61$

Table 3.1: The switching points x_j^o for $\gamma^2 = \pi^4$.

On the basis of obtain results a simulation is implemented using the features of COMSOL Multiphysics 5.0 package and suggested improvements is shown through comparison of the beam deflection in cases of equidistant and non-equidistant dampers. At this, the placements of non-equidistant dampers are found by the above method.

First we place a simply supported elastic beam of 1 m length (the width and the height of the beam cross section are chosen to be 1/50 m and 1/100 m, respectively) on 3 equidistant viscoelastic dampers. The placements of the dampers, therefore are $x_1 = 0.25$ m, $x_2 = 0.5$ m and $x_3 = 0.75$ m. The beam is made of Steel AISI 4340 ($\rho = 7850$ kg/m^3, $\nu = 0.28$, $E = 1.9 \cdot 10^{11}$ N/m^2) and initially is in steady state. A normal constant load P_* starts to move from the left edge of the beam with constant velocity. In Figures 3.2–3.9 the displacement field of points $\xi_1 = 0.65$ m, $\xi_2 = 0.875$ m and $\xi_3 = 0.95$ m of the beam is plotted for different combinations of normal load intensity, velocity and dampers viscoelasticity. The shear and bulk moduli are taken to be $6.16 \cdot 10^7; 2.91 \cdot 10^8$ N/m^2 and $4 \cdot 10^8$ N/m^2, respectively. The intensity of the load is taken to be $P_* = 10^5; 2.5 \cdot 10^5$ N, and its velocity– $\upsilon_* = 0.1; 0.25$ m/s.

For comparison, in Figures 3.18–3.29 we have plotted the displacement field of particular points and the state of the beam for the same set of parameters as above, but in the case of viscoelastic dampers with optimized placements: $x_1 = 0.35$ m, $x_2 = 0.65$ m and $x_3 = 0.85$ m.

The stress state of the beam for both equidistant dampers and dampers with optimized placements are plotted in Figures 3.10–3.17 and 3.26–3.29, respectively.

As a result of simulation it turned out, that in the case of dampers with optimal placements the bending vibrations vanish faster than in the case of equidistant dampers. It is obvious from the first look to presented figures, that the maximal absolute value of the displacement field of particular points of the beam decreases by factor 10^{-2}. For instance, the relative ratio of the maximal absolute value of the point $\xi_2 = 0.875$ m for equidistant dampers and that for dampers with optimized placements is approximately 22. In order to give a quantitative estimate, in Table 3.2 the maximal absolute values of the vertical displacements with respect to time and coordinate are brought for different pairs of normal load intensity and velocity in the case of equidistant dampers, while for comparison in Table 3.3 are brought those in the case of dampers with optimized placements. The shear moduli of the dampers are $6.16 \cdot 10^7$ N/m^2 and $2.91 \cdot 10^8$ N/m^2, and the bulk modulus is $4 \cdot 10^8$ N/m^2. It can be seen from the tables that with increase of the load intensity (for the same value of the velocity) the maximal absolute value of the displacement increases, and with increase of the speed (for the same value of the load intensity) the the maximal absolute value of the displacement decreases.

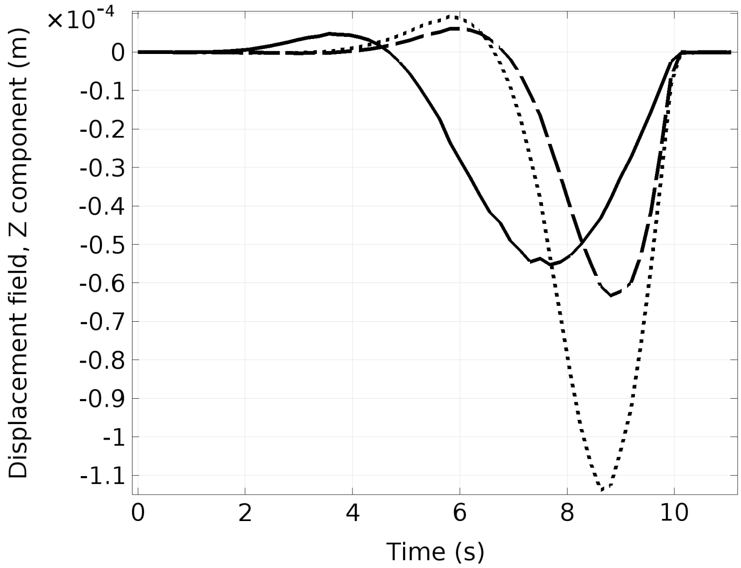

Figure 3.2: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) in the case of $P_* = 10^5$ N, $v_* = 0.1$ m/s and equidistant dampers: $x_1 = 0.25$ m, $x_2 = 0.5$ m and $x_3 = 0.75$ m (distances from the left edge). The shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

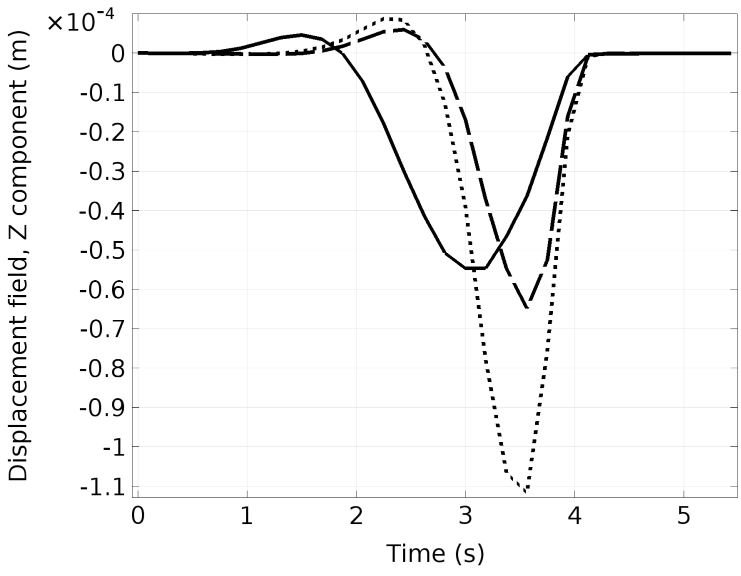

Figure 3.3: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) in the case of $P_* = 10^5$ N, $v_* = 0.25$ m/s and equidistant dampers: $x_1 = 0.25$ m, $x_2 = 0.5$ m and $x_3 = 0.75$ m (distances from the left edge). The shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

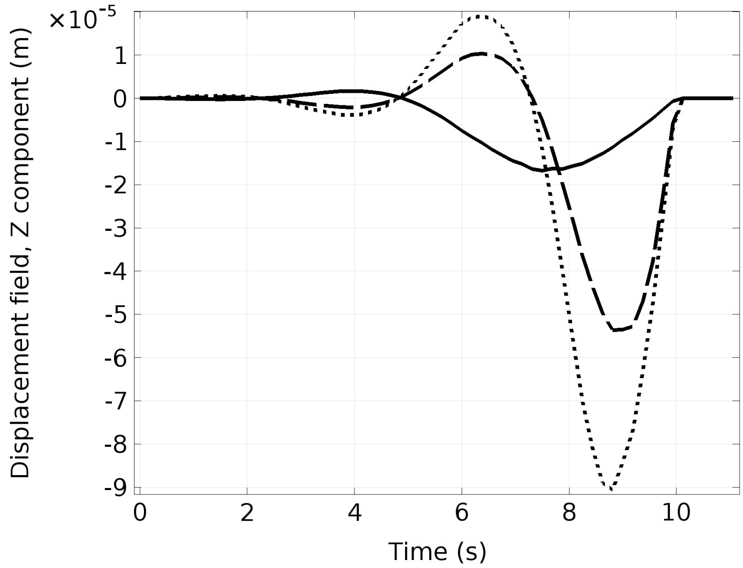

Figure 3.4: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) in the case of $P_* = 10^5$ N, $v_* = 0.1$ m/s and equidistant dampers: $x_1 = 0.25$ m, $x_2 = 0.5$ m and $x_3 = 0.75$ m (distances from the left edge). The shear modulus of the dampers is $2.91 \cdot 10^8$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

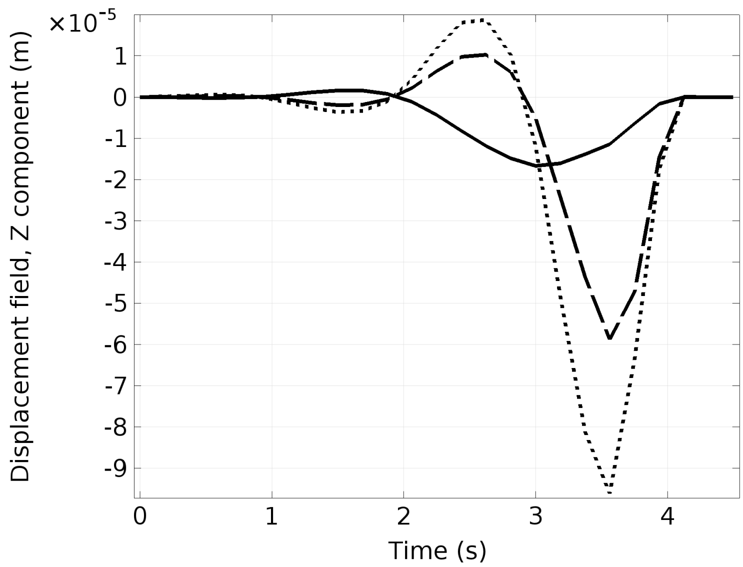

Figure 3.5: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) in the case of $P_* = 10^5$ N, $v_* = 0.25$ m/s and equidistant dampers: $x_1 = 0.25$ m, $x_2 = 0.5$ m and $x_3 = 0.75$ m (distances from the left edge). The shear modulus of the dampers is $2.91 \cdot 10^8$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

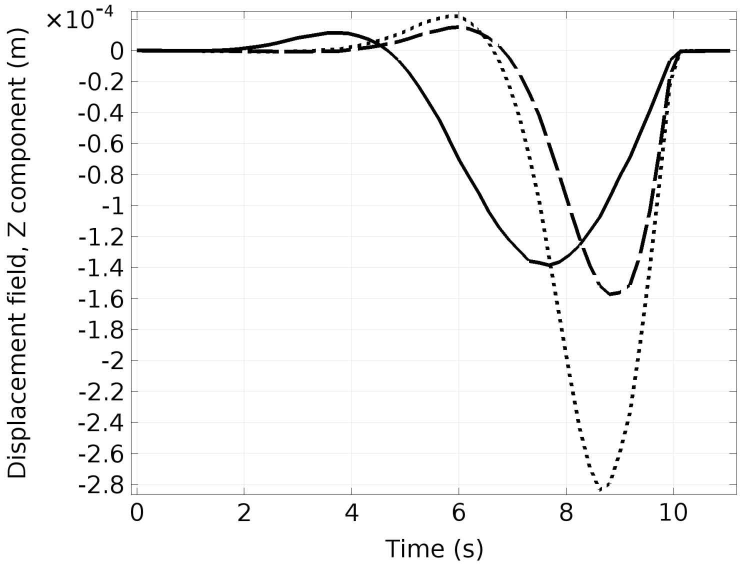

Figure 3.6: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) in the case of $P_* = 2.5 \cdot 10^5$ N, $v_* = 0.1$ m/s and equidistant dampers: $x_1 = 0.25$ m, $x_2 = 0.5$ m and $x_3 = 0.75$ m (distances from the left edge). The shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

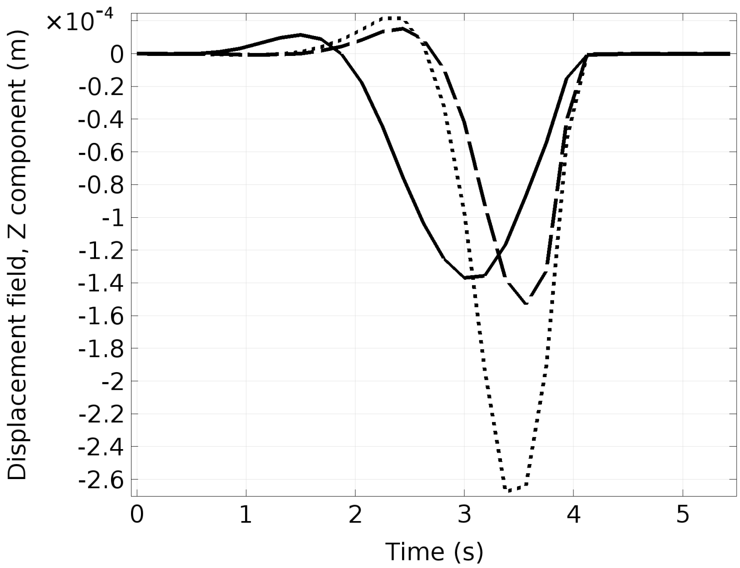

Figure 3.7: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) in the case of $P_* = 2.5 \cdot 10^5$ N, $v_* = 0.25$ m/s and equidistant dampers: $x_1 = 0.25$ m, $x_2 = 0.5$ m and $x_3 = 0.75$ m (distances from the left edge). The shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

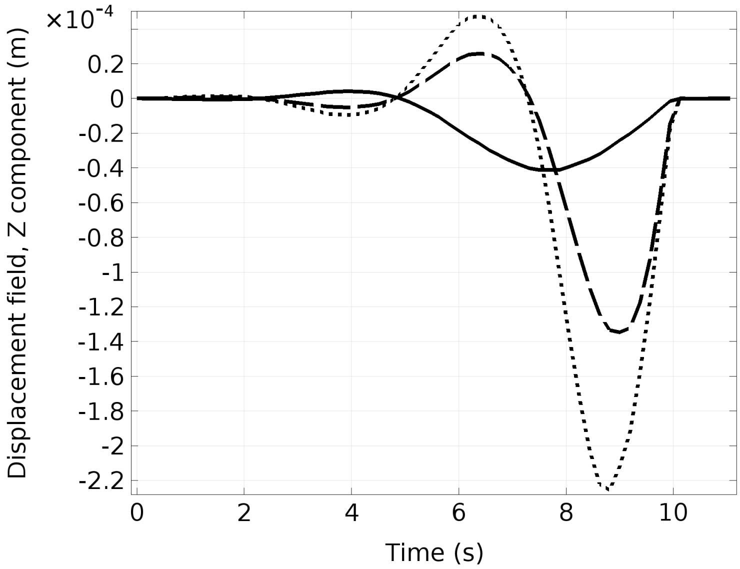

Figure 3.8: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) in the case of $P_* = 10^5$ N, $v_* = 0.1$ m/s and equidistant dampers: $x_1 = 0.25$ m, $x_2 = 0.5$ m and $x_3 = 0.75$ m (distances from the left edge). The shear modulus of the dampers is $2.91 \cdot 10^8$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

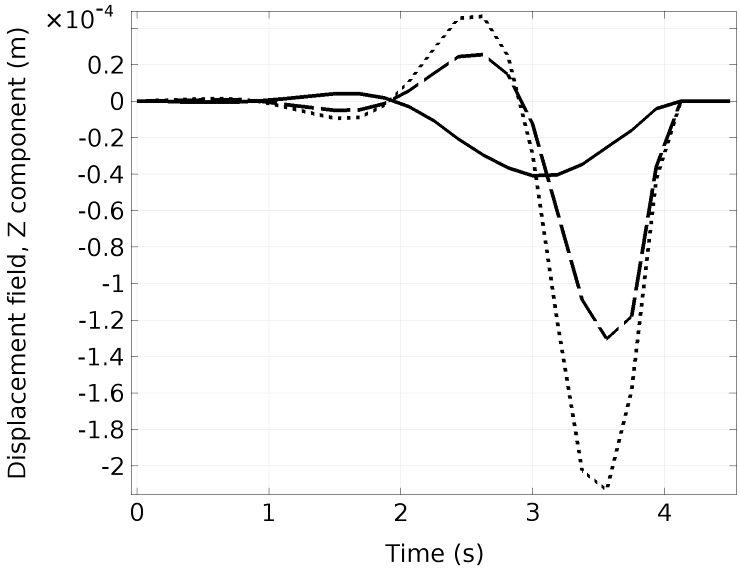

Figure 3.9: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) in the case of $P_* = 10^5$ N, $v_* = 0.25$ m/s and equidistant dampers: $x_1 = 0.25$ m, $x_2 = 0.5$ m and $x_3 = 0.75$ m (distances from the left edge). The shear modulus of the dampers is $2.91 \cdot 10^8$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

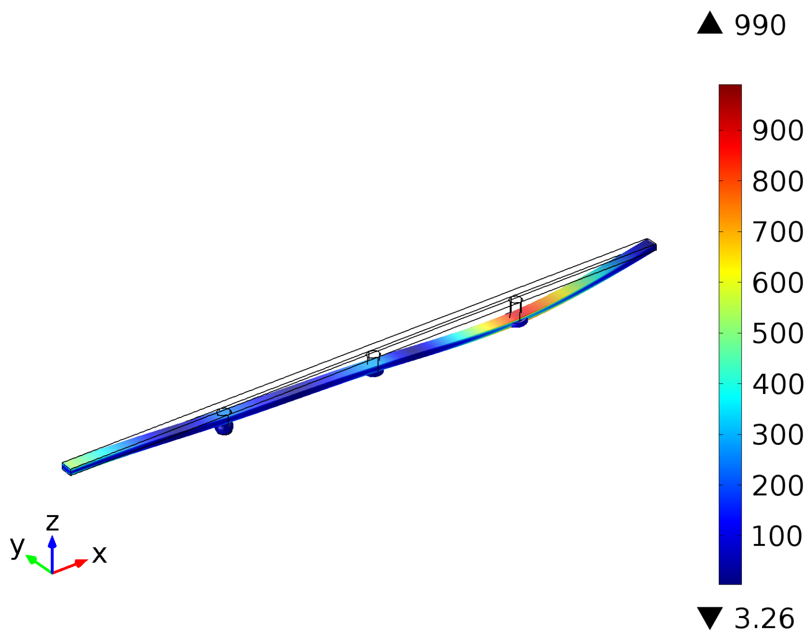

Figure 3.10: The stress state of the beam at $t = 14.25$ s in the case of equidistant dampers. $P_* = 10^5$ N, $v_* = 0.1$ m/s ($\tau_* = 10$ s) and the shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2.

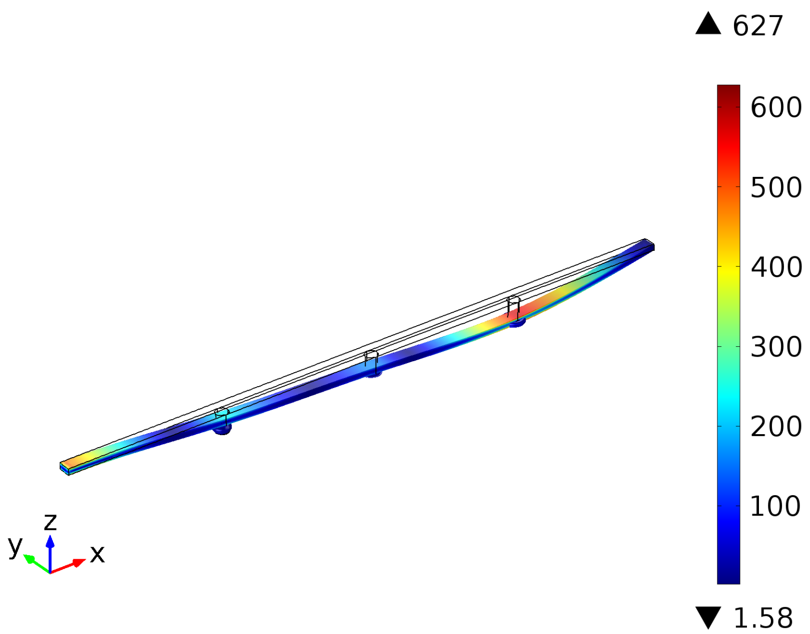

Figure 3.11: The stress state of the beam at $t = 7.5$ s in the case of equidistant dampers. $P_* = 10^5$ N, $v_* = 0.25$ m/s ($\tau_* = 4$ s) and the shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2.

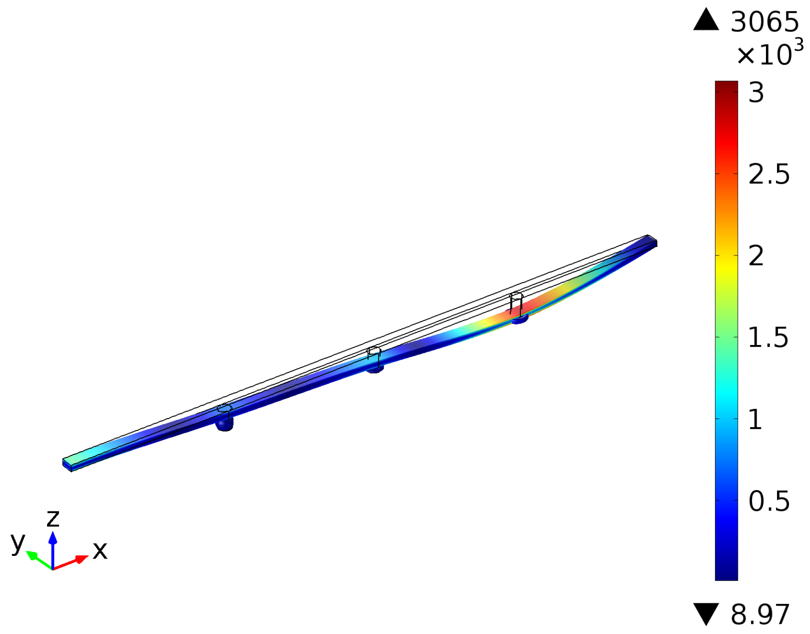

Figure 3.12: The stress state of the beam at $t = 6$ s in the case of equidistant dampers. $P_* = 2.5 \cdot 10^5$ N, $v_* = 0.1$ m/s ($\tau_* = 10$ s) and the shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2.

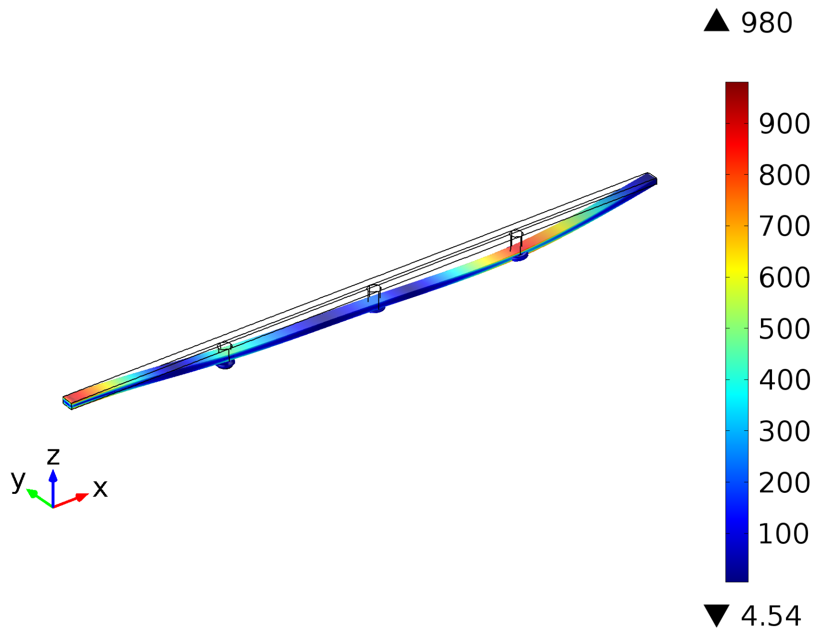

Figure 3.13: The stress state of the beam at $t = 9$ s in the case of equidistant dampers. $P_* = 2.5 \cdot 10^5$ N, $v_* = 0.25$ m/s ($\tau_* = 4$ s) and the shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2.

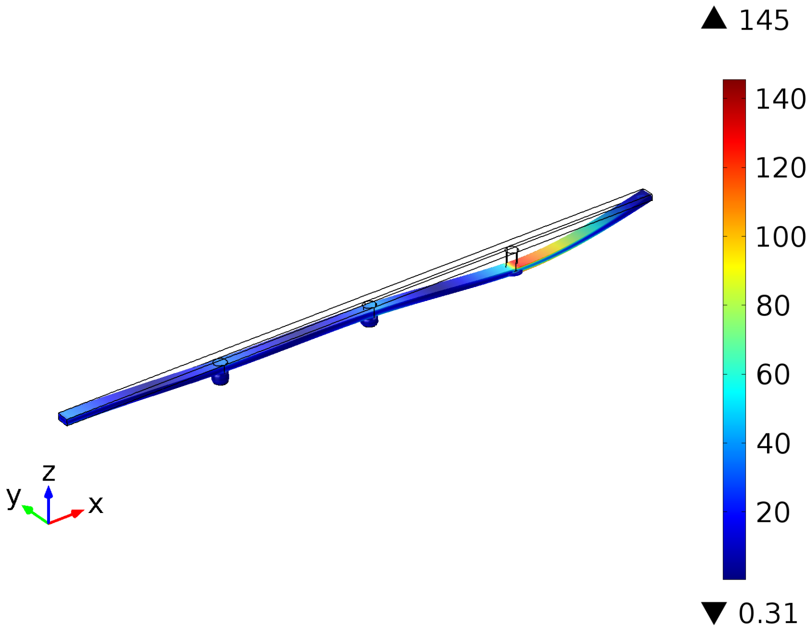

Figure 3.14: The stress state of the beam at $t = 14.25$ s in the case of equidistant dampers. $P_* = 10^5$ N, $v_* = 0.1$ m/s ($\tau_* = 10$ s) and the shear modulus of the dampers is $2.19 \cdot 10^8$ N/m^2.

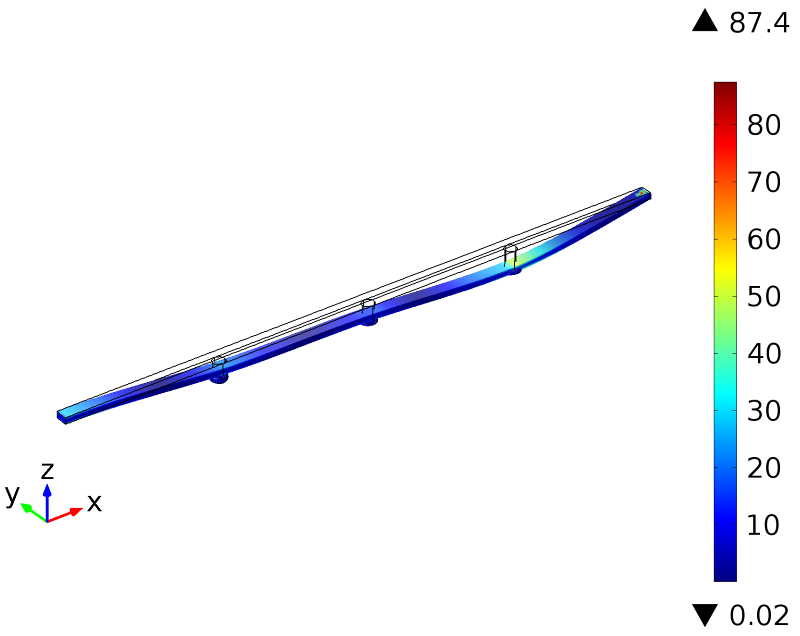

Figure 3.15: The stress state of the beam at $t = 7.5$ s in the case of equidistant dampers. $P_* = 10^5$ N, $v_* = 0.25$ m/s ($\tau_* = 4$ s) and the shear modulus of the dampers is $2.19 \cdot 10^8$ N/m^2.

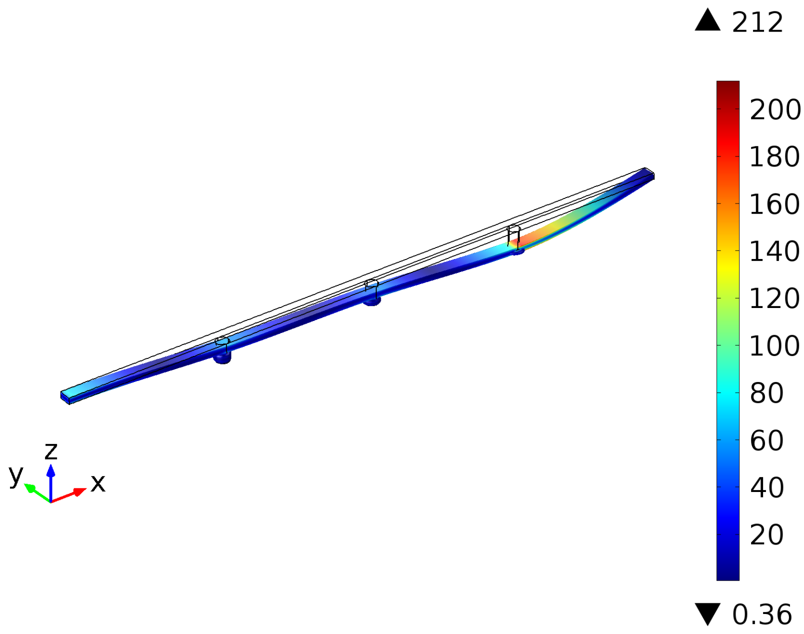

Figure 3.16: The stress state of the beam at $t = 6$ s in the case of equidistant dampers. $P_* = 2.5 \cdot 10^5$ N, $v_* = 0.1$ m/s ($\tau_* = 10$ s) and the shear modulus of the dampers is $2.19 \cdot 10^8$ N/m^2.

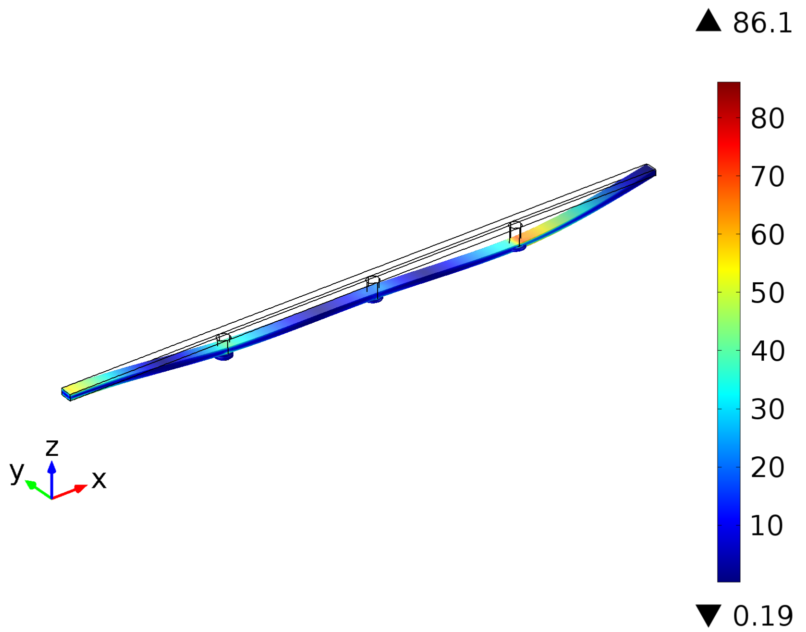

Figure 3.17: The stress state of the beam at $t = 9$ s in the case of equidistant dampers. $P_* = 2.5 \cdot 10^5$ N, $v_* = 0.25$ m/s ($\tau_* = 4$ s) and the shear modulus of the dampers is $2.19 \cdot 10^8$ N/m^2.

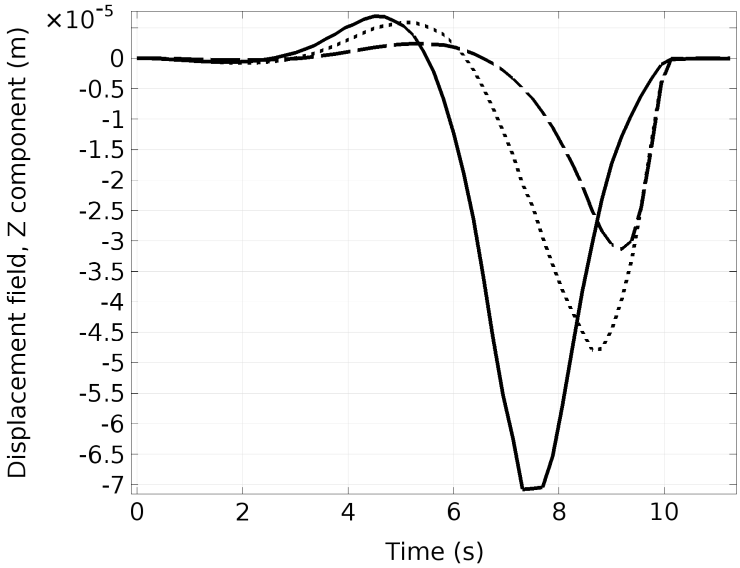

Figure 3.18: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) m in the case of $P_* = 10^5$ N, $v_* = 0.1$ m/s and optimized placements of dampers: $x_1 = 0.35$ m, $x_2 = 0.65$ m and $x_3 = 0.85$ m (distances from the left edge). The shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

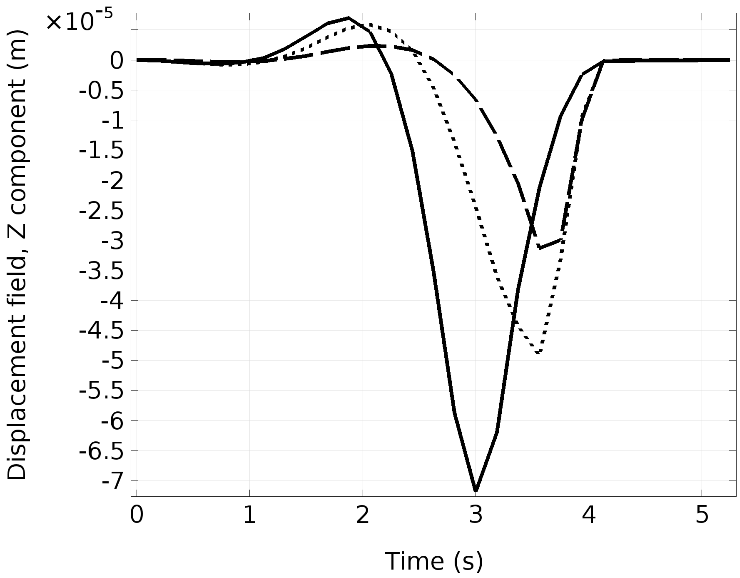

Figure 3.19: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) m in the case of $P_* = 10^5$ N, $v_* = 0.25$ m/s and optimized placements of dampers: $x_1 = 0.35$ m, $x_2 = 0.65$ m and $x_3 = 0.85$ m (distances from the left edge). The shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

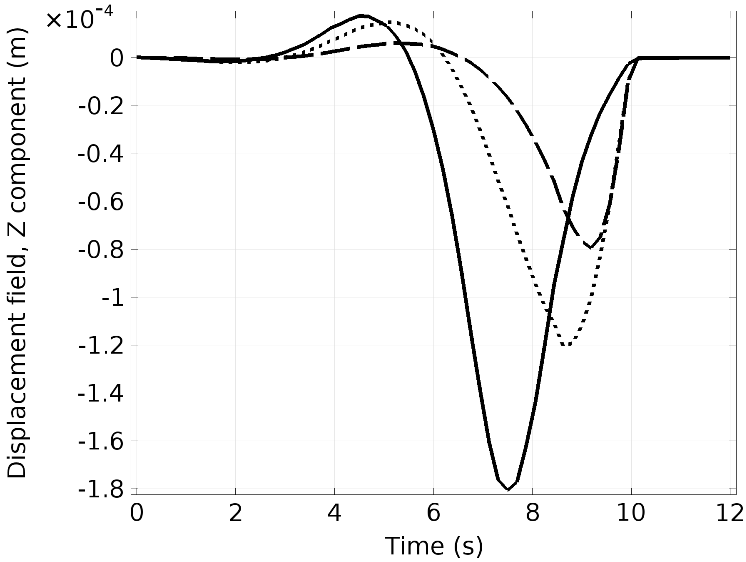

Figure 3.20: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) m in the case of $P_* = 2.5 \cdot 10^5$ N, $v_* = 0.1$ m/s and optimized placements of dampers: $x_1 = 0.35$ m, $x_2 = 0.65$ m and $x_3 = 0.85$ m (distances from the left edge). The shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

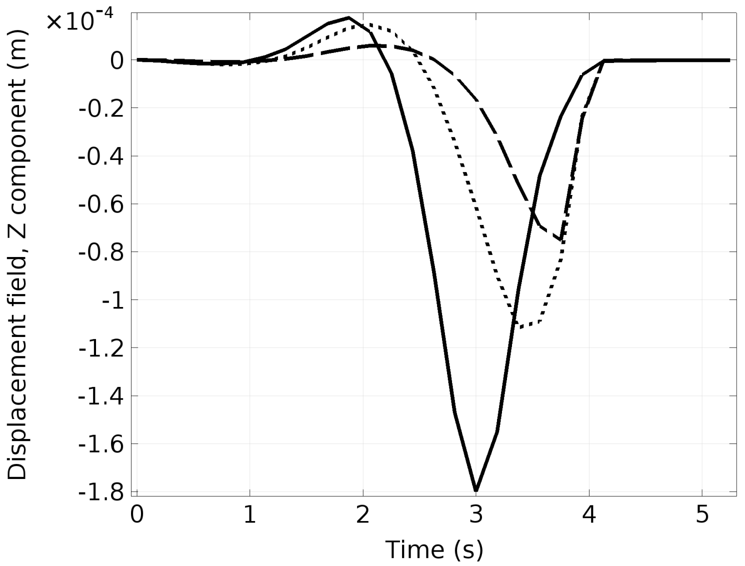

Figure 3.21: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) m in the case of $P_* = 2.5 \cdot 10^5$ N, $v_* = 0.25$ m/s and optimized placements of dampers: $x_1 = 0.35$ m, $x_2 = 0.65$ m and $x_3 = 0.85$ m (distances from the left edge). The shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

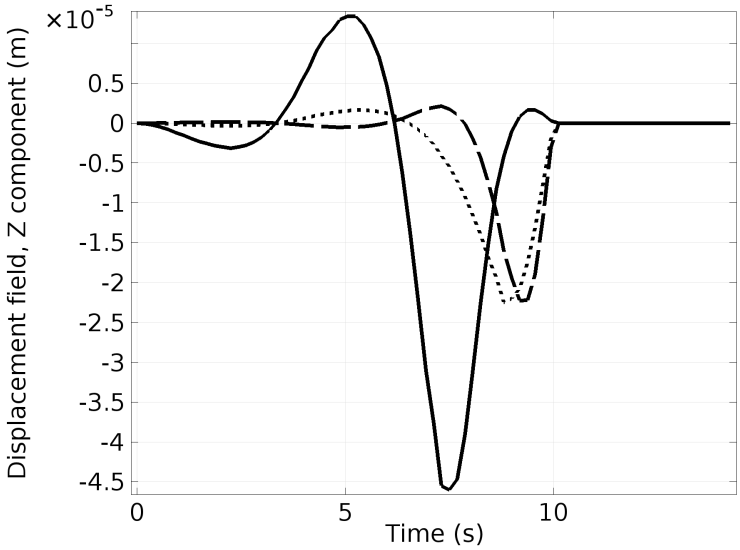

Figure 3.22: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) in the case of $P_* = 10^5$ N, $v_* = 0.1$ m/s and optimized placements of dampers: $x_1 = 0.35$ m, $x_2 = 0.65$ m and $x_3 = 0.85$ m (distances from the left edge). The shear modulus of the dampers is $2.91 \cdot 10^8$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

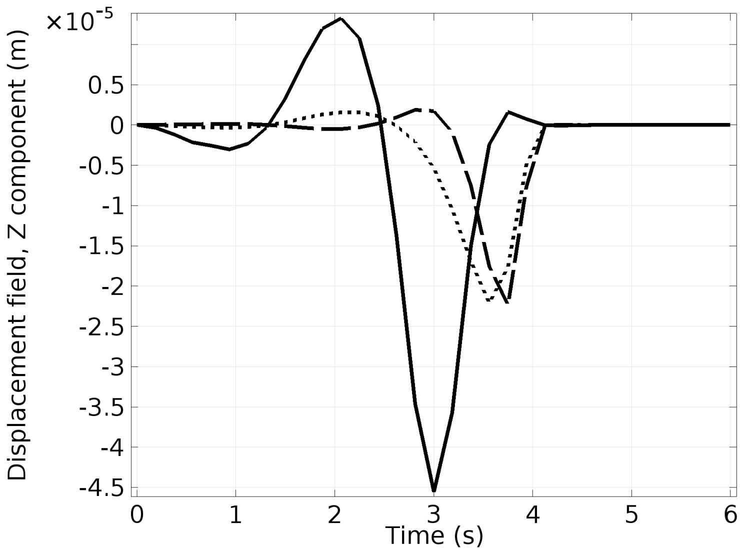

Figure 3.23: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) in the case of $P_* = 10^5$ N, $v_* = 0.25$ m/s and optimized placements of dampers: $x_1 = 0.35$ m, $x_2 = 0.65$ m and $x_3 = 0.85$ m (distances from the left edge). The shear modulus of the dampers is $2.91 \cdot 10^8$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

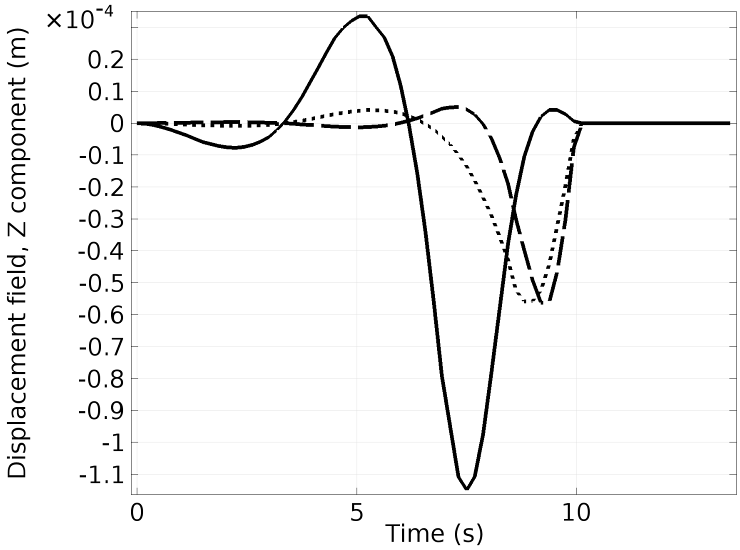

Figure 3.24: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) in the case of $P_* = 2.5 \cdot 10^5$ N, $v_* = 0.1$ m/s and optimized placements of dampers: $x_1 = 0.35$ m, $x_2 = 0.65$ m and $x_3 = 0.85$ m (distances from the left edge). The shear modulus of the dampers is $2.91 \cdot 10^8$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

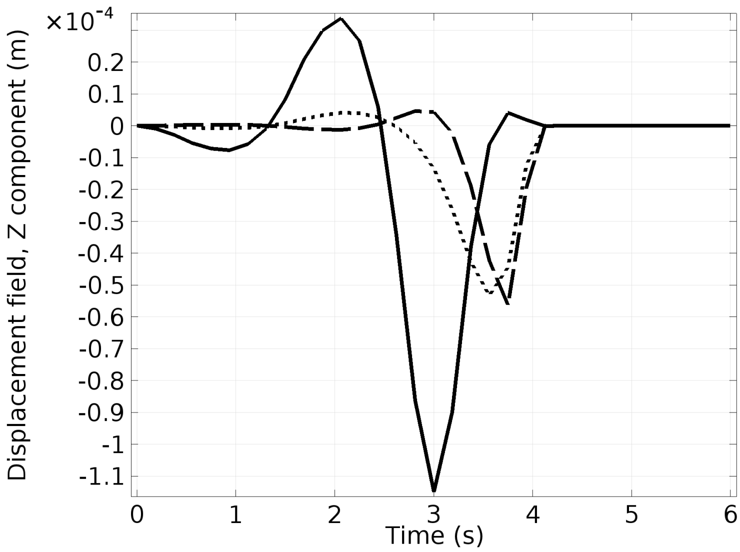

Figure 3.25: The displacement of the points $\xi_1 = 0.65$ m (solid), $\xi_2 = 0.875$ m (dotted) and $\xi_3 = 0.95$ m (dashed) in the case of $P_* = 2.5 \cdot 10^5$ N, $v_* = 0.25$ m/s and optimized placements of dampers: $x_1 = 0.35$ m, $x_2 = 0.65$ m and $x_3 = 0.85$ m (distances from the left edge). The shear modulus of the dampers is $2.91 \cdot 10^8$ N/m^2 and the bulk modulus is $4 \cdot 10^8$ N/m^2.

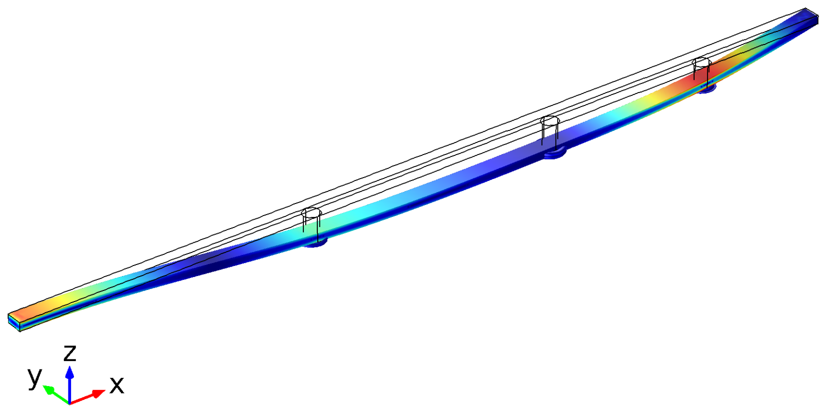

Figure 3.26: The stress state of the beam at $t = 7.5$ s in the case of the dampers optimized positions. $P_* = 10^5$ N, $v_* = 0.25$ m/s ($\tau_* = 4$ s) and the shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2.

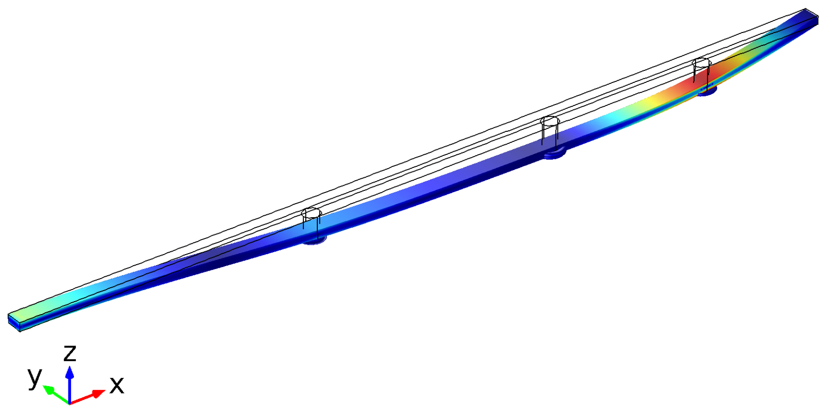

Figure 3.27: The stress state of the beam at $t = 14.25$ s in the case of the dampers optimized positions. $P_* = 2.5 \cdot 10^5$ N, $v_* = 0.1$ m/s ($\tau_* = 10$ s) and the shear modulus of the dampers is $6.16 \cdot 10^7$ N/m^2.

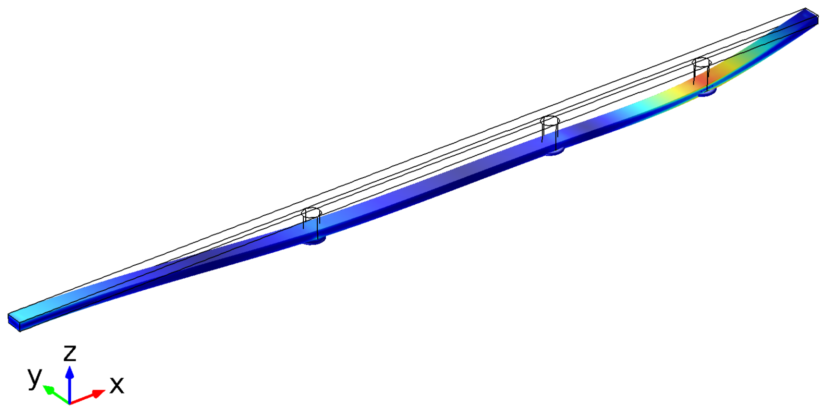

Figure 3.28: The stress state of the beam at $t = 6.375$ s in the case of the dampers optimized positions. $P_* = 10^5$ N,$v_* = 0.25$ m/s ($\tau_* = 4$ s) and the shear modulus of the dampers is $2.91 \cdot 10^8$ N/m^2.

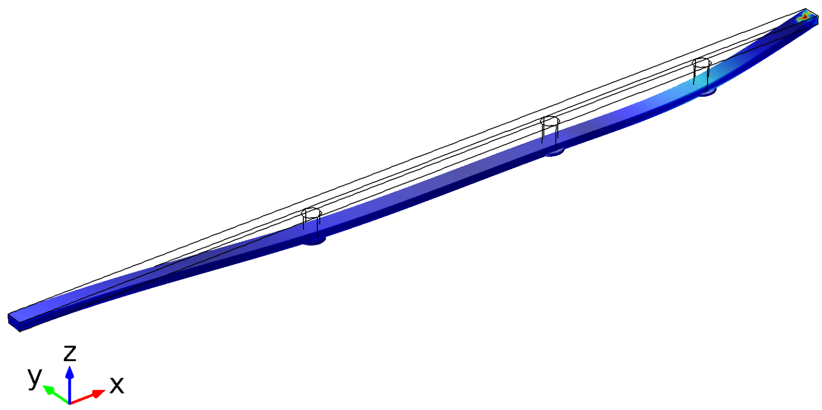

Figure 3.29: The stress state of the beam at $t = 14.5$ s in the case of the dampers optimized positions. $P_* = 2.5 \cdot 10^5$ N,$v_* = 0.1$ m/s ($\tau_* = 10$ s) and the shear modulus of the dampers is $2.91 \cdot 10^8$ N/m^2.

$P_* \cdot 10^5$ [N]	v_* [m/s]	$\max_{(x_*,t_*)} \lvert w_*(x_*,t_*)\rvert \cdot 10^{-4}$ [m]	
1	0.1	1.15	0.91
1	0.15	1.13	0.893
1	0.2	1.13	0.896
1	0.25	1.12	0.97
1.5	0.1	1.72	1.36
1.5	0.15	1.68	1.34
1.5	0.2	1.72	1.36
1.5	0.25	1.64	1.28
2	0.1	2.29	1.82
2	0.15	2.24	1.79
2	0.2	2.29	1.8
2	0.25	2.20	1.72
2.5	0.1	2.86	2.27
2.5	0.15	2.81	2.21
2.5	0.2	2.87	2.26
2.5	0.25	2.75	2.15

Table 3.2: The maximal absolute values of the vertical displacements of the beam with equidistant dampers: $x_1 = 0.25$ m, $x_2 = 0.5$ m and $x_3 = 0.75$ m (distances from the left edge). The shear modulus of the dampers are $6.16 \cdot 10^7$ N/m^2 (the first column) and $2.91 \cdot 10^8$ N/m^2 (the second column) the bulk modulus is $4 \cdot 10^8$ N/m^2.

$P_* \cdot 10^5$ [N]	v_* [m/s]	$\max_{(x_*,t_*)} \lvert w_*(x_*,t_*) \rvert \cdot 10^{-4}$ [m]	
1	0.1	0.73	0.58
1	0.15	0.72	0.58
1	0.2	0.72	0.57
1	0.25	0.71	0.57
1.5	0.1	1.09	0.87
1.5	0.15	1.09	0.87
1.5	0.2	1.08	0.85
1.5	0.25	1.08	0.85
2	0.1	1.46	0.825
2	0.15	1.46	0.825
2	0.2	1.44	0.82
2	0.25	1.44	0.82
2.5	0.1	1.82	1.445
2.5	0.15	1.82	1.44
2.5	0.2	1.8	1.44
2.5	0.25	1.8	1.37

Table 3.3: The maximal absolute values of the vertical displacements of the beam with optimized placements for dampers: $x_1 = 0.35$ m, $x_2 = 0.65$ m and $x_3 = 0.85$ m (distances from the left edge). The shear modulus of the dampers are $6.16 \cdot 10^7$ N/m^2 (the first column) and $2.91 \cdot 10^8$ N/m^2 (the second column) the bulk modulus is $4 \cdot 10^8$ N/m^2.

3.2 Topology Optimization for Elastic Base under Rectangular Elastic Plate Subjected to Moving Load

In this section we are going to extend the technique described in the beginning of this section for a two–dimensional system. For this purpose we will consider an elastic, isotropic, solid rectangular plate of sufficiently small constant thickness $2h$, the middle plane of which in Cartesian system of coordinates occupies the domain

$$\mathcal{O}_* = \{(x_*, y_*);\ x_* \in [-l_1, l_1],\ y_* \in [-l_2, l_2]\} \subset \mathbb{R}^2, \quad \min(l_1, l_2) \gg 2h.$$

Let the plate lies on a linear–elastic one–parametric base, distributed in domain $\mathcal{O}_*$ by a controlled law. The plate is supposed to be simply supported by edges $x_* = \pm l_1$ and $y_* = \pm l_2$, as well as subjected to a constant normal load P_* distributed on the upper surface of the plate by given law $r_* = r_*(y_*)$, supp $r_* \neq \varnothing$, and moving along the upper surface of the plate in x_* direction with constant speed v_*. This section is based one the results of [60, 74].

Implying the same considerations as in last section, we suppose that the load detaches from the plate at a given moment τ_*, so $v_*\tau_* = 2l_1$. In the framework of the Kirchhoff hypothesis, the vertical displacements of the plate middle plane satisfies

$$D\Delta\Delta w_*(x_*, y_*, t_*) + \alpha_*^2 u(x_*, y_*) w_*(x_*, y_*, t_*) + 2\rho h \frac{\partial^2 w_*(x_*, y_*, t_*)}{\partial t_*^2} = f_*(x_*, y_*, t_*), \tag{3.34}$$

$$(x_*, y_*, t_*) \in \mathcal{O}_* \times (0, T),$$

and boundary conditions of simply support

$$\begin{aligned} w_*(\pm l_1, y_*, t_*) = \left.\frac{\partial^2 w_*(x_*, y_*, t_*)}{\partial x_*^2}\right|_{x_* = \pm l_1} &= 0, \qquad (y_*, t_*) \in [-l_2, l_2] \times [0, T], \\ w_*(x_*, \pm l_2, t_*) = \left.\frac{\partial^2 w_*(x_*, y_*, t_*)}{\partial y_*^2}\right|_{y_* = \pm l_2} &= 0, \qquad (x_*, t_*) \in [-l_1, l_1] \times [0, T]. \end{aligned} \tag{3.35}$$

Above u is the intensity of the elastic base distribution in $\mathcal{O}_*$, $f_*(x_*, y_*, t_*)$ characterizes the impact of the moving load on the plate and therefore $0 \leq u$:

$$f_*(x_*, y_*, t_*) = P_* \mathcal{A}_{[0, \tau_*]}[\delta(x_* + l_1 - v_* t_*)] r_*(y_*). \tag{3.36}$$

The initial state of the plate is known:

$$w_*(x_*, y_*, 0) = w_{0*}(x_*, y_*), \quad \left.\frac{\partial w_*(x_*, y_*, t_*)}{\partial t_*}\right|_{t_* = 0} = w_{0*}^1(x_*, y_*), \quad (x_*, y_*) \in \overline{\mathcal{O}}_*. \tag{3.37}$$

Our main purpose is the determination of an admissible control

$$u^o \in \mathcal{U}_\infty = \{u \in L^\infty\left(\overline{\mathcal{O}}_*;\mathbb{R}^2\right); \text{supp } u = \mathcal{O}_*,\ 0 \le u \le 1\}$$

providing the following terminal state in required (fixed) time T:

$$w_*(x_*,y_*,T) = 0, \quad \left.\frac{\partial w_*(x_*,y_*,t_*)}{\partial t_*}\right|_{t_*=T} = 0, \quad (x_*,y_*) \in \overline{\mathcal{O}}_*, \tag{3.38}$$

as well as minimizing the functional

$$\kappa_\infty[u] = \max_{(x_*,y_*)\in\mathcal{O}_*} u(x_*,y_*), \qquad u \in \mathcal{U}, \tag{3.39}$$

which describes the intensity of the elastic base distribution under the plate.

Remark 3.4 *i) It is obvious that the plate vibrations may "vanish" solely after the load detaches from it. Thence restrict ourselves by condition $\tau_* < T \le T_0$ where $v_*\tau_* = 2l_1$, and T_0 is the time in which the vibrations would "vanish" in the case of $u \equiv 1$.*

ii) We realize that, in general, terminal conditions (3.38) may not be provided by any choice of distribution u (even in the case $u \equiv 1$) exactly, because there will remain some residual stresses of very small amplitude inversely proportional to the base stiffness. Thus, we aim to ensure (3.38) approximately [42].

Let us introduce the dimensionless variables and functions

$$w = \frac{w_*}{h},\ r = \frac{r_*}{l_2},\ x = \frac{x_*}{l_1},\ y = \frac{y_*}{d},\ t = \frac{2t_* - T}{T}\vartheta,\ \tau = \frac{2\vartheta}{T}\tau_*,\ v_* = \frac{v}{l_1}\frac{T}{2\vartheta},$$

$$\alpha^2 = \frac{\alpha_*^2 l_1^4}{D},\ \beta^2 = 2\rho h\frac{l_1^4}{D}\left(\frac{2\vartheta}{T}\right)^2,\ \gamma = \frac{l_1}{l_2},\ P_* = \frac{Pdl_1^3}{Dh}.$$

Then, from (3.34), (3.35) we will respectively obtain

$$\mathcal{D}[w] + \alpha^2 u(x,y)w(x,y,t) + \beta^2\frac{\partial^2 w(x,y,t)}{\partial t^2} = f(x,y,t), \quad (x,y,t) \in \mathcal{O}\times(-\vartheta,\vartheta),$$

$$w(\pm 1,y,t) = \left.\frac{\partial^2 w(x,y,t)}{\partial x^2}\right|_{x=\pm 1} = 0, \qquad (y,t) \in [-1,1]\times[-\vartheta,\vartheta],$$

$$w(x,\pm 1,t) = \left.\frac{\partial^2 w(x,y,t)}{\partial y^2}\right|_{y=\pm 1} = 0, \qquad (x,t) \in [-1,1]\times[-\vartheta,\vartheta],$$

$$\mathcal{D}[w] = \frac{\partial^4 w(x,y,t)}{\partial x^4} + 2\gamma^2\frac{\partial^4 w(x,y,t)}{\partial x^2\partial y^2} + \gamma^4\frac{\partial^4 w(x,y,t)}{\partial y^4},\ \mathcal{O} := (-1,1)\times(-1,1).$$

As a result of the variables change, the expression (3.36) is transformed to

$$f(x,y,t) = P\mathcal{A}_{[-\vartheta,\vartheta-\tau]}\left[\delta(x+1-v(t+\vartheta))\right]r(y).$$

Repeating the same procedure as in previous section, with respect to the coefficients of the approximation

$$\overline{w}_{1n}(x,y,\sigma) = \sum_{m,n=1}^{N} \overline{\varepsilon}_{mn}(\sigma)\sin(\pi m x)\sin(\pi n y), \quad (x,y,\sigma) \in \overline{\mathcal{O}}\times\mathbb{R}, \tag{3.40}$$

we will derive the following system of linear algebraic equations for $\overline{\varepsilon}_{mn}(\sigma)$:

$$\sum_{m,n=1}^{N} \Lambda_{mn}^{\mu\nu}(\sigma)\overline{\varepsilon}_{mn}(\sigma) = \Omega_{\mu\nu}(\sigma), \quad \mu,\nu \in \{1;N\},$$

$$\Lambda_{mn}^{\mu\nu}(\sigma) = \Gamma_{mn}\delta_{mn}^{\mu\nu} + \alpha^2 J_{mn}^{\mu\nu}[u], \quad \Gamma_{mn} = \left[(\pi m)^2 + \gamma^2(\pi n)^2\right]^2 - \sigma^2\beta^2,$$

$$\overline{G}(x,y,\sigma) = \overline{f}_1(x,y,\sigma) + \beta^2[\dot{w}_{01}(x,y) - i\sigma w_{01}(x,y)],$$

$$\overline{f}_1(x,y,\sigma) = \frac{P}{v}\left[\theta(x+1)-\theta(x-1)\right]\exp\left[i\sigma\left(\frac{x+1}{v}+\vartheta\right)\right]r(y),$$

$$\Omega_{\mu\nu}(\sigma) = \int_{-1}^{1}\int_{-1}^{1}\overline{G}(x,y,\sigma)\sin(\pi\mu x)\sin(\pi\nu y)dxdy =$$

$$= Y_\mu(\sigma)\int_{-1}^{1} r(y)\sin(\pi\nu y)dxdy + \beta^2\left[K_{\mu\nu} - i\sigma L_{\mu\nu}\right],$$

$$Y_\mu(\sigma) = \frac{(-1)^{\mu+1}\pi\mu\cdot Pv}{\sigma^2+(\pi\mu v)^2}\left[1-\exp\left[\frac{2i\sigma}{v}\right]\right]\exp\left[i\sigma\vartheta\right],$$

$$K_{\mu\nu} = \int_{-1}^{1}\int_{-1}^{1} w_{01}^{1}(x,y)\sin(\pi\mu x)\sin(\pi\nu y)dxdy,$$

$$L_{\mu\nu} = \int_{-1}^{1}\int_{-1}^{1} w_{01}(x,y)\sin(\pi\mu x)\sin(\pi\nu y)dxdy,$$

$$J_{mn}^{\mu\nu}[u] = \int_{-1}^{1}\int_{-1}^{1} u(x,y)\sin(\pi m x)\sin(\pi\mu x)\sin(\pi n y)\sin(\pi\nu y)dxdy.$$

We have taken into account, that $v\tau = 2$. It is obvious, that $J_{mn}^{mn}[u] \geq 0$, $J_{mn}^{\mu\nu}[u] = J_{nm}^{\nu\mu}[u]$, $m,n,\mu,\nu \in \{1;N\}$.

Remark 3.5 *Since the Chebyshev polynomials of the first kind* $\{\mathrm{T}_m(x)\mathrm{T}_n(y)\}_{m,n\in\mathbb{N}}$ *are orthogonal in* $\mathcal{O}$ *and provide more accuracy of approximation compared with that by trigonometric system* $\{\sin(\pi m x)\sin(\pi n y)\}_{m,n\in\mathbb{N}}$ *[91], it is more efficient to use them instead. But we take into account, that the aim is to show that the proposed algorithm works in dimensions two, and not the accuracy of the approximation.*

Consequently, we will derive the system

$$J^{\mu\nu}_{mn}[u] = \mathcal{M}^{\mu\nu}_{mn}, \quad m, n, \mu, \nu \in \{1; 2N\},$$

which follows from $\Delta_1(z) = 0$, as long as $\Delta_0(z) = 0$, where Δ_0 and Δ_{11} are the main and the first auxiliary determinants of the system.

Remark 3.6 *i) It is easy to see from the expression of $\Lambda^{\mu\nu}_{mn}$, the determinant $\Delta_0(\sigma)$ is a polynomial of degree $2N$, therefore $\Delta_0(z) = 0$ admits $2N$ complex roots. From the other hand side* $\text{Re}\ \Delta_0(-\sigma - i\varsigma) = \text{Re}\ \Delta_0(\sigma + i\varsigma)$, $\text{Im}\ \Delta_0(-\sigma - i\varsigma) = \text{Im}\ \Delta_0(\sigma + i\varsigma)$, *which means that $z_k = \sigma_k + i\varsigma_k$ and $z_k = -\sigma_k - i\varsigma_k$ satisfy $\Delta_0(z) = 0$ simultaneously. Then, since* $\text{Re}\ \Delta_{11}(-\sigma - i\varsigma) = \text{Re}\ \Delta_{11}(\sigma + i\varsigma)$, $\text{Im}\ \Delta_{11}(-\sigma - i\varsigma) = \text{Im}\ \Delta_{11}(\sigma + i\varsigma)$, *we conclude that $\Delta_{11}(z) = 0$ contains only N independent constraints.*

ii) Moreover, even though $\text{Re}\ \Delta_0(-\sigma + i\varsigma) = \text{Re}\ \Delta_0(\sigma - i\varsigma) = \text{Re}\ \Delta_0(\sigma + i\varsigma)$, $\text{Im}\ \Delta_0(-\sigma + i\varsigma) = \text{Im}\ \Delta_0(\sigma - i\varsigma) = -\text{Im}\ \Delta_0(\sigma + i\varsigma)$, *nevertheless $\Delta_{11}(\sigma + i\varsigma)$ does not have the same property, therefore the number of independent constraints cannot be reduced further.*

According to Subsection 1.3.1, the solution will be

$$u^o(x, y) = \sum_{j=1}^{J} \left[\theta\left(x - x^o_j, y - y^o_j\right) - \theta\left(x - x^o_{j+1}, y - y^o_{j+1}\right)\right], \quad (x, y) \in \mathcal{O}.$$

The obtained solution describes the distribution law of the elastic base under the plate, and the set of points $\{x^o_j, y^o_j\}^J_{j=1} \in \mathcal{O}$ underlines the domains where the base exists and depends on inner and external parameters $D, r(y), v, \tau, P, \alpha^2, \beta^2, \gamma$.

Applying the Fourier inverse transform to (3.40) and repeating the arguments made in the last subsection, we will obtain

$$w_{1N}(x, y, t) = \sum_{m,n=1}^{N} \varepsilon_{mn}(t) \sin(\pi m x) \sin(\pi n y), \quad (x, y, t) \in \overline{\mathcal{O}} \times \mathbb{R},$$

$$\varepsilon_{mn}(t) = \frac{1}{\pi} \int_0^\infty \left[\text{Re}\ \overline{\varepsilon}_{mn}(\sigma) \cos(\sigma t) + \text{Im}\ \overline{\varepsilon}_{mn}(\sigma) \sin(\sigma t)\right] d\sigma = \mathcal{F}_c^{-1}[\text{Re}\ \overline{\varepsilon}_{mn}] + \mathcal{F}_s^{-1}[\text{Im}\ \overline{\varepsilon}_{mn}],$$

$$\text{Re}\ \overline{\varepsilon}_{mn}(\sigma) = \frac{\text{Re}\ \Delta_m(\sigma)\text{Re}\ \Delta_0(\sigma) + \text{Im}\ \Delta_m(\sigma)\text{Im}\ \Delta_0(\sigma)}{(\text{Re}\ \Delta_0(\sigma))^2 + (\text{Im}\ \Delta_0(\sigma))^2},$$

$$\text{Im}\ \overline{\varepsilon}_{mn}(\sigma) = -\frac{\text{Re}\ \Delta_m(\sigma)\text{Im}\ \Delta_0(\sigma) - \text{Im}\ \Delta_m(\sigma)\text{Re}\ \Delta_0(\sigma)}{(\text{Re}\ \Delta_0(\sigma))^2 + (\text{Im}\ \Delta_0(\sigma))^2}.$$

To demonstrate algorithm we consider an example when $N = 3$. Let

$$w_0(x, y) = \sin(\pi x)\sin(\pi y), \quad w_0^1(x, y) = 0.$$

Let the moving load has a point support: $r(y) = \delta(y - y_0)$, $y_0 \in (-1, 1)$. Then,

$$J_{mn}^{\mu\nu}[u] = \sum_{j=1}^{J} \int_{x_j^o}^{x_{j+1}^o} \sin(\pi m x)\sin(\pi\mu x)dx \int_{y_j^o}^{y_{j+1}^o} \sin(\pi n y)\sin(\pi\nu y)dy,$$

$$\Omega_{\mu\nu}(z) = \frac{(-1)^{\mu+1}\pi\mu \cdot vP}{z^2 + (\pi\mu v)^2}\left[1 - \exp\left[\frac{2iz}{v}\right]\right]\exp\left[iz\vartheta\right]\cdot\sin(\pi\nu y_0) - iz\beta^2\delta_1^\mu\delta_1^\nu,\ K_{\mu\nu} = 0,\ L_{\mu\nu} = \delta_{11}^{\mu\nu}.$$

Only numerical values of $\mathcal{M}_{mn}^{\mu\nu}$ have been computed. The results are combined in Figures 3.30 and 3.31 and in Tables 3.4–3.6.

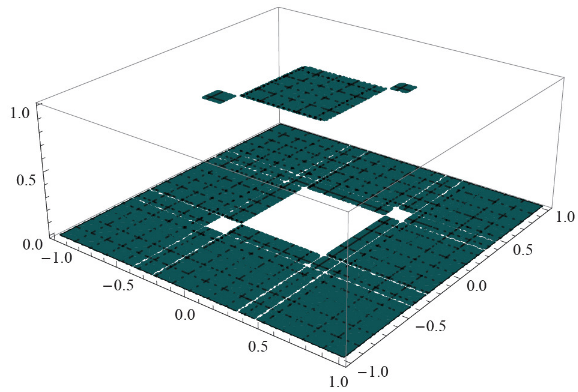

Figure 3.30: u^o when $\beta^2 = 50, \gamma^2 = 0.25, \alpha^2 = 0.05, P = 0.5, v = 0.1$ and $y_0 = 0.25$.

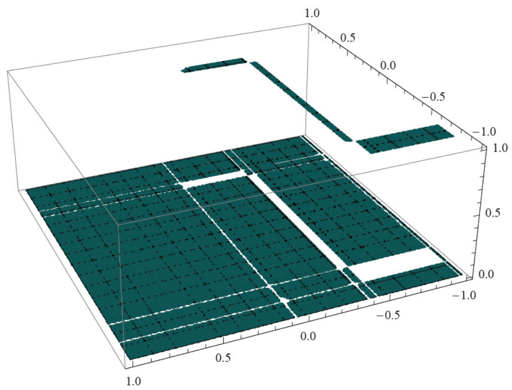

Figure 3.31: u^o when $\beta^2 = 25, \gamma^2 = 1, \alpha^2 = 0.01, P = 0.5, v = 0.05$ and $y_0 = -0.75$.

α^2	P	v	y_0	$x_j^o,\ y_j^o$
0.01	0.1	0.01	–0.25	$x_1^o = 0.1973, x_2^o = 0.2650, x_3^o = 0.2691, x_4^o = 0.3355,\ y_1^o = 0.3307, y_2^o = 0.4931, y_3^o = 0.5330, y_4^o = 0.6699$
0.01	0.5	0.05	0.5	$x_1^o = -0.4444, x_2^o = -0.2989, x_3^o = 0.3074, x_4^o = 0.4349,\ y_1^o = -0.4444, y_2^o = -0.2989, y_3^o = 0.3074, y_4^o = 0.4349$
0.05	0.1	0.01	0.25	$x_1^o = -0.7264, x_2^o = -0.5139, x_3^o = -0.0475, x_4^o = 0.0969,\ y_1^o = -0.7809, y_2^o = -0.4936, y_3^o = -0.0077, y_4^o = 0.1864$
0.05	0.5	0.1	0.25	$x_1^o = -0.4167, x_2^o = -0.2703, x_3^o = 0.3270, x_4^o = 0.4466,\ y_1^o = -0.4167, y_2^o = -0.2703, y_3^o = 0.3270, y_4^o = 0.4466$
0.1	0.1	0.1	–0.5	$x_1^o = -0.7557, x_2^o = 0.1406, x_3^o = 0.5143, x_4^o = 0.7555,\ y_1^o = -0.5619, y_2^o = 0.0554, y_3^o = 0.4113, y_4^o = 0.9799$
0.1	0.5	0.1	0.5	$x_1^o = -0.4371, x_2^o = -0.2963, x_3^o = 0.3042, x_4^o = 0.4282,\ y_1^o = -0.4371, y_2^o = -0.2963, y_3^o = 0.3042, y_4^o = 0.4282$

Table 3.4: The switching points for $\beta^2 = 50$, $\gamma^2 = 0.25$. The base thus exists in $\bigcup_{j=1}^{4} [x_j^o, x_{j+1}^o] \times [y_j^o, y_{j+1}^o]$.

α^2	P	v	y_0	$x_j^o,\ y_j^o$
0.01	0.1	0.01	0.25	$x_1^o = -0.8376, x_2^o = -0.6615, x_3^o = 0.6087, x_4^o = 0.6897,\ y_1^o = -0.9815, y_2^o = -0.4452, y_3^o = -0.3912, y_4^o = 0.0556$
0.01	0.5	0.05	-0.75	$x_1^o = -0.4146, x_2^o = -0.2677, x_3^o = 0.3283, x_4^o = 0.4349,\ y_1^o = -0.4146, y_2^o = -0.2677, y_3^o = 0.3283, y_4^o = 0.4477$
0.05	0.1	0.01	–0.1	$x_1^o = -0.2748, x_2^o = -0.2709, x_3^o = 0.5555, x_4^o = 0.7504,\ y_1^o = -0.3973, y_2^o = -0.2035, y_3^o = 0.2735, y_4^o = 0.3567$
0.05	0.5	0.1	0.1	$x_1^o = -0.4421, x_2^o = -0.2965, x_3^o = 0.2926, x_4^o = 0.4228,\ y_1^o = -0.4421, y_2^o = -0.2965, y_3^o = 0.2926, y_4^o = 0.4228$
0.1	0.1	0.05	0.4	$x_1^o = -0.2065, x_2^o = -0.1324, x_3^o = 0.4080, x_4^o = 0.4781,\ y_1^o = -0.2984, y_2^o = -0.2968, y_3^o = 0.3931, y_4^o = 0.3943$
0.1	0.5	0.1	0.1	$x_1^o = -0.4211, x_2^o = -0.2759, x_3^o = 0.3234, x_4^o = 0.4437,\ y_1^o = -0.4211, y_2^o = -0.2759, y_3^o = 0.3233, y_4^o = 0.4437$

Table 3.5: The switching points for $\beta^2 = 25$, $\gamma^2 = 1$. The base thus exists in $\bigcup_{j=1}^{4} [x_j^o, x_{j+1}^o] \times [y_j^o, y_{j+1}^o]$.

α^2	P	v	y_0	$x_j^o,\ y_j^o$
0.01	0.1	0.01	0.25	$x_1^o = -0.4258, x_2^o = -0.2114, x_3^o = 0.3054, x_4^o = 0.4408,\ y_1^o = -0.3881, y_2^o = -0.2724, y_3^o = 0.4269, y_4^o = 0.4270$
0.01	0.5	0.05	-0.75	$x_1^o = -0.4389, x_2^o = -0.2925, x_3^o = 0.3025, x_4^o = 0.4308,\ y_1^o = -0.4146, y_2^o = -0.2925, y_3^o = 0.3025, y_4^o = 0.4308$
0.05	0.1	0.05	–0.4	$x_1^o = -0.4360, x_2^o = -0.2888, x_3^o = 0.3136, x_4^o = 0.4399,\ y_1^o = -0.4360, y_2^o = -0.2888, y_3^o = 0.3136, y_4^o = 0.4399$
0.05	0.5	0.1	0.4	$x_1^o = -0.4105, x_2^o = -0.1466, x_3^o = 0.4508, x_4^o = 0.5467,\ y_1^o = -0.3583, y_2^o = -0.3567, y_3^o = 0.4778, y_4^o = 0.5478$
0.1	0.1	0.01	0.25	$x_1^o = -0.4456, x_2^o = -0.2521, x_3^o = 0.4542, x_4^o = 0.4566,\ y_1^o = -0.3596, y_2^o = -0.2483, y_3^o = 0.2450, y_4^o = 0.46$
0.1	0.5	0.1	–0.25	$x_1^o = -0.2776, x_2^o = 0.0706, x_3^o = 0.4388, x_4^o = 0.5237,\ y_1^o = -0.1503, y_2^o = -0.1494, y_3^o = 0.5020, y_4^o = 0.5192$

Table 3.6: The switching points for $\beta^2 = 20$, $\gamma^2 = 2$. The base thus exists in $\bigcup_{j=1}^{4} [x_j^o, x_{j+1}^o] \times [y_j^o, y_{j+1}^o]$.

3.3 Structural Optimization for Infinite Non–Homogeneous Elastic Layer

In this subsection we do some structural optimization. In [6,7] many attractive problems of inner structure optimization for elastic deformable solids with various purposes are considered. Some of the purposes are: the maximal rigidity of plates, the maximal strength of three-layered plates, the optimal cross-section of rods under torsion, the minimization of stress intensity factor in plates with holes, the optimal shape of holes in bending plates, the optimal shape of equal stressed holes etc. Optimization problems are concerned isotropic, non-isotropic, as well as hydroelastic systems. Problems of optimization under lack of information and multi-purpose optimization are also considered.

In [92] a plane problem of elastic properties determination for middle reinforcing mono-layer of three-layered elastic infinite layer is considered in order to ensure propagation of harmonic wave with phase speed equals to that of harmonic wave propagating in a homogeneous layer with elastic properties (Young‘s modulus and density) as the upper and the lower mono-layers have. Numerical analysis of the characteristic equation has shown that there exist materials satisfying the demanded conditions. A relevant problem is considered in [11].

In this subsection we study a bit more general problem about structural optimization of elastic transversely non–homogeneous layer in order to provide propagation of waves with given phase speed. Let in Cartesian coordinate system Ox_*z_* the layer occupies the domain

$$\mathcal{O}_* = \{(x_*, z_*); x_* \in \mathbb{R},\ z_* \in [-h, h]\}.$$

The surfaces $z_* = \pm h$ of the layer are supposed to be stress free. Assume, that its density and Young modulus depend on transverse coordinate z_*: $\rho_* = \rho_*(z_*)$, $E_* = E_*(z_*)$, and the Poisson‘s ratio is constant: $\nu = const$. Merely infinitesimal strains of the layer are taken into account, that is to say, the Cauchy relations between strain and displacements are accepted.

We accept the physically obvious assumption that the density and Young modulus vary according to the same law, i.e.

$$\rho_*(z_*) = \rho_0 u(z_*), \quad E_*(z_*) = E_0 u(z_*), \quad z_* \in [-h, h],$$

where ρ_0 and E_0 are some gauge density and Young modulus, $u = u(z_*)$ is the unknown function. This section is based one the results of [106].

Then, the displacement field of the layer obeys the following system of differential equations

$$
\begin{gathered}
(1-2\nu)\frac{\partial}{\partial z_*}\left[E_*(z_*)\frac{\partial w_{1*}}{\partial z_*}\right] + \frac{\partial}{\partial z_*}\left[E_*(z_*)\frac{\partial w_{2*}}{\partial z_*}\right] + 2(1-\nu)E_*(z_*)\frac{\partial^2 w_{1*}}{\partial x_*^2} - \\
-2\nu E_*'(z_*)\frac{\partial w_{2*}}{\partial x_*} = 2(1+\nu)(1-2\nu)\rho_*(z_*)\frac{\partial^2 w_{1*}}{\partial t_*^2}, \\
2(1-\nu)\frac{\partial}{\partial z_*}\left[E_*(z_*)\frac{\partial w_{2*}}{\partial z_*}\right] + \frac{\partial}{\partial z_*}\left[E_*(z_*)\frac{\partial w_{1*}}{\partial z_*}\right] + (1-2\nu)E_*(z_*)\frac{\partial^2 w_{2*}}{\partial x_*^2} - \\
-(1-\nu)E_*'(z_*)\frac{\partial w_{1*}}{\partial x_*} = 2(1+\nu)(1-2\nu)\rho_*(z_*)\frac{\partial^2 w_{2*}}{\partial t_*^2}, \\
(x_*, z_*, t_*) \in \mathcal{O}_* \times \mathbb{R}^+.
\end{gathered}
$$

The boundary conditions corresponding stress free surfaces of the layer are

$$
\begin{gathered}
\sigma_{33}(x_*, \pm h, t_*) = \left[\frac{E_*(z_*)}{(1+\nu)(1-2\nu)}\left(\nu\frac{\partial w_{1*}}{\partial x_*} + (1-\nu)\frac{\partial w_{2*}}{\partial z_*}\right)\right]\Big|_{z_*=\pm h} = 0, \\
\sigma_{13}(x_*, \pm h, t_*) = \left[\frac{E_*(z_*)}{2(1+\nu)}\left(\frac{\partial w_{1*}}{\partial z_*} + \frac{\partial w_{2*}}{\partial x_*}\right)\right]\Big|_{z_*=\pm h} = 0, \\
x_* \in \mathbb{R}, \quad t_* > 0.
\end{gathered}
$$

Above w_{1*} and w_{2*} are the vertical and the horizontal displacements of the layer, σ_{33} and σ_{13} are the normal and the tangential components of the stress tensor.

Introducing dimensionless variables and functions

$$
x = \frac{x_*}{h}, \quad z = \frac{z_*}{h}, \quad t = \frac{c_0 t_*}{h}, \quad w_1 = \frac{w_{1*}}{h}, \quad w_2 = \frac{w_{2*}}{h}, \quad c_0^2 = \frac{E_0}{\rho_0},
$$

we can write the equations of motion in the following form:

$$
\begin{aligned}
&(1-2\nu)\frac{\partial}{\partial z}\left[u(z)\frac{\partial w_1}{\partial z}\right] + \frac{\partial}{\partial z}\left[u(z)\frac{\partial w_2}{\partial z}\right] + 2(1-\nu)u(z)\frac{\partial^2 w_1}{\partial x^2} - \\
&\qquad - 2\nu u'(z)\frac{\partial w_2}{\partial x} = 2(1+\nu)(1-2\nu)u(z)\frac{\partial^2 w_1}{\partial t^2}, \\
&2(1-\nu)\frac{\partial}{\partial z}\left[u(z)\frac{\partial w_2}{\partial z}\right] + \frac{\partial}{\partial z}\left[u(z)\frac{\partial w_1}{\partial z}\right] + (1-2\nu)u(z)\frac{\partial^2 w_2}{\partial x^2} - \\
&\qquad - \frac{1}{2}2(1-\nu)u'(z)\frac{\partial w_1}{\partial x} = 2(1+\nu)(1-2\nu)u(z)\frac{\partial^2 w_2}{\partial t^2}, \\
&\qquad (x, z, t) \in \mathcal{O} \times \mathbb{R}^+,
\end{aligned}
\tag{3.41}
$$

with $\mathcal{O} := \mathbb{R} \times [-1,1]$. The boundary conditions will take the form

$$\left.\begin{aligned} \left[\frac{u(z)}{(1+\nu)(1-2\nu)}\left(\nu\frac{\partial w_1}{\partial x} + (1-\nu)\frac{\partial w_2}{\partial z}\right)\right]\right|_{z=\pm 1} &= 0, \\ \left[\frac{u(z)}{2(1+\nu)}\left(\frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial x}\right)\right]\right|_{z=\pm 1} &= 0, \end{aligned}\right. \quad x \in \mathbb{R}, \quad t > 0. \tag{3.42}$$

Our aim is the determination of admissible control function $u^o \in \mathcal{U}_\infty = \{u \in L^\infty[-1,1];$ supp $u \subseteq [-1,1],\ 0 < u < 1\}$, ensuring the existence of harmonic solution of bilinear system (3.41), (3.42) with given phase speed c equals to that of propagating in a homogeneous layer with ρ_0, E_0, as well as minimizing the functional

$$\kappa_\infty[u] = \sup_{z\in[-1,1]} u(z), \quad u \in \mathcal{U}_\infty. \tag{3.43}$$

It is important to reveal, that the control function u is included as in bilinear system (3.41) by itself and by its first derivative, as well as in boundary conditions (3.42).

According to [97] we the represent the solution of (3.41), (3.42) as follows

$$w_1(x,z,t) = \frac{\partial \Phi_0(x,z,t)}{\partial x} - \frac{\partial \psi_0(x,z,t)}{\partial z}, \quad w_2(x,z,t) = \frac{\partial \Phi_0(x,z,t)}{\partial z} + \frac{\partial \psi_0(x,z,t)}{\partial x},$$

$$(x,z,t) \in \overline{\mathcal{O}} \times \mathbb{R}^+,$$

where

$$\Phi_0(x,z,t) = [a_1 \cosh(\zeta z) + b_1 \sinh(\zeta z)] \exp[ik(x - c_1 t)] := \Phi(z) \exp[ik(x - c_1 t)],$$

$$\psi_0(x,z,t) = [a_2 \cosh(\eta z) + b_2 \sinh(\eta z)] \exp[ik(x - c_1 t)] := \psi(z) \exp[ik(x - c_1 t)],$$

$$\zeta^2 = k^2\left(1 - c_\zeta^2\right), \quad \eta^2 = k^2\left(1 - c_\eta^2\right), \quad c_\zeta = \frac{c}{c_l}, \quad c_\eta = \frac{c}{c_t}, \quad c_1 = \frac{c}{c_0}.$$

Substituting them into (3.41) we will derive

$$\begin{aligned} u'(z)\Gamma_{11}(z;k) + u(z)\Gamma_{12}(z;k) &= u(z)\Gamma_{13}(z;k,c_1), \\ u'(z)\Gamma_{21}(z;k) + u(z)\Gamma_{22}(z;k) &= u(z)\Gamma_{23}(z;k,c_1), \end{aligned} \quad z \in [-1,1], \tag{3.44}$$

where

$$\Gamma_{11}(z;k) = (1-2\nu)\left[2ik\frac{d\Phi(z)}{dz} - \left(\frac{d^2}{dz^2} + k^2\right)\psi(z)\right],$$

$$\Gamma_{12}(z;k) = \left[\frac{d^2}{dz^2} - k^2\right]\left[2ik(1-\nu)\Phi(z) - (1-2\nu)\frac{d\psi(z)}{dz}\right],$$

$$\Gamma_{13}(z;k,c_1) = 2(1-2\nu)(1+\nu)kc_1\left[kc_1\Phi(z) + i\frac{d\psi(z)}{dz}\right],$$

$$\Gamma_{21}(z;k) = \left[2(1-\nu)\frac{d^2}{dz^2} - \nu k^2\right]\Phi(z) + i(1+\nu)k\frac{d\psi(z)}{dz},$$

$$\Gamma_{22}(z;k) = \left[\frac{d^2}{dz^2} - k^2\right]\left[2(1-\nu)\frac{d\Phi(z)}{dz} + i(1-2\nu)k\psi(z)\right],$$

$$\Gamma_{23}(z;k,c_1) = 2(1-2\nu)(1+\nu)kc_1\left[i\frac{d\Phi(z)}{dz} + kc_1\psi(z)\right].$$

Above c_l and c_t are the velocities of longitudinal and transverse wave propagation in the layer, $k = hk_*,\ k_*$ is the wave number, and (from (3.42))

$$a_1 = a_2\frac{a_{22}}{a_{21}}\frac{\cosh\eta}{\cosh\zeta}, \quad b_1 = b_2\frac{a_{22}}{a_{21}}\frac{\sinh\eta}{\cosh\zeta}, \quad a_2 = -b_2\frac{a_{21}a_{12}}{a_{11}a_{22}}\tanh\zeta,$$

$$a_{11} = 2(1-\nu)\zeta^2 + 2\nu k^2, \quad a_{12} = 2i(1-2\nu)k\eta, \quad a_{21} = 2ik\zeta, \quad a_{22} = \eta^2 + k^2.$$

Moreover, c satisfies the dispersion equation

$$a_{11}^2a_{22}^2 - a_{12}^2a_{21}^2 = 0,$$

corresponding to dispersion equation in the case of homogeneous layer with parameters E_0, ρ_0.

It is quite obvious that (3.44) has not non–trivial solutions in the class of ordinary functions. Since supp $w_p(\cdot, z, \cdot) \subseteq [-1,1]$, the coefficients of (3.44) belong to $\mathfrak{T}$, then the integrals of its both sides belong to $\mathfrak{D}$ and thence are equal almost everywhere. Thus, integrating by parts we will derive

$$\int_{-1}^{1} u(z)\Lambda_1(z;k,c_1)dz = \mathcal{M}_1, \quad \int_{-1}^{1} u(z)\Lambda_2(z;k,c_1)dz = \mathcal{M}_2, \tag{3.45}$$

where

$$\Lambda_p(z;k,c_1) = \frac{d\Gamma_{p1}(z;k)}{dz} - \Gamma_{p2}(z;k) + \Gamma_{p3}(z;k,c_1),$$

$$\mathcal{M}_p = [u(z)\Gamma_{p1}(z;k)]\,\Big|_{-1}^{1}, \quad p \in \{1;2\}.$$

Separating the real and the imaginary parts of (3.45), we will obtain the following system of restrictions with respect to unknown control:

$$\begin{aligned}
&\int_{-1}^{1} u(z)\Lambda_1^{\mathrm{Re}}(z;k,c_1)dz = \mathcal{M}_1^{\mathrm{Re}}, \quad \int_{-1}^{1} u(z)\Lambda_2^{\mathrm{Re}}(z;k,c_1)dz = \mathcal{M}_2^{\mathrm{Re}},\\
&\int_{-1}^{1} u(z)\Lambda_1^{\mathrm{Im}}(z;k,c_1)dz = \mathcal{M}_1^{\mathrm{Im}}, \quad \int_{-1}^{1} u(z)\Lambda_2^{\mathrm{Im}}(z;k,c_1)dz = \mathcal{M}_2^{\mathrm{Im}},
\end{aligned} \tag{3.46}$$

where the superscripts Re and Im denote real and imaginary parts of corresponding expression.

Thus, the optimization problem under investigation is reduced to determination of positive function $u^o(z)$ optimal in the sense (3.43) and satisfying (3.46). Since functions $\Lambda_1^{\mathrm{Re}}(z;k,c_1)$, $\Lambda_2^{\mathrm{Re}}(z;k,c_1)$ and $\Lambda_1^{\mathrm{Im}}(z;k,c_1)$, $\Lambda_2^{\mathrm{Im}}(z;k,c_1)$ are bounded in $[-1,1]$, the solution may be constructed, for instance, with help of moments problem. Then, the required optimal control function is piecewise constant with discontinuities in switching points, where its value jumps from one level to another [22, 77]. Denoting those points by $-1 = z_1^o < z_2^o < \dots < z_n^o < z_{n+1}^o = 1$, the optimal control, unlike [59], we represent as follows:

$$u^o(z) = \sum_{j=1}^{J} u_j^o \left[\theta(z - z_j^o) - \theta(z - z_{j+1}^o)\right], \quad z \in [-1,1]. \tag{3.47}$$

It is clear that, $0 < u_j^o < 1$, $j \in \{1;J\}$.

Substituting (3.47) into (3.46) we will derive

$$\sum_{j=1}^{J} u_j^o \int_{z_j^o}^{z_{j+1}^o} \Lambda_1^{\mathrm{Re}}(z;k,c_1)dz = \mathcal{M}_1^{\mathrm{Re}}, \quad \sum_{j=1}^{J} u_j^o \int_{z_j^o}^{z_{j+1}^o} \Lambda_2^{\mathrm{Re}}(z;k,c_1)dz = \mathcal{M}_2^{\mathrm{Re}},$$
$$\sum_{j=1}^{J} u_j^o \int_{z_j^o}^{z_{j+1}^o} \Lambda_1^{\mathrm{Im}}(z;k,c_1)dz = \mathcal{M}_1^{\mathrm{Im}}, \quad \sum_{j=1}^{J} u_j^o \int_{z_j^o}^{z_{j+1}^o} \Lambda_2^{\mathrm{Im}}(z;k,c_1)dz = \mathcal{M}_2^{\mathrm{Im}}.$$

In view of notations made above, this system may be represented also as follows:

$$\begin{aligned} &u_1^o \cdot \Pi_{11}^{\mathrm{Re}} + \sum_{j=2}^{J} u_j^o \Lambda_{1j}^{\mathrm{Re}} + u_J^o \cdot \Pi_{1J}^{\mathrm{Re}} = 0, \quad u_1^o \cdot \Pi_{21}^{\mathrm{Re}} + \sum_{j=2}^{J} u_j^o \Lambda_{2j}^{\mathrm{Re}} + u_J^o \cdot \Pi_{2J}^{\mathrm{Re}} = 0, \\ &u_1^o \cdot \Pi_{11}^{\mathrm{Im}} + \sum_{j=2}^{J} u_j^o \Lambda_{1j}^{\mathrm{Im}} + u_J^o \cdot \Pi_{1J}^{\mathrm{Im}} = 0, \quad u_1^o \cdot \Pi_{21}^{\mathrm{Im}} + \sum_{j=2}^{J} u_j^o \Lambda_{2j}^{\mathrm{Im}} + u_J^o \cdot \Pi_{2J}^{\mathrm{Im}} = 0, \end{aligned} \tag{3.48}$$

where $u_1^o = u^o(-1)$, $u_J^o = u^o(1)$,

$$\Pi_{p1}^{\mathrm{Re}} = \Lambda_{p1}^{\mathrm{Re}} + \Gamma_{p1}^{\mathrm{Re}}(-1;k), \quad \Pi_{pn}^{\mathrm{Re}} = \Lambda_{pn}^{\mathrm{Re}} - \Gamma_{p1}^{\mathrm{Re}}(1;k),$$

$$\Pi_{p1}^{\mathrm{Im}} = \Lambda_{p1}^{\mathrm{Im}} + \Gamma_{p1}^{\mathrm{Im}}(-1;k), \quad \Pi_{pn}^{\mathrm{Im}} = \Lambda_{pn}^{\mathrm{Im}} - \Gamma_{p1}^{\mathrm{Im}}(1;k),$$

$$\Lambda_{pj}^{\mathrm{Re}} = \int_{z_j^o}^{z_{j+1}^o} \Lambda_p^{\mathrm{Re}}(z;k,c_1)dz, \quad \Lambda_{pj}^{\mathrm{Im}} = \int_{z_j^o}^{z_{j+1}^o} \Lambda_p^{\mathrm{Im}}(z;k,c_1)dz, \quad p \in \{1;2\}, \quad j \in \{1;J\}.$$

Thus, the structural optimization problem under consideration is reduced to determination of positive constants u_j^o and switching points $\{z_j^o\}_{j=1}^J \in [-1,1]$, satisfying (3.50). The problem can be solved by efficient numerical methods of nonlinear programming [15].

Let us now consider the procedure of determination of optimal structure for the layer on a certain example. Let $\nu = 0.25$, then $3c_\zeta^2 = c_\eta^2 = 2.5c_1^2$. When, for simplicity, the arbitrary constant $b_2 = 1$ and $c_\eta < 1$, we will obtain

$$\Lambda_1^{\mathrm{Re}}(z;k) = \frac{k}{2}\left[\frac{d^2}{dz^2} - 3k\right]\Phi^{\mathrm{Re}}(z) - k^2\frac{d\psi(z)}{dz},\ \Lambda_1^{\mathrm{Im}}(z;k,c_1) = -\frac{5kc_1}{4}\left[kc_1\Phi^{\mathrm{Re}}(z) + \frac{d\psi(z)}{dz}\right],$$

$$\Lambda_2^{\mathrm{Re}}(z;k,c_1) = \frac{5kc_1}{4}\left[-\frac{d\Phi^{\mathrm{Re}}(z)}{dz} + kc_1\psi(z)\right],$$

$$\Lambda_2^{\mathrm{Im}}(z;k) = \frac{5k^2}{4}\frac{d\Phi^{\mathrm{Re}}(z)}{dz} + \frac{k}{2}\left[\frac{3}{2}\frac{d^2}{dz^2} + k^2\right]\psi(z),$$

$$\Phi^{\mathrm{Re}}(z) = \frac{\eta k}{\cosh^2\zeta}\cdot\frac{\cosh\zeta\cosh\eta\cosh(\zeta z) + \sinh\zeta\sinh\eta\sinh(\zeta z)}{3\zeta^2 + k^2},$$

$$\psi(z) = \frac{2\zeta\eta k^2\tanh\zeta}{(3\zeta^2 + k^2)(\eta^2 + k^2)}\cdot\cosh(\eta z) + \sinh(\eta z).$$

After substitution of these results into (3.48) numerical calculations were done, the main results of which are presented in Tables 3.7–3.9.

k	u_j^o	z_j^o
0.1	$u_1^o = 0.18$, $u_2^o = 1.31$, $u_3^o = 1.43$, $u_4^o = 0.2$	$z_2^o = -0.6$, $z_3^o = -0.5$, $z_4^o = 0.34$
0.5	$u_1^o = 0.4$, $u_2^o = 1.$, $u_3^o = 0.4$	$z_2^o = -0.35$, $z_3^o = 0.2$
1	$u_1^o = 0.2$, $u_2^o = 0.96$, $u_3^o = 0.38$	$z_2^o = -0.42$, $z_3^o = 0.79$
π	$u_1^o = 0.36$, $u_2^o = 1.14$, $u_3^o = 0.7$, $u_4^o = 0.53$	$z_2^o = -0.6$, $z_3^o = 0.51$,$z_4^o = 0.91$
2π	$u_1^o = 0.7$, $u_2^o = 0.85$, $u_3^o = 0.9$, $u_4^o = 0.27$	$z_2^o = -0.8$, $z_3^o = 0.$,$z_4^o = 0.56$
3π	$u_1^o = 0.4$, $u_2^o = 1.$, $u_3^o = 0.75$	$z_2^o = -0.43$, $z_3^o = 0.76$

Table 3.7: Optimal parameters of the layer when $c_\eta = 0.75$.

It should be noted, that, for instance, when $c_\eta = 0.75$, $k = 0.5$ and $c_\eta = 0.87$, $k = 0.1$, the optimal structure of the layer coincides with the structure of that from [92], but without geometric symmetry.

k	u_j^o	z_j^o
0.1	$u_1^o = 0.32$, $u_2^o = 1.1$, $u_3^o = 0.32$	$z_2^o = -0.38$, $z_3^o = 0.47$
1	$u_1^o = 0.72$, $u_2^o = 0.98$, $u_3^o = 0.58$, $u_4^o = 0.84$	$z_2^o = -0.3$, $z_3^o = 0.24$, $z_4^o = 0.77$
π	$u_1^o = 0.83$, $u_2^o = 1.3$, $u_3^o = 1.67$, $u_4^o = 0.91$	$z_2^o = -0.34$, $z_3^o = 0.53$, $z_4^o = 0.6$
2π	$u_1^o = 0.91$, $u_2^o = 1.1$, $u_3^o = 1.5$	$z_2^o = -0.82$, $z_3^o = 0.56$
3π	$u_1^o = 1.$, $u_2^o = 1.4$, $u_3^o = 0.82$, $u_4^o = 0.91$	$z_2^o = -0.3$, $z_3^o = 0.$, $z_4^o = 0.75$

Table 3.8: Optimal parameters of the layer when $c_\eta = 0.87$.

k	u_j^o	z_j^o
0.1	$u_1^o = 0.27$, $u_2^o = 0.5$, $u_3^o = 0.23$	$z_2^o = -0.44$, $z_3^o = 0.85$
0.5	$u_1^o = 0.72$, $u_2^o = 0.92$, $u_3^o = 0.64$, $u_4^o = 1$	$z_2^o = -0.32$, $z_3^o = 0.68$, $z_4^o = 0.97$
2π	$u_1^o = 1.87$, $u_2^o = 1.2$, $u_3^o = 0.91$, $u_4^o = 0.67$, $u_5^o = 1.2$, $u_6^o = 0.12$	$z_2^o = -0.82$, $z_3^o = -0.08$, $z_4^o = 0.13$ $z_5^o = 0.64$, $z_6^o = 0.85$
3π	$u_1^o = 0.75$, $u_2^o = 1.$, $u_3^o = 1.3$ $u_4^o = 0.9$, $u_5^o = 1.5$, $u_6^o = 0.36$	$z_2^o = -0.81$, $z_3^o = 0.$, $z_4^o = 0.13$ $z_5^o = 0.62$, $z_6^o = 0.85$

Table 3.9: Optimal parameters of the layer when $c_\eta = 0.95$.

Conclusion

Three analytical techniques providing explicit solution, as well as efficient and easily implementable numerical scheme are developed in this dissertation and applied for solving several particular problems having an uncertainty in the state operator or boundary conditions.

Butkovskiy‘s generalized method is applied to examine exact controllability and to find explicit control in

- problems of boundary and distributed control of vibrations in elastic non-homogeneous bounded and unbounded systems in dimension one; the mathematical model is equivalent to one-dimensional wave equation with coordinate dependent coefficients in corresponding domains. In the case of finite domain we also deal with the case of boundary and distributed controls containing given constant delay. Countable system of necessary and sufficient conditions for controllability as equality type integral restrictions in unknown function with smooth kernels is derived. A closed form solution requires a particular solution of special Riccati equation containing the variable parameters of the vibrating system. The Riccati equation is solved for several non-homogeneities and due to simplicity of the algorithm, numerical analysis is implemented. As a result the parameters of the optimal control function are computed and plotted.

- problems of control by input signal (boundary control) and signal filter (distributed control) of one–dimensional electromagnetic signal propagating in dispersive finite medium; the mathematical model is equivalent to partial integro–differential equation associated with wave equation. Countable system of necessary and sufficient conditions is derived for boundary and distributed controllability in terms of the Fourier transform of the kernel of the equation. On the basis of obtained results numerical analysis is implemented and the behaviour of the control function in time is plotted for different values of the system parameters.

Butkovskiy‘s generalized method and the Bubnov–Galerkin procedure are applied in turn in distribution optimization problems when the unknown function does not explicitly depend on time parameter. One- and two-dimensional deformable structures are considered. The general mathematical model is equivalent to bilinear equations (respectively one- and two-dimensional) of fourth order. In particular,

- the distribution of viscoelastic material with given parameters under simply supported finite elastic homogeneous beam subjected to concentrated constant normal load moving uniformly along the beam is optimised in order to ensure vanishing the bending vibrations of the beam in required time after the load detaches from the beam. The Kelvin–Voight viscoelastic body model is accepted. Finite system of integral equality type restrictions with smooth kernels on the control function is derived and it is proved that the discrete distribution of the material used for our purpose minimizes the relative volume of the distribution. The determination of the switching points specifying the dampers placements is reduced to a problem of nonlinear programming. Numerical experiment implemented on the basis of obtained solutions reveals the dependencies of the optimal placements against input parameters of the system.

- the in-plane distribution of one–parametric linear–elastic substrate under simply supported elastic isotropic rectangular homogeneous plate of small constant thickness subjected to a constant normal load uniformly moving along the length of the plate in oder to ensure vanishing (in some sense) of the bending vibrations in required time after the load detaches from the plate. The Kirchhoff hypothesis with respect to plate and Winkler hypothesis with respect to the base are accepted. Finite system of integral equality type restrictions with smooth kernels on the control function is derived and it is proved that the piecewise existing distribution of the base minimizes the maximal value of the unknown function. The determination of the switching points indicating the existence areas of the base is reduced to a problem of nonlinear programming. On the basis of explicit formulas numerical analysis is implemented and the main dependencies of the optimal control function from the internal and external parameters of the system is revealed quantitatively, and qualitatively as well.

In both cases the results are presented via tables and figures.

Applying decomposition of *dynamical* problem solution via vector and scalar potentials an efficient numerical scheme is suggested to deal with complicated *linear* equations (even coupled system of them). In particular,

- the problem of structural optimization for an isotropic elastic infinite layer with stress–free boundaries and transverse non-homogeneity is investigated via that approach in order to ensure propagation in the direction perpendicular to its thickness of periodic wave with specified phase speed. Assuming that the density and the Young‘s modulus of the layer are described by the same (unknown) function; the Poisson‘s ratio remains constant. The problem mathematically is formulated as a two–dimensional system of bilinear partial differential equations containing the control function and its first derivative subjected to boundary conditions also containing the control function. Finite system of integral equality with smooth kernels is derived and it is obtained that the piece-wise (layered) non-homogeneity of the layer minimizes the maximal value of the control function. The problem is reduced to that of nonlinear programming with respect to thickness and elastic characteristics (Young‘s modulus and density) of each mono–layer. Numerical analysis is done and the optimal parameters of the layer are calculated for various values of the phase speed and wave number.

Obtained results might be of interest to structural engineers working in the fields such as input wave (signal) and wave source (signal filter) regulation of different wave-fields, optimization of material distribution in prescribed domains subjected to external excitation, structural optimization of wave-guides, etc.

There remain some results on the topics included, which did not find place in the dissertation. There are also some plans concerning our future investigations in this field. Particularly, we aim to

- consider control problems for some particular nonlinear equations, including Korteweg–de Vries equation, equations with square nonlinearity etc.,

- continue the extension of control problems types, which can be solved explicitly. For this purpose, we use Green's function approach. It is motivated by the fact, that it gives the explicit solution of initial–boundary value problems (even nonlinear), the analytical investigation of which is quite complicated,

- continue considering and investigating more attractive and, at the same time, more concrete problems of topology and structural optimization,
- develop an efficient numerical scheme for well–known problem of mobile (scanning) control.

We want to end with remembering that the *method* is much more important that any particular problem, however complicated, that can be solved by that method.

Bibliography

[1] Abramowitz M., Stegun I. A., *Handbook of special functions.* Dover Publications, New York, 1999, 1046 p.

[2] Agrawal O. P., Fractional optimal control of a distributed system using eigenfunctions // ASME Journal of Computational and Nonlinear Dynamics, 2008, vol. 3, issue 2, 6 p.

[3] Appel J. M., Kalitvin A. S., Zabrejko P. P., *Partial integral operators and integro–differential equations.* Marcel Dekker, New York, 2000, 578 p.

[4] Avdonin S. A., Belinskiy B. P., Pandolfi L., Controllability of a nonhomogeneous string and ring under time dependent tension // Mathematical Modeling of Natural Phenomena, 2010, vol. 5, issue 4, pp. 4–31.

[5] Babouskos N. G., Katsikadelis J. T., Optimum design of thin plates via frequency optimization using BEM // Archives of Applied Mechanics, 2015, vol. 85, pp. 1175–1190.

[6] Banichuk N. V., *Introduction to structural optimization.* Springer, Berlin, 2011, 300 p.

[7] Banichuk N. V., Neittaanmaki P. J., *Structural optimization with uncertainties.* Springer, Dordrecht, 2012, 233 p.

[8] Barseghyan V. R., The problem for optimal restoration of the state of the system described by an integro–differential equation in the presence of errors in measurements // Automation and Remote Control, 2012, vol. 73, issue 8, pp. 1365–1370.

[9] Barseghyan V. R., Movsisyan L. A., Optimal control of the vibration of elastic systems described by the wave equation // International Applied Mechanics, 2012, vol. 48, issue 2, pp. 234–239.

[10] Baudouin L., Cerpa E., Crepeau E., Mercado A., On the determination of the principal coefficient from boundary measurements in a KdV equation // Journal of Inverse and Ill-posed Problems, 2013.

[11] Belubekyan M. V., Mheryan D. E., Three-dimensional problem of the surface waves propagation in transversely isotropic elastic medium // Proceedings of NAS of Armenia, 2006, vol. 59, issue 2, pp. 3–9 (in Russian).

[12] Becker R., *Adaptive finite elements for optimal control problems.* Habilitationsschrift, University of Heidelberg, Heidelberg, 2004.

[13] Bellman R., *Dynamic programming.* Dover, New York, 2003, 366 p.

[14] Bendsoe M. P., Sigmund O., *Topology optimization.* Springer, Berlin, 2003, 370 p.

[15] Betts J. T., *Methods of optimal control and estimation using nonlinear programming.* 2nd edition. SIAM, Philadelphia, 2010, 448 p.

[16] Borovskikh A. V., Boundary control formulas for inhomogeneous string. I, II // Differential Equations, 2007, vol. 45, issues 1, 5, pp. 69–95, 656–666.

[17] Bradley M. E., Lenhart S., Bilinear optimal control of a Kirchhoff plate // Systems & Control Letters, 1994, vol. 22, issue 1, pp. 27–38.

[18] Bradley M. E., Lenhart S., Bilinear optimal control of a Kirchhhoff plate via internal controllers. In "*Modelling and Optimization of Distributed Parameter Systems: Applications to Engineering*". Chapman and Hall, London, 1996, pp. 233–240.

[19] Browne P. A., *Topology optimization of linear elastic structures.* Thesis submitted for the PhD degree, Bath, 2013.

[20] Brychkov Yu., Prudnikov A., *Integral transforms of generalized functions.* Gordon and Breach Science Publishing, Amsterdam, 1989, 344 p.

[21] Bungartz H.–J., Schäfer M., *Fluid–structure interaction: modelling, simulation, optimization.* Springer, Berlin, 2006, 402 p.

[22] Butkovskiy A. G., *Methods of control of systems with distributed parameters.* Nauka publ., Moscow, 1975, 675 p. (in Russian).

[23] Butkovskiy A. G., Pustyl'nikov L. M., *Characteristics of distributed parameter systems.* Kluwer Academic, London, 1993, 386 p.

[24] Butkovskiy A. G., Some problems of control of the distributed-parameter systems // Automation and Remote Control, 2011, vol. 72, issue 6, pp. 1237–1241.

[25] Chiang Sh., Numerical optimal unbounded control with a singular integro-differential equation as a constraint // Discrete Continuous Dynamical Systems, 2013, pp. 129–137.

[26] Christensen P. W., Klarbring A., *An introduction to structural optimization.* Springer, Berlin, 2009, 214 p.

[27] Christensen R. M., *Theory of viscoelasticity.* 2nd edition. Dover, New York, 2010, 384 p.

[28] Cortes R. M., Gonnet G. H., Hare D. E. G., Jeffrey J. D., Knuth D. E., On the Lambert W function // Advances in Computational Mathematics, 1996, vol. 5, issue 1, pp. 329–359.

[29] Cosentino C., Bates D., *Feedback control in systems biology.* CRC Press, Boca Raton, 2011, 296 p.

[30] Da Prato G., Ichikawa A., Optimal control for integrodifferential equations of parabolic type // SIAM Journal on Control and Optimization, 1993, vol. 31, issue 5, pp. 1167–1182.

[31] Elliot S., *Signal processing for active control.* Academic Press, New York, 2000, 511 p.

[32] Eschenauer H. A., Olhoff N., Topology optimization of continuum structures: A review // ASME Applied Mechanics Reviews, 2001, vol. 54, issue 4, pp. 331–390.

[33] Evans G., Blackledge J., Yardley P., *Numerical methods for partial differential equations.* Springer, London, 2000, 290 p.

[34] Fardigola L. V., Controllability problems for the string equation on a half–axis with a boundary control bounded by a hard constant // SIAM Journal on Control and Optimization, 2008, vol. 47, issue 4, pp. 2179–2199.

[35] Fardigola L. V., Transformation operators of the Sturm–Liouville problem in controllability problems for the wave equation on a half–axis // SIAM Journal on Control and Optimization, 2013, vol. 51, issue 2, pp. 1781–1801.

[36] Focardi S. M., Fabozzi F. J., *The mathematics of financial modelling and investment management*. John Wiley & Sons, New York, 2004, 800 p.

[37] Freeden W., Gutting M., *Special functions of mathematical (geo–)physics*. Springer, Basel, 2013, 504 p.

[38] Fryba L., *Vibration of solids and structures under moving loads*. 3rd edition. Thomas Telford Ltd, New York, 1999, 500 p.

[39] Fujita K., Moustafa A., Takewaki I., Optimal placement of viscoelastic dampers and supporting members under variable excitations // Earthquakes and Structures, 2010, vol. 1, issue 1, pp. 43–67.

[40] Gakhov F. D., Cherskiy Yu. I., *Equations of convolution type*. Nauka Publ., Moscow, 1978, 296 p. (in Russian).

[41] Garcia M. E., Grigorenko I., Analytical solution of the optimal laser control problem in two-level systems // Journal of Physics B: Atomic, Molecular and Optical Physics, 2004, vol. 37, pp. 2569–2575.

[42] Glowinski R., Lions J.-L., He J., *Exact and approximate controllability for distributed parameter systems: A numerical approach*. Cambridge, 2008.

[43] Grigoryan E. Kh., Solution to problem of finite elastic inclusion, terminating to the boundary of semi-plane // Proceedings of the Yerevan State University. Natural Sciences, 1981, issue 3, pp. 32–43 (in Russian).

[44] Grigoryan E. Kh., On one effective method of solution of a class of mixed problem of elasticity theory // Proceedings of the Yerevan State University. Natural Sciences, 1979, issue 2, pp. 62–71 (in Russian).

[45] Gugat M., Optimal switching boundary control of a string to rest in finite time // ZAMM Journal of Applied Mathematics and Mechanics, 2008, vol. 88, issue 4, pp. 283–305.

[46] Gugat M., Leugering G., Sklyar G., L^p–optimal boundary control for the wave equation // SIAM Journal on Control and Optimization, 2005, vol. 44, issue 1, pp. 49–74.

[47] Haslinger J., Mäkinen R. A. E., *Introduction to shape optimization: theory, approximation, and computation*. SIAM, Philadelphia, 2003, 273 p.

[48] Haslinger J., Mäkinen R. A. E., On a topology optimization problem governed by two-dimensional Helmholtz equation// Computational Optimization and Applications, 2015, vol. 62, issue 2, pp 517–544.

[49] Haslinger J., Málek J., Stebel J., A new approach for simultaneous shape and topology optimization based on dynamic implicit surface function // Control and Cybernetics, 2005, vol. 34, issue 1, pp. 283–303.

[50] Haslinger J., Neittaanmäki P., *Finite element approximation for optimal shape, material and topology design*. 2nd edition. Wiley, New York, 1996, 442 p.

[51] Hillion P., Electromagnetic pulse propagation in dispersive media // Progress in Electromagnetics Research, 2002, vol. 35, 299–314.

[52] Huang X., Xie Y. M., *Evolutionary topology optimization of continuum structures*. Methods and Applications. Springer, Berlin, 2010, 223 p.

[53] Il'in V.A., Moiseev E.I., Optimization of boundary controls of string vibrations // Uspekhi Mat. Nauk, 2005, vol. 60, issue 6, pp. 89–114.

[54] Il'in V. A., Moiseev E. I., Boundary control of string vibrations that minimizes the integral of power $p \geq 1$ of the module of control or its derivative // Automation and Remote Control, 2007, vol. 68, issue 2, pp. 313–319.

[55] *IUTAM Symposium on Topological Design Optimization of Structures, Machines and Materials: Status and Perspectives.* Solid Mechanics and its Applications, Vol. 137. Edited by Bendsoe M. P., Olhoff N., Sigmund O. Amsterdam, Springer, 2006, 608 p.

[56] Jeffreys H., Jeffreys (Swirles) B., *Methods of mathematical physics.* 3rd edition. Cambridge Mathematical Library, Cambridge, 2013, 730 p.

[57] Jilavyan S. H., Khurshudyan As. Zh., Optimal control of thermo-elastic vibrations in elastic infinite layer // Proceedings of the III international conference "Topical Problems of Deformable Body Mechanics" dedicated to the centenary of academician N. Kh. Arutyunian. Institute of Mechanics, NAS of Armenia, Yerevan, 2012, vol. 1, pp. 228–232.

[58] Jilavyan S. H., Khurshudyan As. Zh., Optimal control of anisotropic layer-plate vibrations in view of transverse shear // Proceedings of International Conference on "Information Control, Transmission and Transfer", 2012, Saratov, Russia, pp. 219–228.

[59] Jilavyan S. H., Khurshudyan As. Zh., Sarkisyan A. S., On adhesive binding optimization of elastic homogeneous rod to a fixed rigid base as a control problem by coefficient // A Quarterly of Polish Academy of Sciences. Archives of Control Sciences, 2013, vol. 23 (LIX), issue 4, pp. 413–425.

[60] Jilavyan S. H., Khurshudyan As. Zh., Topology optimization for elastic base under rectangular plate subjected to moving load // A Quarterly of Polish Academy of Sciences. Archives of Control Sciences, 2015, vol. 25 (LXI), issue 3, pp. 289–305.

[61] Kerbal S., Jiang Y., General integro–differential equations and optimal controls in Banach spaces // Journal of Industrial and Management Optimization, 2007, vol. 3, issue 1, pp. 119–128.

[62] Khalina K. S., On the Neumann boundary controllability for the non–homogeneous string on a segment // Journal of Mathematical Physics, Analysis and Geometry, 2011, vol. 7, issue 4, pp. 333-351.

[63] Khalina K. S., Boundary controllability problems for the equation of oscillation of an inhomogeneous string on a semiaxis // Ukrainian Mathematical Journal, 2012, vol. 64, issue 4, pp. 594–615.

[64] Khludnev A., Negri M., Optimal rigid inclusion shapes in elastic bodies with cracks // ZAMM Journal of Applied Mathematics and Mechanics, 2013, vol. 63, pp. 179–191.

[65] Khurshudyan As. Zh., On optimal impulsive null-compactly supported control of non-homogeneous rod vibrations // Proceedings of VII International Scientific Conference "Applied Mathematics, Informatics, Mechanics", 2012, Voronezh, Russia, vol. 1, pp. 393–399 (in Russian).

[66] Khurshudyan As. Zh., On optimal boundary null-controllability for non-homogeneous string vibrations under impulsive boundary perturbations // Proceedings of IV Russian-Armenian Colloquium on "Mathematical Physics, Complex Analysis and Related Topics", 2012, Krasnoyarsk, Russia, pp. 81–84.

[67] Khurshudyan As. Zh., On optimal boundary control of non–homogeneous string vibrations under impulsive concentrated perturbations with delay in controls // Mathematical Bulletin of T. Shevchenko Scientific Society, 2013, vol. 10, pp. 203–209.

[68] Khurshudyan As. Zh., Arakelyan Sh. Kh., Delaying control of non–homogeneous string forced vibrations under mixed boundary conditions // IEEE Proceedings on Control and Communication, 2013, vol. 10, pp. 1–5.

[69] Khurshudyan As. Zh., Generalized control with compact support of wave equation with variable coefficients // International Journal of Dynamics and Control, 2015, DOI: 10.1007/s40435-015-0148-3.

[70] Khurshudyan As. Zh., Generalized control with compact support for systems with distributed parameters // A Quarterly of Polish Academy of Sciences. Archives of Control Sciences, 2015, vol. 25 (LXI), issue 1, pp. 5–20.

[71] Khurshudyan As. Zh., On optimal boundary and distributed control of partial integro–differential equations // A Quarterly of Polish Academy of Sciences. Archives of Control Sciences, 2014, vol. 24 (LX), issue 1, pp. 5–25.

[72] Khurshudyan As. Zh., Bubnov–Galerkin procedure in bilinear control problems // Automation and Remote Control, 2015, vol. 76, issue 8, pp. 1361–1368.

[73] Khurshudyan Am. Zh., Khurshudyan As. Zh., Optimal distribution of viscoelastic dampers under elastic finite beam under moving load // Proceedings of NAS of Armenia, 2014, vol. 67, issue 3, pp. 56–67 (in Russian).

[74] Khurshudyan As. Zh., On some problems of designs structural and topological optimization // Proceedings of International Summer School-Conference "Advanced Problems in Mechanics". Institute for Problems in Mechanical Engineering. RAS. St. Petersburg, 2014, pp. 310–320.

[75] Klarbring A., Haslinger J., On almost constant contact stress distributions by shape optimization // Structural Optimization, 1993, vol. 5, pp. 213–216.

[76] Kočvara M., Stingl M., Solving stress constrained problems in topology and material optimization // Structural and Multidisciplinary Optimization, 2012, vol. 46, pp. 1–15.

[77] Krasovskiy N. N., *Motion control theory.* Nauka publ., Moscow, 1968, 476 p. (in Russian).

[78] Krasovskiy N. N., Krasovskiy A. N., *Control under lack of information.* Systems & control: Foundations & applications. Birkhaäuser, Berlin, 1995, 324 p.

[79] Krotov V. F., Bulatov A. V., Baturina O. V., Optimization of linear systems with controllable coefficients // Automation and Remote Control, 2011, vol. 72, issue 6, pp. 1199–1212.

[80] Leugering G., Exact boundary controllability of an integrodifferential equation // Applied Mathematics & Optimization, 1987, vol. 15, pp. 223–250.

[81] Liang M., Bilinear optimal control for a wave equation // Mathematical Models and Methods in Applied Sciences, 1999, vol. 9, issue 1, pp. 45–68.

[82] Lin Zh., Wang X., Ren Y., Topology optimization design of micro-mass sensors for maximizing detection sensitivity // Acta Mechanica Sinica, 2015, vol. 31, issue 4, pp 536–544.

[83] Lions J.–L., *Optimal control of systems governed by partial differential equations*. 2nd edition. Softcover reprint of the original 1st edition. Springer, Berlin, 2011, 416 p.

[84] Longuski J. M., Guzman J. J., Prussing J. E., *Optimal control with aerospace applications*. Springer, Berlin, 2014, 273 p.

[85] Lukyanov A. T., Serovayskiy S. Ya., Optimal control for a bilinear hyperbolic system // Soviet Mathematics (Izvestiya VUZ. Matematika), 1983, vol. 27, issue 10, pp. 46–48.

[86] Lukyanov A. T., Serovayskiy S. Ya., The method of successive approximations in the problem of optimal control of a nonlinear parabolic system // USSR Computational Mathematics and Mathematical Physics, 1984, vol. 24, issue 6, pp. 23–30.

[87] Lurie K. A., On material optimization in continuum dynamics // Journal of Optimization Theory and Applications, 2015, vol. 167, pp. 147–160.

[88] Martinez–Rodrigo M. D., Museros P., Optimal design of passive viscous dampers for controlling the resonant response of orthotropic plates under high–speed moving loads // Journal of Sound and Vibration, 2011, vol. 330, issue 7, pp. 1328–1351.

[89] Melnikova I. V., Filinkov A., *Abstract Cauchy problems: Three Approaches.* Chapman & Hall/CRC, Boca Raton, Florida, 2001, 246 p.

[90] Miguel Let. F. F., Miguel Lea. F. F., Lopez R. H., A firefly algorithm for the design of force and placement of friction dampers for control of man-induced vibrations in footbridges // Optimization and Engineering, 2015, vol. 16, issue 3, pp. 633–661.

[91] Mikhlin S. G., *Error Analysis in Numerical Processes.* John Wiley & Sons Ltd, New York, 1991, 284 p.

[92] Mol'kov V. A., Ryabcev A. V., Harmonic wave propagation in reinforced layer // Elasticity and Inelasticity. Part 1. MSU publ., Moscow, 1993, pp. 46–54 (in Russian).

[93] Movsisian L. A., Gabrielian M. S., On one problem of motion control for thermoelastic plate-layer // Proceedings of NAS of Armenia, 1995, vol. 48, issue 3, pp. 15–22 (in Russian).

[94] Museros P., Moliner E., Martinez–Rodrigo M. D., Free vibrations of simply supported beam bridges under moving loads. Maximum resonance, cancelation and resonant vertical acceleration // Journal of Sound and Vibration, 2013, vol. 332, issue 2, pp. 326–345.

[95] Museros P., Martinez–Rodrigo M. D., Vibration control of simply supported beams under moving loads using fluid viscous dampers // Journal of Sound and Vibration, 2007, vol. 300, issue 1–2, pp. 292–315.

[96] Neches L. C., Cisilino A. P., Topology optimization of 2D elastic structures using boundary elements // Engineering Analysis with Boundary Elements, 2008, vol. 32, issue 7, pp. 533–544.

[97] Nowacki V., *Theory of elasticity.* Mir, Moscow, 1975, 872 p. (in Russian)

[98] *Optimal control of coupled systems of partial differential equations.* Edited by K. Kunisch, J. Sprekels, G. Leugering, F. Tröltzsch. Birkhäuser, Berlin, 2009, 345 p.

[99] Ouzahra M., Controllability of the wave equation with bilinear controls // European Journal of Control, 2014, vol. 20, issue 2, pp. 57–63.

[100] Pardalos P. M., Yatsenko V., *Optimization and control of bilinear systems.* Springer, Berlin, 2008, 370 p.

[101] Pontryagin L. S., *Mathematical theory of optimal processes.* CRC Press, Boca Raton, 1987, 360 p.

[102] Pytlak R., *Numerical methods for optimal control problems with state constraints.* Springer, Berlin, 1999, 218 p.

[103] Qu Ji–ting, Li Hong–nan, Study on optimal placement and reasonable number of viscoelastic dampers by improved weight coefficient method // Mathematical Problems of Engineering, 2013, vol. 2013, ID 358709, 10 pages.

[104] Rasina I. V., Baturina O. V., Control optimization in bilinear systems // Automation and Remote Control, 2013, vol. 74, issue 5, pp. 802–810.

[105] Sachs E. W., Strauss A. K., Efficient solution of partial integro–differential equation in finance // Applied Numerical Mathematics, 2008, vol. 58, issue 11, pp. 1687–1703.

[106] Sarkisyan S. V., Jilavyan S. H., Khurshudyan As. Zh., Structural optimization for infinite non homogeneous layer in periodic wave propagation problems // Composite Mechanics, 2015, vol. 51, issue 3, pp. 277–284.

[107] Sarkisyan V. S., Geghamyan B. P., *Theory of optimal design of structures and applications*. YSU publ., Yerevan, 1990, 324 p. (in Armenian).

[108] Serovayskiy S. Ya., A control problem in coefficients for equations of parabolic type // Soviet Mathematics (Izvestiya VUZ. Matematika), 1982, vol. 26, issue 12, pp. 44–52.

[109] Shilov G. E., *Mathematical analysis. Second special course*. 2nd edition. MSU publ., Moscow, 1984, 208 p. (in Russian).

[110] de Silva C. W., *Vibration damping, control and design*. CRC Press, Boca Raton, 2007, 634 p.

[111] Sklyar G. M., Fardigola L. V., The Markov trigonometric moment problem in controllability problems for the wave equation on a half–axis // Journal of Mathematical Physics, Analysis and Geometry, 2002, vol. 9, issue 2, pp. 233–242.

[112] Soares R. M., del Prado Z., Gonsalves P. B., On the vibration control of beams using a moving absorber and subjected to moving loads // Mècanica Computacioal, 2010, vol. 29, pp. 1829–1840.

[113] Schwartz L., *Thèorie des distributions*. Vol. 1, 2. Paris, Hermann & Cie, 1950–1951.

[114] Tenenbaum R. A., Fernandes K. M., Stutz L. T., Silva Neto A. J., Damage identification in bars with a wave propagation approach and a hybrid optimization method // Shock and Vibration, 2012, vol. 19, pp. 301–321.

[115] Teodorescu P., Kecs W., Toma A., *Distribution theory: with applications in engineering and physics.* Wiley–VCH, 2013, 394 p.

[116] *Unsolved problems in mathematical systems and control theory.* Edited by Blondel V. D. and Megretski A. Princeton University Press, Princeton, 2004, 352 p.

[117] Vinogradova M. B., Rudenko O. V., Sukhorukov A. P., *Theory of waves.* 2nd edition. Moscow, Nauka publ., 1990, 432 p. (in Russian).

[118] Vladimirov V. S., *Methods of the theory of generalized functions. Analytical methods and special functions.* CRC Press, London–New York, 2002, 328 p.

[119] Vladimirov V. S., *Equations of mathematical physics.* 5th edition. Nauka, Moscow, 1988, 512 p. (in Russian).

[120] Voronin A. F., Necessary and sufficient well–posedness conditions for a convolution equation of the second kind with even kernel on a finite interval // Siberian Mathematical Journal, 2008, vol. 48, issue 4, pp. 601–611.

[121] Wein F., Kaltenbacher M., Stingl M., Topology optimization of a cantilevered piezoelectric energy harvester using stress norm constraints // Structural and Multidisciplinary Optimization, 2013, vol. 48, issue 1, pp. 173–185.

[122] Yang T., *Impulsive control theory.* Berlin, Springer, 2001, 367 p.

[123] Zaitsev V. F., Polyanin A. D., *Handbook of exact solutions for ordinary differential equations.* 2nd edition. CRC Press, Boca Raton, 2003, 816 p.

[124] Zemanian A. H., *Distribution theory and transform analysis: An introduction to generalized functions, with applications.* Dover Publications, New York, 2010, 400 p.