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USAGE OF LINES IN GC SETS

Dissertation

By Specialty

A.01.01 ”Mathematical Analysis”

For PhD in Mathematics and Physics

Scientific adviser

Doctor of Physical and Mathematical Sciences

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YEREVAN 2019

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INTRODUCTION

Actuality of the subject. It is a classic problem in mathematics to estimate the value of a function at certain points from known values at other points. The process of reconstructing a function, curve, surface from certain known data is called interpolation. In some sense, polynomials are the simplest type of interpolants to work with, as their definition only involves a finite number of additions, subtractions, and multiplications. Also they can be easily differentiated or integrated.

Univariate polynomial interpolation is a classical subject, with a long history and a well settled theory. It dates back to the Newton and the Lagrange fundamental solutions of the interpolation problem.

Compared to that, the multivariate counterpart is much more complicated. It has only been systematically considered in the second half of the 20th century due to the development of computers. Another reason for the interest of multivariate problems was the emergence of new mathematical methods, as cubature formulae, finite element methods. Another connection is Algebraic Geometry, since the solvability of a multivariate interpolation problem relies on the fact that the interpolation points do not lie on an algebraic surface of a certain degree.

These days, applications of multivariate polynomial interpolation range over many different fields of pure and applied mathematics. Interpolation theory finds applications in many problems, as numerical differentiation and integration, numerical solution of differential equations, evaluation of transcendental functions, typography, and the computer-aided geometric design of cars, ships, and airplanes.

Purpose and goals of the thesis. The thesis consists of two parts. The first part is dedicated to the factorization of the fundamental polynomials of two and three variables.

In the second part we provide an adjustment to the formulation and then prove a conjecture proposed by V. Bayramyan and H. Hakopian in a recent paper. The conjecture characterizes the number of usages of a line in a GC set, provided that the Gasca-Maeztu conjecture is true.

The object of research. Multivariate polynomial spaces, n-independent and n-poised sets, GC_n sets, maximal lines, Gasca-Maeztu conjecture, general principal lattices, Chung-Yao lattices, Carnicer-Gasca lattices.

The methods of research. The methods of univariate and multivariate polynomial interpolations are used. Also some methods of linear algebra and algebraic geometry are used.

Scientific novelty. Necessary and sufficient conditions are provided for the factorization of fundamental polynomials of node sets in $\mathbb{R}^2$ of cardinality $\leq 2n + [n/2] + 1$ and of node sets in $\mathbb{R}^3$ of cardinality $\leq 3n + 1$. A new simple proof of the Gasca-Maeztu conjecture for the case of $n = 4$ is provided. Then a correct formulation of a property established by V. Bayramyan and H. Hakopian on the usage of n-node lines in GC_n sets is provided and then proved. Finally, an adjustment to the formulation of a conjecture proposed by V. Bayramyan and H. Hakopian on the usage of k-node lines in GC_n sets, $2 \leq k \leq n + 2$, is provided and then proved. Also counterexamples for the cases when the lines are not used at all are presented.

Practical significance. Main results in the thesis are of theoretical nature but at the same time some of them can have practical application. For instance finding fundamental polynomials in the simplest possible form is significant from the point of view of applications. The characterization of independent and poised sets can be used in the mathematical problems where multivariate polynomial interpolation is considered.

The following provisions are presented for the defence.

- Necessary and sufficient conditions are provided for the factorization of fundamental polynomials of node sets in $\mathbb{R}^2$ of cardinality not exceeding $2n + [n/2] + 1$ as a product of factors of at most second degree.

- Independence of node sets with $3n+1$ nodes in $\mathbb{R}^3$. Necessary and sufficient conditions are provided for the factorization of fundamental polynomials of such node sets as a product of linear factors.

- A correction of a property on the usage of n-node lines in GC_n sets established by V.

Bayramyan and H. Hakopian.

- An adjustment of the formulation of a conjecture proposed by V. Bayramyan and H. Hakopian on the usage of k-node lines in GC_n sets, $2 \leq k \leq n+2$, is provided and then proved. Namely, by assuming that the Gasca-Maeztu conjecture is true, we prove that any k-node line ℓ is not used at all, or it is used by exactly $\binom{s}{2}$ nodes, where s satisfies the condition $2k-n-1 \leq s \leq k$.

The approbation of obtained results. The results of the thesis were reported in

- the scientific seminars held in the department of Numerical Analysis of the Faculty of Informatics and Applied Mathematics of Yerevan State University,

- the International Conference Dedicated to 90th Anniversary of Sergey Mergelyan, 20-25 May 2018, Yerevan, Armenia, Abstracts, pp. 37-38 ([30]).

- the Emil Artin International Conference, Dedicated to 120th Anniversary Emil Artin, 29 May - 02 June, Yerevan, Armenia, Abstracts p. 67, Yerevan, 2018 ([31]).

- the International Conference Harmonic Analysis and Approximations VII, Dedicated to 90th Anniversary of Alexander Talalyan, 16-22 September, 2018, Tsaghkadzor, Armenia, Abstracts pp. 45-47. ([32]).

Publications. The results of the thesis were published in 4 scientific articles [25], [27], [28], [29] and reported in 3 international conferences [30], [31], [32].

The structure and the content of thesis. The thesis consists of introduction, two parts each of which contains three chapters, summary and bibliography. The puplications of the author are [25], [27], [28], [29], [30], [31], [32]. The paper [26] is accepted for publication. The number of references is 32. The content of the thesis is 102 pages.

THE CONTENT OF THE THESIS

In Chapter 1 we present univariate and multivariate interpolation and some basic known facts.

Denote

$$\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d, \ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}_+^d$$

$$\mathbf{x}^\alpha = {x_1}^{\alpha_1} \cdot {x_2}^{\alpha_2} \cdot \cdots \cdot {x_d}^{\alpha_d}, \ |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d.$$

Let Π_n^d be the space of polynomials of d variables of total degree at most n :

$$\Pi_n^d = \left\{ \sum_{|\alpha| \le n} a_\alpha \mathbf{x}^\alpha : a_\alpha \in \mathbb{R}^d \right\}.$$

The dimension of this space is given by

$$N(n, d) := \dim \Pi_n^d = \binom{n+d}{d}.$$

Below we state the Lagrange multivariate interpolation problem. Suppose we have a set of s distinct nodes $\mathcal{X}_s = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\} \subset \mathbb{R}^d$ and s arbitrary values $c_1, c_2, \ldots c_s$. The problem of finding a (unique) polynomial satisfying the following conditions:

$$p(\mathbf{x}_i) = c_i, \quad i = 1, 2, \ldots s, \tag{1.2.1}$$

is called interpolation problem. The conditions (1.2.1) are called interpolation conditions.

A polynomial $p \in \Pi_n^d$ is called an n-fundamental polynomial for a node $A = \mathbf{x}_k \in \mathcal{X}_s$ if $p(\mathbf{x}_i) = \delta_{ik}, \quad i = 1, \ldots, s$, where δ is the Kronecker symbol. We denote this fundamental polynomial by $p_k^\star = p_A^\star = p_{A,\mathcal{X}_s}^\star$.

Definition 1.2.1. A set of nodes $\mathcal{X}$ is called *n-independent* if all its nodes have fundamental polynomials. Otherwise, $\mathcal{X}$ is called *n-dependent.*

Fundamental polynomials are linearly independent. Therefore, a necessary condition of n-independence is $|\mathcal{X}| \le N(n, d)$. Having fundamental polynomials of all nodes of $\mathcal{X}$ we get a solution of the interpolation problem (1.2.1) by using the Lagrange formula:

$$p(\mathbf{x}) = \sum_{i=1}^{s} c_i p_i^\star(\mathbf{x}). \tag{1.2.2}$$

Thus we get that the node set $\mathcal{X}_s$ is n-independent if and only if it is n-solvable, meaning that for any data $\{c_1, \ldots, c_s\}$ there exists a (not necessarily unique) polynomial $p \in \Pi_n^d$ satisfying the conditions (1.2.1).

Definition 1.2.2. The set of nodes $\mathcal{X}_s$ is called *n-poised* if for any data $\{c_1, \ldots, c_s\}$ there exists a unique polynomial $p \in \Pi_n^d$, satisfying the conditions (1.2.1).

A necessary condition for n-poisedness of the set $\mathcal{X}_s$ is

$$s = |\mathcal{X}_s| = N(n, d). \tag{1.2.3}$$

Let us mention that in the univariate case the condition (1.2.3) is also sufficient for n-poisedness. In the multivariate case the condition is no longer sufficient, unless the case $n = 0$.

We also have that a set $\mathcal{X}_s$, with $s = N(n, d)$ is n-poised if and only if it is n-independent.

In Section 1.2.2 we present basic known results on bivariate interpolation.

For the brevity we denote the space of bivariate polynomials of total degree at most n by $\Pi_n := \Pi_n^2$. Similarly we set

$$N := N(n, 2) = \binom{n+2}{2}.$$

We have

Proposition 1.2.3. The set of nodes $\mathcal{X}_N$ is n-poised if and only if the following condition holds:

$$p \in \Pi_n,\ p(x_i, y_i) = 0,\ i = 1, \ldots, N \implies p = 0. \tag{1.2.4}$$

A plane algebraic curve is the zero set of some bivariate polynomial. To simplify notation, we shall use the same letter, say p, to denote the polynomial p of degree $n \geq 1$ and the curve (of the same degree) given by the equation $p(x, y) = 0$.

We get from Proposition 1.2.3 the following geometric interpretation of n-poisedness.

Proposition 1.2.4. The set of nodes $\mathcal{X}_N$ is not n-poised if and only if there is an algebraic plane curve of degree n passing through all the nodes of $\mathcal{X}_N$.

As it follows from Proposition 1.2.4 the construction of a set $\mathcal{X}_N$ which is not n-poised is very easy. One just needs to choose an algebraic curve p of degree n and locate all the nodes of $\mathcal{X}_N$ in that curve. While the construction of an n-poised set $\mathcal{X}_N$ is more difficult, since the n-poisedness means that there is no curve p of degree n passing through all the nodes of $\mathcal{X}_N$.

A general construction of n-poised sets was introduced by Berzolari [3] and Radon [23]. This construction is described in Section 1.2.2.

Next, we bring some known results on n-independence.

In Chapter 2 we start the presentation of the results of this thesis.

First the factorization of fundamental polynomials is studied. In the case of univariate interpolation fundamental polynomials can be presented as products of linear factors. But in the case of bivariate interpolation this is not always possible. Here we characterize n-independent node sets for which all fundamental polynomials are products of lines or conics.

Theorem 2.1.1 ([27]). Let $\mathcal{X}$ be an n-independent set of nodes with $|\mathcal{X}| \leq 2n+1$. Then for each node of $\mathcal{X}$ there is an n-fundamental polynomial, which is a product of lines. Moreover, this statement is not true in general for n-independent node sets $\mathcal{X}$ with $|\mathcal{X}| \geq 2n+2$.

In Section 2.3 we present the main result of Chapter 2:

Theorem 2.3.1 ([27]). Let $\mathcal{X}$ be an n-independent set of nodes with $|\mathcal{X}| \leq 2n + [n/2] + 1$. Then for each node of $\mathcal{X}$ there is an n-fundamental polynomial, which is a product of lines and conics. Moreover, this statement is not true in general for n-independent node sets $\mathcal{X}$ with $|\mathcal{X}| \geq 2n + [n/2] + 2$ and $n \geq 3$.

We bring a counterexample to prove the "Moreover" part of this Theorem.

In Chapter 3 we present our results concerning factorization of trivariate fundamental polynomials. We provide the following

Proposition 3.1.1 ([28]). Let $\mathcal{X}$ be a set of knots in $\mathbb{R}^2$ with $|\mathcal{X}| = 3n - k$, where $k, n \geq 1$ and $A \in \mathcal{X}$. Suppose that the following three conditions hold:

i) No $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

ii) If $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear and are lying in a line α, $A \notin \alpha$, then no n knots of $\mathcal{X} \setminus \alpha$ are collinear together with A;

iii) No $2n+1$ knots of $\mathcal{X} \setminus \{A\}$ belong to an irreducible conic together with A.

Then there exists a fundamental polynomial of A of form $p^\star_{A,\mathcal{X}} = \alpha_1\alpha_2...\alpha_k q$, where $\alpha_1, ..., \alpha_k$ are lines and $q \in \Pi_{n-k}$.

We denote by $\Pi_n(L)$ the set of restrictions of polynomials of total degree at most n on a plane L.

Bellow we present the main results of Chapter 3.

Proposition 3.2.2 ([28])**.** Let $\mathcal{X}$ be a set of knots in $\mathbb{R}^3$ with $|\mathcal{X}| \le 3n+1$ and $A \in \mathcal{X}$. Then the knot A has an n-fundamental polynomial, which is a product of linear factors, if and only if the following two assertions hold:

i) No $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

ii) If at least $2n+1$ of $\mathcal{X} \setminus \{A\}$ knots are lying in a plane L_A passing through A then all the knots of $\mathcal{X} \cap L_A$ different from A lie in n lines not passing through A.

In Section 3.3 we prove the following

Theorem 3.3.1 ([28])**.** Let $\mathcal{X}$ be a set of non-coplanar knots in $\mathbb{R}^3$ with $|\mathcal{X}| \le 3n+1$. Then $\mathcal{X}$ is n-independent if and only if the following three statements hold:

i) No $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

ii) There are no $2n+2$ coplanar knots of $\mathcal{X}$, which belong to a conic(reducible or irreducible);

iii) If $3n$ knots belong to a plane L, then there are no curves $\sigma_n \in \Pi_n(L)$ and $\gamma \in \Pi_3(L)$ such that $\mathcal{X} \cap L = \sigma_n \cap \gamma$.

At the end of Section 3.3 we present the following

Corollary 3.3.4 ([28])**.** Let $\mathcal{X}$ be a set of knots in $\mathbb{R}^3$ with $|\mathcal{X}| \le 3n+1$. Then $\mathcal{X}$ is n-independent if and only if for any plane L the set $\mathcal{X} \cap L$ is n-independent.

Part II of the thesis is devoted to GC_n sets, i.e, n-poised sets where each node possesses a fundamental polynomial which is a product of n lines.

In Chapter 4 we define GC_n sets, present their classification and bring some known results.

Definition 4.1.1. Given an n-poised set $\mathcal{X}$. We say that a node $A \in \mathcal{X}$ *uses a line* $\ell \in \Pi_1$, if $p_A^\star = \ell q,$ where $q \in \Pi_{n-1}$.

Next, we bring the following proposition concerning the factorization of polynomials vanishing at some points of lines.

Proposition 4.1.2. Suppose that a polynomial $p \in \Pi_n$ vanishes at $n+1$ points of a line ℓ. Then we have that $p = \ell r,$ where $r \in \Pi_{n-1}$.

As it follows from Proposition 4.1.2, at most $n+1$ nodes of an n-poised set $\mathcal{X}$ can be collinear. Thus we arrive to the following

Definition 4.1.3 ([4])**.** A line passing through $n+1$ nodes is called a *maximal line.*

Clearly, in view of Proposition 4.1.2 a maximal line λ is used by all the nodes in $\mathcal{X} \setminus \lambda$.

In Section 4.1, we define *GC* sets introduced by K.C. Chung and T.H. Yao.

Definition 4.1.6 ([14])**.** An n-poised set $\mathcal{X}$ is called GC_n *set* (or *GC* set) if the n-fundamental polynomial of each node $A \in \mathcal{X}$ is a product of n linear factors.

So, GC_n sets are n-poised sets such that each of its nodes uses exactly n lines.

Next, the Gasca-Maeztu conjecture, briefly called GM conjecture is presented:

Conjecture 4.1.7 ([16], Sect. 5)**.** Any GC_n set possesses a maximal line.

Until now, this conjecture has been confirmed to be true for the degrees $n \leq 5$ (see [5], [19]). For a generalization of the Gasca-Maeztu conjecture to maximal curves see [20].

The following important result is due to Carnicer and Gasca

Theorem 4.1.8 ([8], Thm. 4.1)**.** If the Gasca-Maeztu conjecture is true for all $k \leq n$, then any GC_n set possesses at least three maximal lines.

This yields, in view of Proposition 4.1.2, that each node of a GC_n set $\mathcal{X}$ uses at least one maximal line.

Denote by $\mu = \mu(\mathcal{X})$ the number of maximal lines of a node set $\mathcal{X}$. Thus, we have for any GC_n set $\mathcal{X}$:

$$3 \leq \mu(\mathcal{X}) \leq n+2, \tag{4.1.2}$$

where for the first inequality it is assumed that GM conjecture is true.

In Section 4.2, we start consideration of the results of Carnicer, Gasca, and Godés, concerning the classification of GC_n sets according to the number of maximal lines the sets possess.

Theorem 4.2.1 ([12])**.** Let $\mathcal{X}$ be a GC_n set with $\mu(\mathcal{X})$ maximal lines. Suppose also that GM conjecture is true for the degrees not exceeding n. Then $\mu(\mathcal{X}) \in \{3, n-1, n, n+1, n+2\}$.

We define $\mathcal{N}_\ell$ and $\mathcal{X}_\ell$ sets and present some known results which will be used in the sequel.

Definition 4.3.1 ([7])**.** Given an n-poised set $\mathcal{X}$ and a line ℓ. Then

(i) $\mathcal{X}_\ell$ is the subset of nodes of $\mathcal{X}$ which use the line ℓ;

(ii) $\mathcal{N}_\ell$ is the subset of nodes of $\mathcal{X}$ which do not use the line ℓ and do not lie in ℓ.

It is easy to see that

$$\mathcal{X}_\ell \cup \mathcal{N}_\ell = \mathcal{X} \setminus \ell. \tag{4.3.1}$$

Note that the previously mentioned statement on maximal lines can be expressed as follows

$$\mathcal{X}_\ell = \mathcal{X} \setminus \ell, \text{ if } \ell \text{ is a maximal line.} \tag{4.3.2}$$

Suppose that λ is a maximal line of $\mathcal{X}$ and $\ell \neq \lambda$ is any line. Then we have that

$$\mathcal{X}_\ell \setminus \lambda = (\mathcal{X} \setminus \lambda)_\ell. \tag{4.3.3}$$

Let $\mathcal{X}$ be an n-poised set and ℓ be a line with $|\ell \cap \mathcal{X}| \leq n$. We call a maximal line λ *ℓ-disjoint* if

$$\lambda \cap \ell \cap \mathcal{X} = \emptyset. \tag{4.3.4}$$

Let $\mathcal{X}$ be an n-poised set and ℓ be a line with $|\ell \cap \mathcal{X}| \leq n$. We call two maximal lines λ', λ'' *ℓ-adjacent* if

$$\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}. \tag{4.3.6}$$

Next, we introduce the concept of an ℓ-reduction of a GC_n set and ℓ-proper GC_m subsets.

Definition 4.3.5. Let $\mathcal{X}$ be a GC_n set, ℓ be a k-node line, $k \geq 2$. We say that a set $\mathcal{Y} \subset \mathcal{X}$ is an ℓ-reduction of $\mathcal{X}$, and briefly denote this by $\mathcal{X} \searrow_{\ell} \mathcal{Y}$, if

$$\mathcal{Y} = \mathcal{X} \setminus (\mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_s),$$

where

(i) $\mathcal{C}_0$ is an ℓ-disjoint maximal line of $\mathcal{X}$, or $\mathcal{C}_0$ is the union of a pair of ℓ-adjacent maximal lines of $\mathcal{X}$;

(ii) $\mathcal{C}_i$ is an ℓ-disjoint maximal line of the GC set $\mathcal{Y}_i := \mathcal{X} \setminus (\mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{i-1})$, or $\mathcal{C}_i$ is the union of a pair of ℓ-adjacent maximal lines of $\mathcal{Y}_i,\ i = 1, \ldots s$;

(iii) ℓ passes through at least 2 nodes of $\mathcal{Y}$.

Definition 4.3.6. Let $\mathcal{X}$ be a GC_n set, ℓ be a k-node line, $k \geq 2$. We say that the set $\mathcal{X}_\ell$ is an ℓ-proper GC_m subset of $\mathcal{X}$ if there is a GC_{m+1} set $\mathcal{Y}$ such that

(i) $\mathcal{X} \searrow_{\ell} \mathcal{Y}$;

(ii) The line ℓ is a maximal line in $\mathcal{Y}$.

The following result immediately follows from Definitions 4.3.5 and 4.3.6.

Proposition 4.3.7. Suppose that $\mathcal{X}$ is a GC_n set. If $\mathcal{X} \searrow_{\ell} \mathcal{Y}$ and $\mathcal{Y}_\ell$ is an ℓ-proper GC_m subset of $\mathcal{Y}$ then $\mathcal{X}_\ell$ is an ℓ-proper GC_m subset of $\mathcal{X}$.

At the end of Chapter 4 we present a new, simple proof of the Gasca-Maeztu conjecture for the case $n = 4$. The Conjecture was proposed in 1981 by Gasca and Maeztu [16]. Until now, this has been confirmed only for the values $n \leq 5$. The case $n = 5$ was proven in 2014 by Hakopian, Jetter, and Zimmermann, in [19]. So far that is the only proof for $n = 5$. In addition, it is very long and complicated. In our opinion a simple proof of the Gasca-Maeztu conjecture for smaller values of n greatly simplifies its generalization for higher values. We believe that this is a way in trying to prove the Conjecture for the values $n \geq 6$.

In Chapter 5 we consider the main result of the paper [1] by V. Bayramyan and H. Hakopian, stating that any n-node line of GC_n set is used either by exactly $\binom{n}{2}$ nodes or by exactly $\binom{n-1}{2}$ nodes, provided that the Gasca-Maeztu conjecture is true.

Here we show that this result is not correct in the case $n = 3$. Namely, we bring an example of a GC_3 set and a 3-node line there which is not used at all. The proof of the result in [1] is inductive and based on the case $n = 3$. For this reason we needed to consider a new proof of the result. Then we were able to establish that this is the only possible counterexample, i.e., the above mentioned result is true for all $n \geq 4$.

Theorem 5.1.1 ([25])**.** Assume that Conjecture 4.1.7 holds for all degrees up to n. Let $\mathcal{X}$ be a GC_n set, $n \geq 4$, and ℓ be an n-node line. Then we have that

$$|\mathcal{X}_\ell| = \binom{n}{2} \quad \text{or} \quad \binom{n-1}{2}.$$

Moreover, the following hold:

(i) $|\mathcal{X}_\ell| = \binom{n}{2}$ if and only if there is a maximal line λ_0 such that $\lambda_0 \cap \ell \cap \mathcal{X} = \emptyset$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda_0)$. Hence it is a GC_{n-2} set;

(ii) $|\mathcal{X}_\ell| = \binom{n-1}{2}$ if and only if there are two maximal lines λ', λ'', such that $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda' \cup \lambda'')$. Hence it is a GC_{n-3} set.

We also characterize the exclusive case $n = 3$ and present some new results on the maximal lines and the usage of n-node lines in GC_n sets.

Proposition 5.1.3 ([25])**.** Let $\mathcal{X}$ be a GC_3 set and ℓ be a 3-node line. Then we have that

$$|\mathcal{X}_\ell| = 3, \quad 1, \quad \text{or} \quad 0.$$

Moreover, the following hold:

(i) $|\mathcal{X}_\ell| = 3$ if and only if there is a maximal line λ_0 such that $\lambda_0 \cap \ell \cap \mathcal{X} = \emptyset$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda_0)$. Hence it is a GC_1 set.

(ii) $|\mathcal{X}_\ell| = 1$ if and only if there are two maximal lines λ', λ'', such that $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda' \cup \lambda'')$;

(iii) $|\mathcal{X}_\ell| = 0$ if and only if there are exactly three maximal lines in $\mathcal{X}$ and they intersect ℓ at three distinct nodes.

Furthermore, if the node set $\mathcal{X}$ possesses exactly three maximal lines then any 3-node line ℓ is either used by exactly three nodes or is not used at all:

$$|\mathcal{X}_\ell| = 3 \quad \text{or} \quad 0.$$

In Section 5.2, we present the proof of Theorem 5.1.1.

Next, we present the following simple but interesting by itself proposition.

Proposition 5.2.2 ([25]). Let $\mathcal{X}$ be a GC_n set and ℓ be an n-node line, where $n \geq 4$. Suppose that there are n maximal lines passing through n distinct nodes in ℓ. Then there exists at least one more maximal line in $\mathcal{X}$.

At the end of Section 5.2 we present the following

Corollary 5.2.4 ([25]). Assume that Conjecture 4.1.7 holds for all degrees up to n. Let $\mathcal{X}$ be a GC_n set with exactly three maximal lines, where $n \geq 4$. Then there are exactly three n-node lines in $\mathcal{X}$ and each of them is used by exactly $\binom{n}{2}$ nodes from $\mathcal{X}$.

Remark 5.2.5. It is worth mentioning that Corollary 5.2.4 is not valid in the case $n = 3$. Indeed, the GC_3 set $\mathcal{X}^\star$ and the 3-node lines ℓ_1, ℓ_2, ℓ_3 and $\ell^\star$ (see Fig. 5.1.1) give us a counterexample for this.

In Chapter 6 we consider a conjecture proposed in the paper [1] by V. Bayramyan and H. Hakopian, concerning the usage of any k-node line in GC_n sets, $2 \leq k \leq n+1$. Here we make an adjustment in the mentioned conjecture and then prove it. Namely, by assuming that the Gasca-Maeztu conjecture is true, we prove that for any GC_n set $\mathcal{X}$ and any k-node line ℓ the following statement holds:
The line ℓ is not used at all, or it is used by exactly $\binom{s}{2}$ nodes of $\mathcal{X}$, where s satisfies the condition $k - \delta \leq s \leq k,\ \delta = n + 1 - k$. If in addition $k - \delta \geq 3$ and $\mu(\mathcal{X}) > 3$ then the first case here is excluded, i.e., the line ℓ is necessarily a used line.

In Section 6.2, we formulate an adjusted version of the conjecture proposed by V. Bayramyan and H. Hakopian in [1] (Conj. 3.7) as:

Theorem 6.2.1 ([26], [31])**.** Let $\mathcal{X}$ be a GC_n set, and ℓ be a k-node line, $k \geq 2$. Assume that GM Conjecture holds for all degrees up to n. Then we have that

$$\mathcal{X}_\ell = \emptyset, \text{ or} \tag{6.2.1}$$

$$\mathcal{X}_\ell \text{ is an } \ell\text{-proper } GC_{s-2} \text{ subset of } \mathcal{X}, \text{ hence } |\mathcal{X}_\ell| = \binom{s}{2}, \tag{6.2.2}$$

for some $k - \delta \leq s \leq k$ and $\delta = n + 1 - k$.
Moreover, if $k - \delta \geq 3$ and $\mu(\mathcal{X}) > 3$ then $\mathcal{X}_\ell \neq \emptyset$, i.e., (6.2.2) holds with $s \geq 2$. Furthermore, in the case $\mathcal{X}_\ell \neq \emptyset$ we have for any maximal line λ :
$|\lambda \cap \mathcal{X}_\ell| = 0$ or $|\lambda \cap \mathcal{X}_\ell| = s - 1$.

Next, we present the following proposition, which shows that Theorem 6.2.1 is true for the node set $\mathcal{X}$ if $\mu(\mathcal{X}) = 3$.

Proposition 6.2.5 ([26], [31])**.** Let $\mathcal{X}$ be a GC_n set with $\mu(\mathcal{X}) = 3$ and ℓ be a k-node line, $k \geq 2$. Assume that GM Conjecture holds for all degrees up to $n - 3$. Then we have that

$$\mathcal{X}_\ell = \emptyset, \text{ or } \mathcal{X}_\ell \text{ is an } \ell\text{-proper } GC_{k-2} \text{ subset of } \mathcal{X}, \text{ hence } |\mathcal{X}_\ell| = \binom{k}{2}.$$

Moreover, if $k \leq n$ and $\mathcal{X}_\ell \neq \emptyset$ then for a maximal line λ_1 of $\mathcal{X}$ we have that $\lambda_1 \cap \ell \notin \mathcal{X}$ and $|\lambda_1 \cap \mathcal{X}_\ell| = 0$.
For the remaining two maximal lines we have that $|\lambda \cap \mathcal{X}_\ell| = k - 1$.
Furthermore, if the line ℓ intersects each maximal line at a node then $\mathcal{X}_\ell = \emptyset$.

At the end of Section 6.2 we provide a result on the presence and usage of $(n-1)$-node lines in GC_n sets with $\mu(\mathcal{X}) = n - 1$.

Proposition 6.2.6 ([26], [31])**.** Let $\mathcal{X}$ be a GC_n set with $\mu(\mathcal{X}) = n-1$, and ℓ be an $(n-1)$-node line, where $n \geq 4$. Assume also that through each node of ℓ there passes exactly one maximal line. Then we have that either $n = 4$ or $n = 5$. Moreover, in both these cases we have that $\mathcal{X}_\ell = \emptyset$.

We characterize the case $k-\delta = 2,\ \mu(\mathcal{X}) > 3$. For each n and k, with $k-\delta = 2k-n-1 = 2$, we bring two constructions of GC_n sets and a non-used k-node line in each case. The following proposition shows that these are the only constructions with the mentioned property.

Proposition 6.4.1 ([26], [31])**.** Let $\mathcal{X}$ be a GC_n set and ℓ be a k-node line with $k - \delta := 2k - n - 1 = 2$ and $\mu(\mathcal{X}) > 3$. Suppose that the line ℓ is a non-used line. Then we have that either $\mathcal{X} = \bar{\mathcal{X}}^*,\ \ell = \bar{\ell}_4^*$, or $\mathcal{X} = \bar{\mathcal{Y}}^*,\ \ell = \bar{\ell}_3^*$.

Part I

FACTORIZATION OF FUNDAMENTAL POLYNOMIALS

Chapter 1

MULTIVARIATE INTERPOLATION

1.1 Univariate Polynomial Interpolation

Let π_n be the space of univariate polynomials of total degree at most n :

$$\pi_n = \left\{ \sum_{i \leq n} a_i x^i : a_i \in \mathbb{R} \right\}.$$

We have that

$$\dim \pi_n = n + 1.$$

Consider a set of distinct points on the number line

$$\mathcal{X}_s = \{x_0, \dots, x_s\} \subset \mathbb{R}.$$

The problem of finding a polynomial $p \in \pi_n$ which satisfies the conditions

$$p(x_i) = c_i, \quad i = 0, \dots s, \tag{1.1.1}$$

where $c_i \in \mathbb{R},\ i = 0, \dots s$ are arbitrary numbers, is called a univariate interpolation problem.

The polynomial p is called interpolation polynomial, and the points $x_0, \dots, x_s$ interpolation points.

The conditions (1.1.1) form a system of $s + 1$ linear equations in $n + 1$ variables, where the variables are the coefficients of the polynomial.

A necessary and sufficient condition for unisolvence is $s = n$.

Theorem 1.1.1. *For arbitrary chosen values $c_0, \dots, c_n$, there is a unique polynomial $p \in \pi_n$, such that*

$$p(x_i) = c_i, \quad i = 0, \dots n. \tag{1.1.2}$$

Newton and Lagrange have the two main approaches for the solution of interpolation problem.

A polynomial $p \in \pi_n$ is called an n-fundamental polynomial for a node $x_k \in \mathcal{X}_s$ if

$$p(x_i) = \delta_{ik}, \quad i = 0, \dots, s,$$

where δ is the Kronecker symbol. We denote this fundamental polynomial by p_k. It is easy to see that the fundamental polynomial of the point x_k can be presented by the following formula:

$$p_k(x) = \prod_{i=0, i \neq k}^{n} \frac{x - x_i}{x_k - x_i}, \quad k = 0, \dots, n.$$

Having fundamental polynomials for all the points we can write the Lagrange formula for the the interpolation polynomial:

$$p(x) = \sum_{k=0}^{n} c_k p_k(x).$$

In view of Theorem 1.1.1 we can state that this is a unique solution of the interpolation problem.

The Newton solution is based on the following representation of interpolation polynomial:

$$p(x) = \gamma_0 + \gamma_1(x - x_0) + ... + \gamma_n(x - x_0)(x - x_1)...(x - x_{n-1}).$$

Here $\gamma_i, \quad i = 0, \dots, n$, are consecutively chosen constants such that the corresponding i-th condition of (1.1.2), i.e., the condition at x_i is satisfied.

Let us define divided differences by the following recurrent formula:

$$[x_0, x_1, \dots, x_n]f = \frac{[x_1, x_2, \dots, x_n]f - [x_0, x_1, \dots, x_{n-1}]f}{x_n - x_0}.$$

$$[x_0]f = f(x_0),$$

For a given function f and knots $x_i,\ i = 0, \dots, n$, the polynomial $p \in \pi_n$ satisfying the following conditions

$$p(x_i) = f(x_i)$$

is called an interpolation polynomial of function f and is denoted by $p_f(x)$.

The Newton formula is the following:

$$p_f(x) = \sum_{i=0}^{n} [x_0, x_1, \dots, x_i] f(x - x_0) \dots (x - x_{i-1}).$$

1.2 Multivariate Polynomial Interpolation

Let us start with some notation. Denote

$$\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d,\ \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$$

$$\mathbf{x}^\alpha = {x_1}^{\alpha_1} \cdot \dots \cdot {x_d}^{\alpha_d},\ |\alpha| = \alpha_1 + \dots + \alpha_d.$$

Let Π_n^d be the space of polynomials of d variables of total degree at most n :

$$\Pi_n^d = \left\{ \sum_{|\alpha| \le n} a_\alpha \mathbf{x}^\alpha : a_\alpha \in \mathbb{R}^d \right\}.$$

We have that

$$N(n, d) := \dim \Pi_n^d = \binom{n+d}{d}.$$

Consider a set of distinct nodes (points)

$$\mathcal{X}_s = \{\mathbf{x}_1, \dots, \mathbf{x}_s\} \subset \mathbb{R}^d.$$

Let $c_i, \quad i = 1, \dots s$, be pregiven numbers. The problem of finding a polynomial $p \in \Pi_n^d$ which satisfies the conditions

$$p(\mathbf{x}_i) = c_i, \quad i = 1, \dots s, \tag{1.2.1}$$

is called interpolation problem.

1.2.1 Poised and Independent sets

A polynomial $p \in \Pi_n^d$ is called an n-fundamental polynomial for a node $A = \mathbf{x}_k \in \mathcal{X}_s$ if

$$p(\mathbf{x}_i) = \delta_{ik}, \quad i = 1, \ldots, s,$$

where δ is the Kronecker symbol. We denote the fundamental polynomial of the node A by $p_k^\star = p_A^\star = p_{A,\mathcal{X}_s}^\star$. Sometimes we call fundamental also a polynomial that vanishes at all nodes of $\mathcal{X}$ but one, since it is a nonzero constant times a fundamental polynomial.

Definition 1.2.1. A set of nodes $\mathcal{X}$ is called *n-independent* if all its nodes have fundamental polynomials. Otherwise, $\mathcal{X}$ is called *n-dependent.*

Fundamental polynomials are linearly independent. Therefore a necessary condition of n-independence is $|\mathcal{X}| \leq N(n, d)$. Having fundamental polynomials of all nodes of $\mathcal{X}$ we get a solution of general interpolation problem (1.2.1) by using the Lagrange formula:

$$p(\mathbf{x}) = \sum_{i=1}^{s} c_i p_i^\star(\mathbf{x}). \tag{1.2.2}$$

Thus we get readily that the node set $\mathcal{X}_s$ is n-independent if and only if it is n-solvable, meaning that for any data $\{c_1, \ldots, c_s\}$ there exists a (not necessarily unique) polynomial $p \in \Pi_n^d$ satisfying the conditions (1.2.1).

Definition 1.2.2. The set of nodes $\mathcal{X}_s$ is called *n-poised* if for any data $\{c_1, \ldots, c_s\}$ there exists a unique polynomial $p \in \Pi_n^d$, satisfying the conditions (1.2.1).

The interpolation conditions (1.2.1) give a system of s linear equations with $N(n, d)$ unknowns, which are the coefficients of polynomial p. Therefore a necessary condition for n-poisedness of the set $\mathcal{X}_s$ is

$$s = |\mathcal{X}_s| = N(n, d). \tag{1.2.3}$$

Thus from now on we will consider sets $\mathcal{X} = \mathcal{X}_{N(n,d)}$ when n-poisedness is studied.

Let us mention that in the univariate case the condition (1.2.3) is also sufficient for n-poisedness. In the multivariate case the condition is no longer sufficient, unless the case $n = 0$ stands.

We also have that a set $\mathcal{X}_{N(n,d)}$ is n-poised if and only if it is n-independent.

1.2.2 Bivariate interpolation

In this section we present some known basic results on bivariate interpolation.

For the brevity we denote the space of bivariate polynomials of total degree at most n by

$$\Pi_n := \Pi_n^2.$$

Similarly we set

$$N := N(n, 2) = \binom{n+2}{2}.$$

The following proposition is based on an elementary Linear Algebra argument.

Proposition 1.2.3. *The set of nodes $\mathcal{X}_N$ is n-poised if and only if the following condition holds:*

$$p \in \Pi_n, \; p(x_i, y_i) = 0, \; i = 1, \ldots, N \implies p = 0. \tag{1.2.4}$$

A plane algebraic curve is the zero set of some bivariate polynomial. To simplify notation, we shall use the same letter, say p, to denote the polynomial p of degree $n \geq 1$ and the curve (if the same degree) given by the equation $p(x, y) = 0$. More precisely, suppose p is a polynomial without multiple factors. Then the algebraic curve defined by the equation $p(x, y) = 0$ shall also be denoted by p. So lines, conics, and cubics are equivalent to polynomials of degree 1, 2, and 3, respectively; a reducible conic is a pair of lines, and a reducible cubic is a triple of lines or consists of a line and an irreducible conic. We denote lines, conics and cubics by α, β and γ, respectively.

Thus, (1.2.4) gives the following geometric interpretation of n-poisedness.

Proposition 1.2.4. *The set of nodes $\mathcal{X}_N$ is not n-poised if and only if there is an algebraic plane curve of degree n passing through all the nodes of $\mathcal{X}_N$.*

As it follows from Proposition 1.2.4 the construction of a set $\mathcal{X}_N$ which is not n-poised is very easy. One just needs to choose an algebraic curve p of degree n and locate all the nodes of $\mathcal{X}_N$ in that curve. While the construction of an n-poised set $\mathcal{X}_N$ is more difficult, since the n-poisedness means that there is no curve p of degree n passing through all the nodes of $\mathcal{X}_N$.

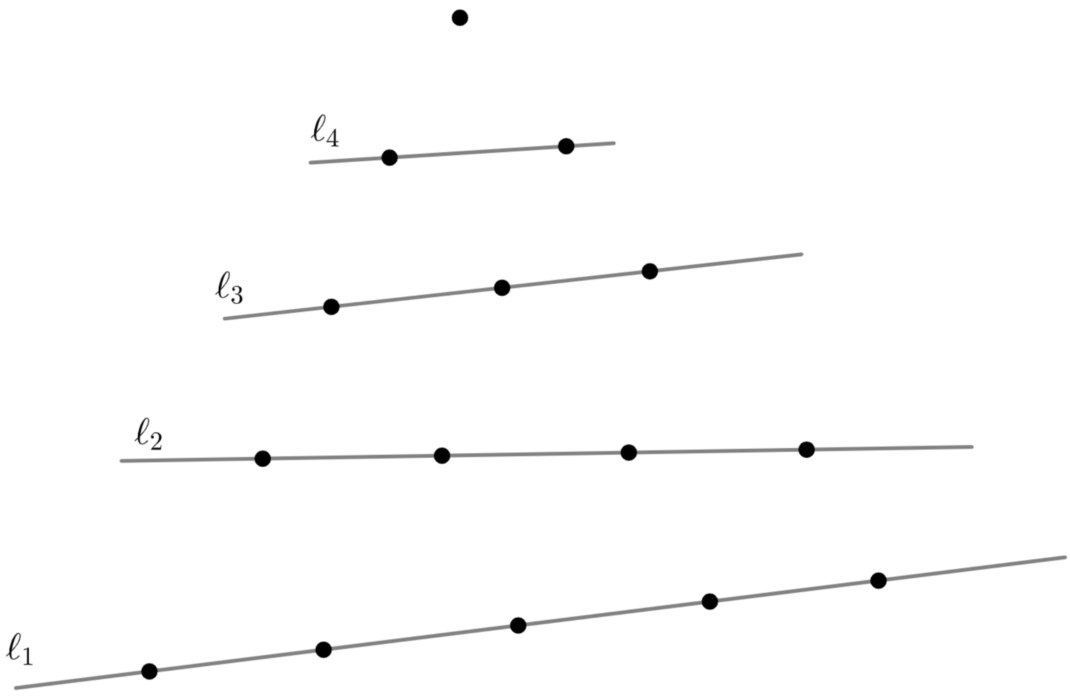

Figure 1.2.1: Berzolari-Radon construction for $n = 4$

A general construction of n-poised sets was introduced by Berzolari [3] and Radon [23] (see Fig. 1.2.5).

Definition 1.2.5. Suppose $\ell_1, ..., \ell_{n+1}$ are distinct lines. The set of nodes N is called Berzolari-Radon set, if

$n + 1$ nodes are lying on ℓ_1,

n nodes are lying on $\ell_2 \setminus \ell_1$,

$\vdots$

1 node is lying on $\ell_{n+1} \setminus (\ell_1 \cup \cdots \cup \ell_n)$.

Proposition 1.2.6. *A set of nodes $\mathcal{X}_N$ satisfying the Berzolari-Radon construction is an n-poised set.*

Let us mention that the proof of this proposition is based on the forthcoming Lemma 1.2.12.

Now let us bring some results on n-independence we shall use in the sequel. Let us start with the following important result of Severi:

Theorem 1.2.7 ([24]). *Any set $\mathcal{X}$, with $|\mathcal{X}| \leq n+1$, is n-independent.*

It is worth mentioning the following

Remark 1.2.8. For each node $A \in \mathcal{X}$ here we can find n-fundamental polynomial which is a product of $|\mathcal{X}| - 1 \leq n$ lines, each of which passes through a respective node of $\mathcal{X} \setminus \{A\}$ and does not pass through A.

Next two results extend the Severi theorem to the cases of sets with no more than $2n+1$ and $3n$ nodes, respectively.

Theorem 1.2.9 ([15], Prop. 1). *Any set $\mathcal{X}$, with $|\mathcal{X}| \leq 2n+1$, is n-dependent, if and only if $n+2$ nodes of $\mathcal{X}$ are collinear.*

Theorem 1.2.10 ([17], Thm. 5.3). *Let $\mathcal{X}$ be a set of nodes with $|\mathcal{X}| \leq 3n$. Then the set $\mathcal{X}$ is n-dependent if and only if, at least one of the following assertions holds:*

i) $n+2$ nodes of $\mathcal{X}$ are collinear,

ii) $2n+2$ nodes of $\mathcal{X}$ are lying on a conic,

iii) $|\mathcal{X}| = 3n$ and there are curves $\gamma \in \Pi_3$ and $p \in \Pi_n$ such that $\gamma \cap p = \mathcal{X}$.

This result was generalized to the case of $\mathbb{R}^k$ in [22].

Theorem 1.2.11 ([22]). *Let $\mathcal{X}$ be a set of knots in $\mathbb{R}^k$, with $|\mathcal{X}| \leq 3n$. Then the set $\mathcal{X}$ is n-dependent if and only if one the following conditions holds:*

i) $n+2$ knots of $\mathcal{X}$ are collinear,

ii) $2n+2$ coplanar knots of $\mathcal{X}$ are lying on a conic,

iii) if $|\mathcal{X}| = 3n$, then $\mathcal{X}$ is coplanar and there are curves $\gamma \in \Pi_3$ and $\sigma_n \in \Pi_n$ such that $\gamma \cap \sigma_n = \mathcal{X}$.

Next, we bring the following proposition concerning the factorization of polynomials vanishing at some points of lines.

Proposition 1.2.12 ([18] Prop. 1.3)**.** *Suppose that α is a line. Then for any polynomial $p \in \Pi_n$ vanishing at $n+1$ points of α we have that*

$$p = \alpha q, \quad \textit{where} \quad q \in \Pi_{n-1}.$$

Next two lemmas concern the similar factorization in the case of reducible and irreducible conics, respectively.

Lemma 1.2.13 ([17])**.** *Suppose that $\alpha_i,\ i = 1, 2,$ are two lines. Then for any polynomial $p \in \Pi_n$ vanishing at $n+1$ points of α_1 and n points of $\alpha_2 \setminus \alpha_1$ we have that*

$$p = \alpha_1 \alpha_2 q, \quad \textit{where} \quad q \in \Pi_{n-2}.$$

Lemma 1.2.14 ([17])**.** *Suppose that β is an irreducible conic. Then for any polynomial $p \in \Pi_n$ vanishing at $2n+1$ points of β we have that*

$$p = \beta q, \quad \textit{where} \quad q \in \Pi_{n-2}.$$

Following theorem is a special case of Cayley-Bacharach theorem from [15].

Theorem 1.2.15 ([15])**.** *Let $\gamma \in \Pi_3$ and $\sigma_n \in \Pi_n$ be curves such that $\gamma \cap \sigma_n = \mathcal{X}$ and $|\mathcal{X}| = 3n$. Then any curve of degree n containing all but one knot of $\mathcal{X}$, contains all of $\mathcal{X}$.*

Chapter 2

FACTORIZATION OF BIVARIATE FUNDAMENTAL POLYNOMIALS

In view of the Lagrange formula (1.2.2) it is important to find n-independent sets for which the fundamental polynomials have the simplest possible forms. In Section 2.1 we characterize n-independent sets for which all fundamental polynomials are products of lines. It is worth mentioning that for the natural lattice, introduced by Chung and Yao in [14], the fundamental polynomials have the mentioned forms. But in this case the configuration of nodes is very special. Namely, they are intersection points of some $n+2$ given lines in general position. In our characterization (see forthcoming Theorem 2.1.1, Proposition 2.1.2) the restrictions on the node set are minimal. In Sections 2.2-2.5 we consider a more involved problem. There we characterize n-independent node sets for which all fundamental polynomials are products of lines or conics.

2.1 The fundamental polynomials as products of lines

Theorem 2.1.1 ([27]). *Let $\mathcal{X}$ be an n-independent set of nodes with $|\mathcal{X}| \leq 2n+1$. Then for each node of $\mathcal{X}$ there is an n-fundamental polynomial, which is a product of lines. Moreover, this statement is not true in general for n-independent node sets $\mathcal{X}$ with $|\mathcal{X}| \geq 2n+2$.*

First, let us present a counterexample for the last statement. Let $\mathcal{X}$ consist of $2n+2$ nodes such that they all do not belong to a conic and no three are collinear. For example, we can take

$2n+1$ nodes on an irreducible conic and a node outside the conic such that it does not belong to any line passing through two nodes on the conic. Then, according to Theorem 1.2.10, $\mathcal{X}$ is n-independent. Now, notice that no node $A \in \mathcal{X}$ has an n-fundamental polynomial, which is a product of lines. Indeed, each line passes through at most two nodes of $\mathcal{X}$, hence n lines pass through at most $2n$ nodes of $\mathcal{X}$. But there are $2n+1$ nodes in $\mathcal{X} \setminus \{A\}$.

The first statement of Theorem follows from the following result, which covers wider a setting.

Proposition 2.1.2 ([27]). *Let $\mathcal{X}$ be a set of nodes with $|\mathcal{X}| \leq 2n+1$ and $A \in \mathcal{X}$. Then the following three statements are equivalent:*

i) The node A has an n-fundamental polynomial;

ii) The node A has an n-fundamental polynomial, which is a product of linear factors;

iii) No $n+1$ nodes of $\mathcal{X} \setminus \{A\}$ are collinear together with the node A.

Proof. Evidently ii) implies i). In view of Proposition 1.2.12 we get that i) implies iii). Indeed, suppose by way of contradiction that a line passes through A and $n+1$ other nodes of $\mathcal{X}$. Then the fundamental polynomial $p_{A,\mathcal{X}}$ vanishes at those $n+1$ nodes. Therefore, by Proposition 1.2.12, it vanishes at all points of the line including A, which is a contradiction. Thus it remains to prove the implication iii) $\Rightarrow$ ii). We will use induction on n. The case $n=1$, is evident. Now suppose that the statement is true for $n-1$, and let us prove it for n. Suppose α_A is a line passing through A and k other nodes of $\mathcal{X}$, with maximal possible k, where $1 \leq k \leq n$. Consider a node $B \neq A$ from α_A and a node C from outside of α_A. Note, that if there is no such node then we have that $|\mathcal{X}| \leq k+1 \leq n+1$. Therefore the statement follows from Remark 1.2.8.

Denote by α the line passing through B and C. Denote also $\mathcal{X}' := \mathcal{X} \setminus \alpha$. We have that $|\mathcal{X}'| \leq 2(n-1)+1$. Let us show that the condition iii) of Proposition is satisfied for the node set $\mathcal{X}'$ and $n-1$, i.e., no $n=(n-1)+1$ nodes of $\mathcal{X}'$ are collinear together with the node A.

Assume to the contrary that there is a line α'_A passing through A and n other nodes of $\mathcal{X}'$. Then $k=n$, according to the choice of the line α_A. Therefore the number of nodes of $\mathcal{X}$ on the lines α_A and α'_A is $k+n+1=2n+1$, where 1 stands for the node A. Now notice that we took the node C from outside of α_A is outside of α'_A too, since otherwise we would have A and $n+1$ other nodes on α'_A. Hence, the number of nodes of $\mathcal{X}$ is at least $2n+2=(2n+1)+1$, where the last 1 stands for the node C. This is a contradiction. Thus the condition iii) is satisfied for $\mathcal{X}'$ and $n-1$ and, by using the induction hypothesis, we can take $p^\star_{A,\mathcal{X}} = \alpha p^\star_{A,\mathcal{X}'}$, where $p^\star_{A,\mathcal{X}'}$ is an $(n-1)$-fundamental polynomial in form of product of lines. □

2.2 The fundamental polynomials as products of lines and conics

Let us start with the following result of Radon:

Lemma 2.2.1 ([23]). *Let $\mathcal{X}$ be a set of 6 nodes, three of which belong to a line α and other three are outside of α and noncollinear. Then the node set $\mathcal{X}$ is 2-poised, i.e., there is no conic passing through all the nodes.*

Indeed, suppose, in view of Proposition 1.2.3, that $p \in \Pi_2$ vanishes at these 6 nodes. Then, by using Lemma 1.2.13 we get $p = \alpha\alpha_1$, where $\alpha_1 \in \Pi_1$ vanishes at the three noncollinear nodes. Therefore $\alpha_1 = 0$ and hence $p = 0$. □

The following lemma will be used frequently in the sequel.

Lemma 2.2.2 ([27]). *The following statements hold:*

i) There is a conic through any 5 points;

ii) Any 5 points determine a unique conic passing through them if and only if no 4 of them are collinear;

iii) Any 5 points determine an irreducible conic passing through them if and only if no 3 of them are collinear.

Proof. Notice, for i), that the conditions $p \in \Pi_2,\ p(x_i, y_i) = 0,\ i = 1, \ldots, 5$, give a system of 5 linear homogeneous equations with 6 unknowns (the coefficients of p), which has a nontrivial solution. Suppose, for ii), a set of 5 points $\mathcal{X}$ is given, 4 of which are collinear. Then evidently there are infinitely many conics, in form of pairs of lines, passing through all 5 points. For the inverse implication, for ii), suppose, a set of 5 points $\mathcal{X}$ is given, no 4 of which are collinear. First, let us show that we can add a sixth point A to $\mathcal{X}$ such that the conditions of Lemma 2.2.1 are satisfied. Indeed, if there are three collinear points in $\mathcal{X}$, lying in a line α then we take a point A outside of α such that A and the two points of $\mathcal{X} \setminus \alpha$ are not collinear. Next, suppose there are no three collinear points in $\mathcal{X}$. Consider the set of 10 lines passing through any two points of $\mathcal{X}$. Then we add A such that it belongs to only one such line. Now, suppose by way of contradiction, that there are two different conics: β_1 and β_2 that pass through all 5 points of $\mathcal{X}$. Then we can choose a scalar c such that the conic $\beta_1 + c\beta_2$ passes through the point A, besides the points of $\mathcal{X}$. This is a contradiction in view of Proposition 1.2.3. Finally, for iii) notice that if a reducible conic $\beta = \alpha_1\alpha_2$ passes through 5 points then 3 of them belong to a line $\alpha_i,\ i = 1, 2$. Conversely, if a conic β passes through 5 points, 3 of which belong to a line α then, in view of Proposition 1.2.12, α is a factor of β. □

Lemma 2.2.3 ([27])**.** *Suppose* 5 *points: A, B, C, D and E determine a unique conic which does not pass through a point F. Then the points B, C, D, E and F determine a conic, which does not pass through the point A.*

Proof. Clearly it suffices to verify only that the set of points $\mathcal{X} := \{B, C, D, E, F\}$ determines a conic, since then the second statement is obvious. Suppose by way of contradiction, that there are two different conics: β_1 and β_2 that pass through all 5 points of $\mathcal{X}$. Then we can choose a scalar c such that the conic $\beta_1 + c\beta_2$ passes through the point A besides the points of $\mathcal{X}$. This is a contradiction, since there is just one conic passing through the points A, B, C, D, E, which is not passing through F. □

At the end we bring a lemma which will be used later in Section 2.4 (Step 2).

Lemma 2.2.4 ([27])**.** *Let $\mathcal{X}$ be a set of* 6 *points, no* 5 *of which are collinear. Then we can divide the point set $\mathcal{X}$ into two noncollinear triples.*

Proof. We will divide $\mathcal{X}$ into two triples by pointing out only the nodes of a noncollinear triple. In each case it is evident that the remaining three nodes also are noncollinear. First, suppose that there are 4 nodes in $\mathcal{X}$ belonging to a line α. Then we divide $\mathcal{X}$ by choosing a triple in the following way: we take two nodes from α and one from outside. Now suppose that no 4 nodes of $\mathcal{X}$ are collinear. Suppose also that 3 nodes of $\mathcal{X}$ belong to a line α_1 and other 3 nodes belong to another line α_2. In this case we choose a triple in the following way: we take two nodes from α_1 and one node from α_2. Then, suppose that there is a line α that contains exactly 3 nodes of $\mathcal{X}$ and the other 3 nodes are not collinear. Then we take two nodes from α and one node A from outside. Besides, if there is a line passing through the third node of α and two nodes from outside of α then we take one of these two as A. Finally, suppose that no 3 nodes of $\mathcal{X}$ are collinear. In this case we divide $\mathcal{X}$ into any two triples. $\square$

2.3 The main theorem and proposition

Theorem 2.3.1 ([27]). *Let $\mathcal{X}$ be an n-independent set of nodes with $|\mathcal{X}| \leq 2n + [n/2] + 1$. Then for each node of $\mathcal{X}$ there is an n-fundamental polynomial, which is a product of lines and conics. Moreover, this statement is not true in general for n-independent node sets $\mathcal{X}$ with $|\mathcal{X}| \geq 2n + [n/2] + 2$ and $n \geq 3$.*

First, let us present a counterexample for the last statement. Suppose that $\mathcal{X}$ consists of $2n + [n/2] + 2$ nodes no three of which are collinear and no six of which belong to a conic. In the cases $n = 3, 4$ in addition we assume that all the nodes of $\mathcal{X}$ do not belong to any cubic. Then, according to Theorem 1.2.10, $\mathcal{X}$ is n-independent. Now let us verify that no node has an n-fundamental polynomial, which is a product of lines and conics.

Suppose by way of contradiction that a fundamental polynomial of a node $A \in \mathcal{X}$ is a product of ν lines and μ conics with $\nu + 2\mu \leq n$. Each line passes through at most two nodes of $\mathcal{X}$, and each conic passes through at most five nodes of $\mathcal{X}$. Hence $p_{A,\mathcal{X}}$ vanishes at most at $2\nu + 5\mu$ nodes. We will show that $2\nu + 5\mu \leq |\mathcal{X}| - 2$, which is a contradiction.

Indeed, in case of even $n = 2k$ we have that $|\mathcal{X}| = 5k + 2$ and $2\nu + 5\mu \leq 2(\nu + 2\mu) + \mu \leq 2n + \mu \leq 4k + k = 5k = |\mathcal{X}| - 2$. While in case of odd $n = 2k + 1$ we have that $|\mathcal{X}| = 5k + 4$ and $2\nu + 5\mu \leq 2(\nu + 2\mu) + \mu \leq 2n + \mu \leq (4k + 2) + k = 5k + 2 = |\mathcal{X}| - 2$.

The first statement of Theorem follows from the following result, which covers a wider setting.

Proposition 2.3.2 ([27]). *Let $\mathcal{X}$ be a set of nodes with $|\mathcal{X}| \leq 2n + [n/2] + 1$ and $A \in \mathcal{X}$. Then the following three statements are equivalent:*

i) The node A has an n-fundamental polynomial;

ii) The node A has an n-fundamental polynomial, which is a product of lines and conics;

iii) a) No $n+1$ nodes of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

b) If $n+1$ nodes of $\mathcal{X} \setminus \{A\}$ are collinear and are lying in a line α then no n nodes of $\mathcal{X} \setminus (A \cup \alpha)$ are collinear together with A;

c) No $2n+1$ nodes of $\mathcal{X} \setminus \{A\}$ are lying on an irreducible conic together with A.

Evidently the condition ii) implies i). Then, i) implies the condition a) of iii), in the same way as in Proposition 2.1.2. Now, let us show that i) implies the condition b) of iii). Indeed, suppose by way of contradiction that a line α passes through $n+1$ nodes of $\mathcal{X}$ and another line α_A passes through A and n nodes of $\mathcal{X} \setminus \alpha$. Then the fundamental polynomial $p_{A,\mathcal{X}}$ vanishes at these $n+1$ and n nodes. Therefore, by Lemma 1.2.13, it vanishes at all the points of the lines α and α_A, including A, which is a contradiction. Finally, let us verify that i) implies the condition c) of iii). Indeed, assume on the contrary that an irreducible conic β passes through A and $2n+1$ other nodes of $\mathcal{X}$. Then the fundamental polynomial $p_{A,\mathcal{X}}$ vanishes at those $2n+1$ nodes and therefore, by Lemma 1.2.14, it vanishes at all the points of β, including A, which is a contradiction.
Thus to prove Proposition 2.3.2 it remains to prove the implication iii) $\Rightarrow$ ii).

Remark 2.3.3. From now on, without loss of generality, we may assume that $|\mathcal{X}| = 2n + [n/2] + 1$ in the conditions of Proposition 2.3.2.

Indeed, it suffices to verify that if $|\mathcal{X}| < 2n + [n/2] + 1$ then we can add a node B to $\mathcal{X}$ such that the set $\mathcal{X}' := \mathcal{X} \cup \{B\}$ satisfies the conditions a), b), and c), with $\mathcal{X}$ replaced with $\mathcal{X}'$. For this purpose consider the set of the lines passing through any two nodes of $\mathcal{X}$ and the set of the conics passing through any five nodes of $\mathcal{X}$, no 4 of which are collinear. Note that the both sets are finite. Then one can check readily that as a desired node B can be chosen as any node which does not belong to these lines and conics. $\square$

From now on, to indicate that a fundamental polynomial is a product of lines and conics, we will use the notation

$$p_A^{\emptyset} := p_A^{\star}.$$

To show that a fundamental polynomial is a product of lines and conics we either bring a precise formula for it, or, more often, we use the so called 1-, $1^{\star}$-, and 2- reductions presented below.

Definition 2.3.4 ([27])**.** We say that a line α is a 1-*reduction* of the node set $\mathcal{X}$ with respect to $A \in \mathcal{X} \setminus \alpha$ and n if it is passing through at least 3 nodes of $\mathcal{X}$ and for the set $\mathcal{X}' := \mathcal{X} \setminus \alpha$ the conditions a), b), and c) are satisfied with n replaced by $n-1$, i.e.,

- a') No n nodes of $\mathcal{X}'$ are collinear together with A;
- b') If n nodes of $\mathcal{X}' \setminus \{A\}$ are collinear and are lying in a line α then no $n-1$ nodes of $\mathcal{X}' \setminus \alpha$ are collinear together with A, α then no n nodes of $\mathcal{X} \setminus (A \cup \alpha)$ are collinear together with A;
- c') No $2n-1$ nodes of $\mathcal{X}'$ are lying in an irreducible conic together with A.

We say that a line α, with $A \notin \alpha$, is an $1^{\star}$-*reduction* of the node A with respect to $\mathcal{X}$, if the condition a') holds and it is passing through at least $[n/2]+2$ nodes of $\mathcal{X}$, i.e., we have that $|\mathcal{X}'| \leq 2n-1$.

Definition 2.3.5 ([27])**.** We say that a conic β is a 2-*reduction* of the node set $\mathcal{X}$ with respect to $A \in \mathcal{X} \setminus \beta$ and n if it is passing through at least 5 nodes of $\mathcal{X}$ and for the set $\mathcal{X}'' := \mathcal{X} \setminus \beta$ the conditions a), b), and c) are satisfied with n replaced by $n-2$, i.e.,

- a") No $n-1$ nodes of $\mathcal{X}''$ are collinear together with A;
- b") If $n-1$ nodes of $\mathcal{X}'' \setminus \{A\}$ are collinear and are lying in a line α, then no $n-2$ nodes of $\mathcal{X} \setminus \alpha$ are collinear together with A;
- c") No $2n-3$ nodes of $\mathcal{X}''$ are lying in an irreducible conic together with A.

Below we describe how one can apply the 1-, $1^{\star}$-, and 2- reductions.

Lemma 2.3.6 ([27]). *Suppose that $\mathcal{X}$ is a node set with $|\mathcal{X}| \leq 2n' + [n'/2] + 1$ and Proposition 2.3.2 is true for any node set and $n \leq n' - 1$. Then Proposition 2.3.2 is true for the node set $\mathcal{X}$ and n' if at least one of the following two conditions holds:*

a) There is a line α which is an 1-reduction, or an $1^\star$-reduction, of the node set $\mathcal{X}$ with respect to $A \in \mathcal{X}$ and $n = n'$.

b) There is a conic β which is a 2-reduction of the node set $\mathcal{X}$ with respect to $A \in \mathcal{X}$ and n'.

Proof. Recall that we need only to verify the implication iii) $\Rightarrow$ ii) of Proposition 2.3.2. Now, if the first condition holds, set $\mathcal{X}' := \mathcal{X} \setminus \alpha$. Notice that we have $|\mathcal{X}'| \leq |\mathcal{X}| - 3 \leq 2n' + [n'/2] + 1 - 3 \leq 2(n'-1) + [(n'-1)/2] + 1$. Hence, we get

$$p^\star_{A,\mathcal{X}} = \alpha p^{\emptyset}_{A,\mathcal{X}'},$$

where the right hand side $(n'-1)$-fundamental polynomial we have by Proposition 2.3.2, or Proposition 2.1.2, if the line α is an 1-reduction, or a $1^\star$-reduction, respectively.

If the second condition holds, set $\mathcal{X}'' := \mathcal{X} \setminus \beta$. Notice that we have $|\mathcal{X}''| \leq |\mathcal{X}| - 5 \leq 2n' + [n'/2] + 1 - 5 \leq 2(n'-2) + [(n'-2)/2] + 1$. Hence, we get

$$p^{\emptyset}_{A,\mathcal{X}} = \beta p^{\emptyset}_{A,\mathcal{X}''},$$

where the right hand side $(n'-2)$-fundamental polynomial we have by Proposition 2.3.2, since the conic β is a 2-reduction. □

Now, we are in a position to start the proof of the main Proposition 2.3.2.

Let us mention that, along with 1- and $1^\star$- reductions, we will use mainly 2-reduction, for which two points are important:

∘ Finding a conic β determined by 5 nodes of $\mathcal{X}$, which is not passing through the node $A \in \mathcal{X}$.

∘ For the node set $\mathcal{X}'' = \mathcal{X} \setminus \beta$ the conditions a''), b''), and c'') of Definition 2.3.5 are satisfied, meaning that there are no lines, reducible and irreducible conics passing through the node A and "too many" other nodes of $\mathcal{X}''$.

For this purpose we identify such lines and conics in $\mathcal{X}$ beforehand and call them critical. Thus a critical line α_A passes through A and at least $n-1$ nodes of $\mathcal{X}$. A critical reducible conic has form $\alpha\alpha_A$, where the line α, with $A \notin \alpha$, passes through at least $n-1$ other nodes of $\mathcal{X}$ and the line α_A passes through A and other $n-2$ nodes of $\mathcal{X} \setminus \alpha$. Finally, a critical irreducible conic β_A passes through A and at least $2n-3$ nodes of $\mathcal{X}$. Then, we are trying to choose the 5 nodes determining the conic β on the critical lines and conics, in order to neutralize them, by decreasing the number of nodes of the set $\mathcal{X}'' = \mathcal{X} \setminus \beta$ they pass through. In this way we are trying to make the conic β 2-reduction, i.e., to fulfill the above mentioned conditions a"), b"), and c").

To neutralize a critical line α or irreducible conic β we should choose necessarily $1+s_\alpha$ or $1+s_\beta$ nodes from α and β, respectively, where s_α and s_β stand for a number of extra nodes in the critical line and conic. That is why we are assuming that α_A and β_A pass through $n-1+s_\alpha$ and $2n-3+s_\beta$ nodes, respectively. Note that, according to the conditions a) and c) of Proposition 2.3.2, we have that $s_\alpha \leq 1$ and $s_\beta \leq 3$. Note also that to neutralize a reducible conic it is enough to neutralize only a line component of it.

2.4 The proof of main proposition for $n \leq 4$

Lemma 2.4.1 ([27])**.** *Proposition* 2.3.2, *i.e., the implication iii) $\Rightarrow$ ii), is valid for the values* $n = 1, 2, 3, 4$.

The case $n = 1$ is evident. Let $n = 2$. Assume, in view of Remark 2.3.3, that $|\mathcal{X}| = 6$. According to the condition c) iii) of Proposition 2.3.2, the six nodes do not belong to any irreducible conic. Consider the case when they belong to a reducible conic β , where $\beta = \alpha_A\alpha$ and $A \in \alpha_A$. Then we get readily from the condition a) that $|\{\mathcal{X} \cap \alpha_A\}| \leq 3$ and therefore $A \notin \alpha$. Now, according to *b*), $\mathcal{X}' := \mathcal{X} \setminus \alpha$ contains at most one node B of $\mathcal{X}$ different of A. So we can take $p^{\emptyset}_{A,\mathcal{X}} = \alpha\alpha'$, where the last line passes through B but not A. Now, we may assume that the six nodes do not belong to any conic. So we can take $p^{\emptyset}_{A,\mathcal{X}} = \beta$, where β is the conic passing through the five nodes of $\mathcal{X} \setminus \{A\}$.

Next, let $n = 3$. Then we have $|\mathcal{X}| = 8$. Suppose first that there is a line α, $A \notin \alpha$, passing through 4 nodes of $\mathcal{X}$. Then we get readily that α is a $1^\star$-reduction of $\mathcal{X}$. The condition a') is satisfied in view of the condition b).

Now let us choose a conic β_A determined by A and other 4 nodes of $\mathcal{X}$ in the following way. Suppose that there are exactly k lines passing through A and at least two other nodes of $\mathcal{X}$ and $k \geq 1$. Notice that there can be at most three such lines, i.e., $k \leq 3$.

Now, we choose A and all the nodes on these k lines except one node on each line. The number of all chosen nodes is less than or equal to 5 if $2 \leq k \leq 3$, and 3 if $k = 1$. We add nodes to complete the 5 arbitrarily.

Then, in view of Lemma 2.2.2 ii), the conic β_A is determined by these nodes, i.e., by A and other 4 nodes. According to the condition c), the set $\mathcal{X} \setminus \beta_A$ contains at least one node B. Now, consider the conic β passing through B and the 4 nodes. In view of Lemma 2.2.3, β does not pass through A. Finally, we can take $p^\star_{A,\mathcal{X}} = \alpha\beta$, where α is a line passing through the remaining two nodes different from A. Note that these nodes do not belong to the same line from the k mentioned. Hence the line α is not passing through A.

Finally assume that $n = 4$. Then $|\mathcal{X}| = 11$. The proof of this case consists of 6 steps.

Step 1. Suppose that there is a line α, $A \notin \alpha$, passing through at least 5 nodes of $\mathcal{X}$. Then, in view of the condition b), we get readily that α is an $1^\star$-reduction of $\mathcal{X}$.

Step 2. Suppose that there is a line α_A passing through A and at least 4 other nodes of $\mathcal{X}$. In view of the condition a) α_A passes through exactly 4 other nodes and the set $\mathcal{X}_1 := \mathcal{X} \setminus \alpha_A$ contains exactly 6 nodes. In view of Step 1 and the condition b), we have that no 5 of these 6 nodes are collinear. Then, by using Lemma 2.2.4, we can divide the node set $\mathcal{X}_1$ into two noncollinear triples. Now, we can take $p^\star_{A,\mathcal{X}} = \beta\beta'$, where β is a conic passing through two nodes of $\alpha_A \setminus \{A\}$ and a noncollinear triple and β' is the conic passing through the remaining 2 nodes of $\alpha_A \setminus \{A\}$ and another noncollinear triple. Note that, in view of Lemma 2.2.1, the conics β and β' do not pass through A.

Step 3. Suppose that there is a line α, $A \notin \alpha$, passing through exactly 4 nodes of $\mathcal{X}$. Then, in view of Step 2, α is an $1^\star$-reduction of $\mathcal{X}$.

Step 4. Now suppose that there is a line α, $A \notin \alpha$, passing through exactly 3 nodes of $\mathcal{X}$, denoted by B, C, D. Denote also $\mathcal{X}' := \mathcal{X} \setminus \alpha$, $|\{\mathcal{X}'\}| = 8$. Notice that, in view of Steps 2 and 3, the conditions a') and b') of Definition 2.3.4 are satisfied, respectively, for the line α and node set $\mathcal{X}'$. Now, if the condition c') is satisfied too, then the line α is a 1-reduction.

Thus, it remains to consider the case where the condition c') is not satisfied, i.e., all 8 nodes of the set $\mathcal{X}'$ belong to an irreducible conic, which we denote by β_A. Then there are no 4 collinear nodes in $\mathcal{X} \setminus \{A\}$ and therefore, in view of Lemma 2.2.2 ii), any five nodes of $\mathcal{X}'$ determine a conic uniquely. Next, according to c), at most one node from the three of α belongs to β_A. Suppose first that a node, say B, belongs to β_A. Then we take $p^{\emptyset}_{A,\mathcal{X}} = \beta_1\beta_2$, where β_1 is a conic passing through 4 nodes from $\beta_A \setminus A$ and C, and β_2 is the conic passing through 4 other nodes from $\beta_A \setminus A$ and D. Note that in view of Lemma 2.2.3, these conics do not pass through A.

It remains to consider the case where no node from α belongs to β_A. Then let us fix any three nodes E, F, G on β_A different from A. Next, we show that we can choose a node from $\{C, D\}$ such that the conic passing through this node and the nodes E, F, G, B, does not pass through A. Indeed, consider the conic β_3 passing through A, E, F, G, B. Then, in view of Lemma 2.2.1, β_3 does not pass through both C and D. Suppose, without loss of generality, that β_3 does not pass through C. Finally, in this case we can take $p^{\emptyset}_{A,\mathcal{X}} = \beta_4\beta_5$, where the conic β_5 passes through the 4 nodes of $\beta_A \setminus \{A\}$ and D. Note that, in view of Lemma 2.2.3, these conics do not pass through A.

Step 5. Suppose that there is an irreducible conic β_A passing through A and k other nodes of $\mathcal{X}$, where $k \geq 5$. According to the condition c) there can be at most eight such nodes, i.e., $k \leq 8$. Let us call a conic of s-type, $1 \leq s \leq 4$, if it passes through at least s nodes of $\beta_A \setminus \{A\}$ and $5 - s$ nodes from outside of β_A.

In this case we can take $p^{\emptyset}_{A,\mathcal{X}} = \beta_1\beta_2$, where β_1 is of $(k-4)$-type conic, $k = 5, 6, 7, 8$ and therefore β_2 is of 4-type conic passing the remaining 5 nodes of $\mathcal{X} \setminus \{A\}$. Notice that, in view of Lemma 2.2.3, 4-type conics do not pass through A. Thus, we only need to choose the nodes of the $s := (k-4)$-type conic such that it does not pass through A, where $k = 5, 6, 7$, i.e., $s = 1, 2, 3$.

For this purpose consider first a conic β_3, which passes through A, and certain 4 nodes. Namely, $s-1$ nodes of β_A and $5-s$ nodes from outside of β_A. Let A, B, C, D, E be the 5 nodes determining β_3.

According to the condition c) there is a node $F \in \beta_A$ such that β_3 does not pass through it. Therefore, according to Lemma 2.2.3, the conic passing through the nodes $\{B, C, D, E, F\}$ does not pass through A. It remains to notice that this latter conic is of s-type.

Step 6. Let us divide the set $\mathcal{X} \setminus \{A\}$ into 2 sets of 5 nodes: $\mathcal{X} \setminus \{A\} = \mathcal{X}_1 \cup \mathcal{X}_2$. We may assume that none of them contains 3 nodes which are collinear together with A. Indeed, if both they contain such a triple then we just exchange one node from them. If only one set, say $\mathcal{X}_1$, contains such triple, say $\mathcal{T}$, then we exchange a node $B \in \mathcal{T}$, with a node $C \in \mathcal{X}_2$, requiring only that the triple $[\mathcal{X}_1 \cup \{B\}] \setminus \mathcal{T}$ is not collinear together with A. Now, in view of Lemma 2.2.2 ii), the sets $\mathcal{X}_1$ and $\mathcal{X}_2$ determine the above mentioned conics β_1 and β_2, respectively. Notice that both these conics are irreducible. Indeed, any reducible conic, in view of Steps 2 and 4, has a line component passing through A and exactly 3 other nodes. Note that, in view of Step 5, these conics do not pass through A.

2.5 The proof of the main proposition for $n \geq 5$

We will use induction on n. First step of induction is included in Lemma 2.4.1. Now suppose that $n \geq 5$ and the statement is true for all natural numbers not exceeding $n-1$. Let us prove it for n.

First, in the subsequent 3 subsections, we will prove 3 lemmas, respectively.

2.5.1 The case of a critical reducible conic

Lemma 2.5.1 ([27]). *Proposition* 2.3.2 *is valid if there is a critical reducible conic, i.e., a pair of two lines α and α_A, where α is passing through at least $n-1$ nodes of $\mathcal{X}$ and not passing through A, while α_A is passing through A and at least $n-2$ other nodes of $\mathcal{X} \setminus \alpha$.*

Proof. Step 1. We show that there is a line that passes through A and n other nodes of $\mathcal{X} \setminus \alpha$. Indeed, otherwise, in view of Definition 2.3.4, the line α is a $1^\star$-reduction, since $n-1 \geq [n/2]+2$, if $n \geq 5$. Notice that, according to the condition b), the line α passes through at most n nodes of $\mathcal{X}$.

Thus from now on we may assume that the line α_A passes through exactly n nodes of $\mathcal{X} \setminus \{A\}$.

Step 2. Let us verify that there is no irreducible critical conic β_A, i.e., passing through A and $2n-3$ other nodes of $\mathcal{X}$. Indeed, otherwise we have at least $1+n+(n-1)+(2n-3)-1-2 = 4n-6$ nodes in $\mathcal{X}$, where -1 and -2 stand for a possible intersection nodes of β_A with α_A and α, respectively. But we have that

$$2n + [n/2] + 1 < 4n - 6 \text{ if } n \geq 5. \tag{2.5.1}$$

Step 3. Let us show that there can be at most one line α'_A, $\alpha'_A \neq \alpha_A$, passing through A and at least $n-2$ other nodes of $\mathcal{X} \setminus \alpha$. Indeed, otherwise, if there are 2 such lines, then there would be at least $1 + 2(n-2) + n + (n-1) = 4n-4$ nodes in $\mathcal{X}$, where 1 stands for A. This, in view of (2.5.1), is a contradiction.

Step 4. Next, let us verify that if there is a line α'_A then it passes through A and at most $n-1$ other nodes of $\mathcal{X}$. Indeed, otherwise, if it passes through n other nodes of $\mathcal{X}$, then we have at least $1 + 2n + (n-1) - 1 = 3n-1$ nodes in $\mathcal{X}$, where second -1 stands for a possible intersection node of α'_A and α. But we have that

$$2n + [n/2] + 1 < 3n - 1 \text{ if } n \geq 5. \tag{2.5.2}$$

Step 5. Next, we consider the case when there is a second line $\alpha' \neq \alpha$ passing through at least $n-1$ nodes and not passing through A. Then we have at least $1+n+2(n-1)-1 = 3n-2$ nodes in $\mathcal{X}$, where -1 stands for a possible intersection node of α and α'. We do not count a possible intersection of α_A and α', since if these lines intersect at a node from $\mathcal{X}$ then, as in Step 1, the line α' is a $1^\star$-reduction, or there is another line α'_A passing through A and n other nodes, which was excluded in Step 4.

Now, we have that $2n + [n/2] + 1 < 3n-2$ if $n \geq 7$ and $2n + [n/2] + 1 = 3n-2$ if $n = 5, 6$. From here we conclude that $n = 5$ or 6 and in each of this cases all the nodes of $\mathcal{X}$ are located on the lines α_A, α, and α'. The line α_A passes through A and other n nodes, while the lines α and α' pass through $n-1$ nodes and intersect in a node of $\mathcal{X}$. In this case we set $p^{\emptyset}_{A,\mathcal{X}} = \beta p^{\emptyset}_{A,\mathcal{X}'}$, where $\mathcal{X}' := \mathcal{X} \setminus \beta$ and β is a conic passing through two nodes from α_A, different from A, and 3 noncollinear nodes from $\alpha \cup \alpha'$. In other words, here β is a 2-reduction.

Step 6. Thus in this final step we have to deal with the critical line α_A, possibly also α'_A and the line α. In this case we set $p^{\star}_{A,\mathcal{X}} = \beta p^{\star}_{A,\mathcal{X}'}$, where $\mathcal{X}' := \mathcal{X} \setminus \beta$ and β is a conic passing through two nodes from α_A, different from A, 2 nodes from α, and a node from α'_A, different from A, if there is such a line. Otherwise, if there is no such line then the fifth node of β is any node outside of α_A and α. There is a such node since otherwise there are at most $1 + n + n$ nodes in $\mathcal{X}$, which is a contradiction in view of Remark 2.3.3. $\square$

2.5.2 The case of a critical irreducible conic

From now on we may assume that there is no reducible critical conic for $\mathcal{X}$.

Lemma 2.5.2 ([27]). *Proposition* 2.3.2 *is valid if there is an irreducible critical conic, i.e., a conic passing through* A *and at least* $2n - 3$ *other nodes of* $\mathcal{X}$.

Proof. First, assume that there is exactly 1 irreducible conic, denoted by β_A.

Step 1. Let us show that there is at most one critical line α_A - passing through A and at least $n-1$ other nodes of $\mathcal{X}$. Indeed, otherwise, if there are 2 such lines, then there are at least $1 + 2(n-1) + (2n-3) - 2 = 4n - 6$ nodes in $\mathcal{X}$, where -2 stands for two possible intersection nodes of the conic and the two lines. This, in view of (2.5.1), is a contradiction.

Step 2. Let us verify that there can be at most 2 extra nodes in the critical conic β_A and the possible critical line α_A, except those mentioned, i.e., $n-1$ and $2n-3$. Indeed, otherwise there are at least $1 + (n-1) + (2n-3) - 1 + 3 = 3n - 1$ nodes in $\mathcal{X}$, where -1 stands for a possible intersection node of the conic and the line. This, in view of (2.5.2), is a contradiction.

Step 3. Finally, in this case let us consider a 2-reduction conic β - passing through 3 nodes from $\beta_A \setminus \alpha_A$, and 2 nodes from α_A, different from A, if there is such a line. If not, then we make the conic β pass through 4 nodes from the conic β_A different from A and a node from outside. Note that, according to Lemmas 2.2.2 and 2.2.3, the conic β does not pass through A.

Next, assume that there are exactly 2 critical irreducible conics, denoted by β_A and β'_A.

Let us verify that except the node A and $2n-3$ nodes on each of β_A and β'_A there can be at most one extra node (on the conics or outside). Indeed, if there are 2 such extra nodes, then there are at least $1+2(2n-3)+2-3=4n-6$ nodes, where 3 stands for intersection nodes of the two conics. This, in view of (2.5.1), is a contradiction. Then, notice that there is no critical line. Indeed, any line passing through A passes through at most 3 ($\leq n-2$ if $n \geq 5$) other nodes, 2 of which are intersection nodes with the conics and 1 is the possible extra node. In this case let us consider that we have a 2-reduction conic β - passing through 4 nodes from $\beta_A \setminus \{A\}$, and a node from $\beta_A \setminus \beta'_A$. Note that, in view of Lemma 2.2.3, β does not pass through A.

Finally, assume that there are at least 3 critical irreducible conics, denoted by β_A, β'_A, and β''_A.

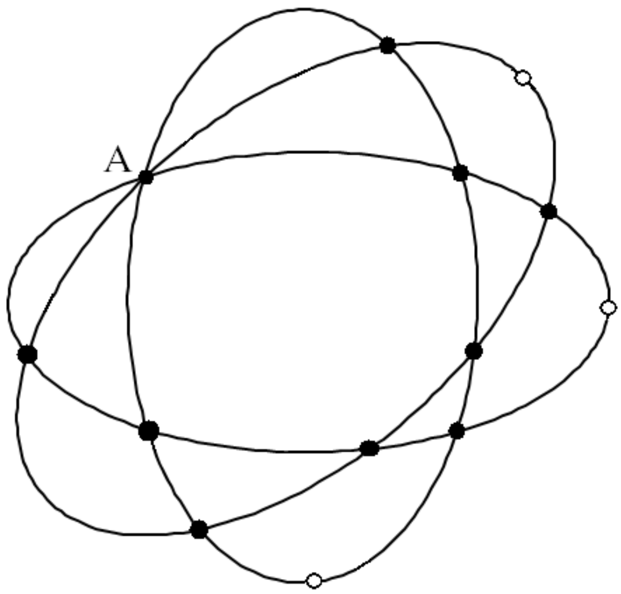

Figure 2.5.1: The case of 3 conics

In this case let us verify first that $n \leq 5$. Indeed, there are at least $1+3(2n-3)-9$ nodes, where $9=3+3+3$ stands for the maximal number of pairwise intersection nodes of the conics, different from A. Therefore

$$1+3(2n-3)-9 \leq 2n+[n/2]+1. \tag{2.5.3}$$

This reduces to $4n \leq [n/2]+18$ and hence we get $n \leq 5$. In view of Lemma 2.4.1 we may consider the case $n=5$ only. Then we have that $|\mathcal{X}|=13$. Notice that we have equality in (4.3.5) when $n=5$. Therefore in this case any two conics must intersect in exactly 3 nodes, different from A and all 9 such intersection nodes must be distinct (see Fig. 2.5.1). Also, we get that each conic passes through A and exactly $7(=2n-3)$ other nodes. Notice that $6(=3+3)$ out of these are intersection nodes with 2 other conics and seventh is not such a node. Now, let us show that there is no other, i.e., fourth critical irreducible conic passing through A and 7 other nodes. Assume on the contrary that there is such a conic $\tilde{\beta}''_A$. Consider the three conics β_A, β'_A and $\tilde{\beta}''_A$. According to what we showed above $\tilde{\beta}''$, as well as β''_A, does not pass through the 3 intersection nodes of the conics β_A and β'_A, different from A. There are 5 nodes of $\mathcal{X}$ outside of β''_A, which include the mentioned 3 nodes. Therefore $\tilde{\beta}''_A$ can pass through at most 2 nodes outside of β''_A. So the remaining 6 nodes of $\tilde{\beta}''_A$ belong to β''_A, thus according to Lemma 2.2.2 iii) $\tilde{\beta}''_A$ coincides with β''_A.

Then, notice that there is no critical line. Indeed, any line passing through A passes through at most 3 ($\leq n-2$ if $n \geq 5$) other nodes, namely the possible intersection nodes of the line with the 3 conics.

Finally let us find a 2-reduction conic β, by choosing a set of 5 nodes. Note that to neutralize the 3 critical conics it is enough to choose at least a node from each conic, since there are no extra nodes. For this purpose, we choose any 4 intersection nodes, different from A, of one of the conics, say β_A, and a node from outside. Note that at most 3 of these 4 belong to one of the conics β'_A and β''_A. Therefore the 4 nodes include at least one node from each of the 3 conics. To complete the proof notice that, in view of Lemma 2.2.3, β does not pass through A. $\square$

2.5.3 The case of a critical line

Now we may assume that there is no critical conic (reducible or not) for $\mathcal{X}$.

Lemma 2.5.3 ([27]). *Proposition* 2.3.2 *is valid if there is a critical line, i.e., a line α_A passing through A and at least $n-1$ other nodes of $\mathcal{X}$.*

Proof. Step 1. Let us start by showing that there can be at most three critical lines. Indeed, otherwise, if there are 4 such lines, then there are at least $1+4(n-1)=4n-3$ nodes in $\mathcal{X}$, where 1 stands for A. In view of (2.5.1), this is a contradiction.

Step 2. Let us verify that if there are three critical lines $\alpha_A, \alpha'_A, \alpha''_A$, then there is at most one extra node. Indeed, otherwise we have at least $1+3(n-1)+2=3n-1$ nodes in $\mathcal{X}$, which, in view of (2.5.2), is a contradiction. If there is an extra node in the lines then we will assume that it is in α_A.

Step 3. Next, let us find a 2-reduction conic β, determined by a set of 5 nodes of $\mathcal{X} \setminus \{A\}$. Assume first that there are at least 2 critical lines. Then for the set of 5 nodes we choose two nodes from each of α_A and α'_A, and a node from the third critical line α''_A, if there is such a line. Otherwise, we choose the fifth node from outside of α_A and α'_A. Note that there is such a node in view of Remark 2.3.3. Finally assume that there is just one critical line: α_A. Then we choose two nodes from this line and any triple of noncollinear nodes from outside. If there is no such triple, then all the nodes of $\mathcal{X} \setminus \alpha_A$ belong to a line α which clearly is a $1^\star$-reduction. It remains to notice that, in view of Lemma 2.2.1, β does not pass through the node A. $\square$

2.5.4 Completion of the proof

Now, we may suppose, in view of Lemmas 5.1-5.3, that there are no critical lines or conics for $\mathcal{X}$. Thus, to find a $2-$ reduction, it suffices to choose a set of five nodes in $\mathcal{X} \setminus \{A\}$, such that the conic β determined by them does not pass through A.

For this purpose first we choose 3 nodes such that no 2 of them are collinear together with A. There are such 3 nodes since otherwise all the nodes of $\mathcal{X}$ belong to 2 lines passing through A. This contradicts the condition iii) a) of Proposition 2.3.2. Next, we may suppose that these 3 nodes are not collinear. Indeed, if they belong to a line α then clearly the latter is a 1-reduction. Now, we choose any fourth node from $\mathcal{X} \setminus \{A\}$. Then, in view of Lemma 2.2.1, these 4 nodes together with A determine a conic β_A. Now, to complete the proof, we choose a node B outside of the conic β_A. Note that, in view of Lemma 2.2.3, the node B together with the 4 chosen nodes makes the desired set. $\square$

Chapter 3

FACTORIZATION OF TRIVARIATE FUNDAMENTAL POLYNOMIALS

In this Chapter we bring necessary and sufficient conditions for the set $\mathcal{X} \in \mathbb{R}^3$ with cardinality not exceeding $3n+1$, such that all its knots have n-fundamental polynomials in form of products of linear factors. We bring also necessary and sufficient conditions for n-independence of non-coplanar knot sets in $\mathbb{R}^3$ of the mentioned cardinality.

Let Π_n^3 be the space of polynomials of three variables and total degree at most n :

$$\Pi_n^3 = \left\{ \sum_{i+j+k\leq n} a_{ijk} x^i y^j z^k : a_{ijk} \in \mathbb{R} \right\}.$$

We have that

$$N := \dim \Pi_n = \binom{n+3}{3}.$$

In case of two variables the corresponding space we denote by Π_n.

We denote by $\Pi_n(L)$ the set of restrictions of polynomials of total degree at most n on a plane L. Notice that if the plane L is not perpendicular to the XY coordinate plane then we may assume that polynomial $p \in \Pi_n(L)$ is given by the equation $q(x,y) = 0$, where $q \in \Pi_n$. Indeed, in this case L is given by an equation $z = ax + by + c$ and we have for the restriction of $p \in \Pi_n^3$ on L: $p|_L = p(x, y, ax + by + c) =: q(x, y)$.

3.1 A result on factorization of bivariate fundamental polynomials

In this section we prove the following, interesting in itself, proposition which in the Section 3.3 will be used to establish a result on n-independence of knot sets in $\mathbb{R}^3$.

Proposition 3.1.1 ([28]). *Let $\mathcal{X}$ be a set of knots in $\mathbb{R}^2$ with $|\mathcal{X}| = 3n - k$, where $k, n \geq 1$ and $A \in \mathcal{X}$. Suppose that the following three conditions hold:*

i) No $n + 1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

ii) If $n + 1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear and are lying in a line α, $A \notin \alpha$, then no n knots of $\mathcal{X} \setminus \alpha$ are collinear together with A;

iii) No $2n + 1$ knots of $\mathcal{X} \setminus \{A\}$ belong to an irreducible conic together with A.

Then there exists a fundamental polynomial of A of form $p^\star_{A,\mathcal{X}} = \alpha_1\alpha_2...\alpha_k q$, where $\alpha_1, ..., \alpha_k$ are lines and $q \in \Pi_{n-k}$.

Proof. The cases $n = 1, 2$ are evident. Let us prove the case $n = 3$.

First assume that $k = 1$. In this case we have 8 knots in $\mathcal{X}$. Therefore $|\mathcal{X}| = 1 + 2n + [n/2]$. Thus according to the conditions i), ii), iii) of the Lemma and Proposition 2.3.2, A has a fundamental polynomial which is a product of lines and conics. Since n is odd there is a line factor.

For the cases $k \geq 2$ we have $|\mathcal{X}| \leq 1 + 2n$ and so according to Theorem 2.1.1 the knot A has a fundamental polynomial which is a product of lines.

Thus we may assume from now on that $n \geq 4$.

Let us now use induction on $n+k$. Assume that the statement is true for all natural numbers n' and k', such that $n' + k' \leq n + k - 1$. Let us prove it for $n' = n$ and $k' = k$. Notice that it is enough to find a line α_0, $A \notin \alpha_0$ passing through two knots of $\mathcal{X}$ such that the conditions i),ii) and iii) hold for the knot set $\mathcal{X}' := \mathcal{X} \setminus \alpha_0$, with $n - 1$.

Indeed, after the choice of such a line we will have at most $3(n - 1) - (k - 1) = 3n - k - 2$ knots in $\mathcal{X}'$. Therefore in view of induction assumption there exists a fundamental polynomial of form $p^{\star}_{A,\mathcal{X}',n-1} = \alpha_1 ... \alpha_{k-1} q$, where α_i is a line and $q \in \Pi_{n-k}$. Thus we get a desired fundamental polynomial $p^{\star}_{A,\mathcal{X},n} = \alpha_0 \alpha_1 ... \alpha_{k-1} q$.

Notice that if there is a line α passing through $n+1$ knots of $\mathcal{X} \setminus \{A\}$, then it can be taken as a desired line. Thus we may assume from now on that any line not passing through A, passes through at most n knots of $\mathcal{X}$.

Now assume that the set $\mathcal{X}$ satisfies the conditions i), ii) and iii) with n replaced by $n-1$, i.e.,

- i') no n knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A,
- ii') if n knots of $\mathcal{X} \setminus \{A\}$ are collinear and are lying in a line α, $A \notin \alpha$, then no $n-1$ knots of $\mathcal{X} \setminus \alpha$ are collinear together with A,
- iii') no $2n-1$ knots of $\mathcal{X} \setminus \{A\}$ belong to a conic together with A.

Then it is easily seen that in this case as desired line we can take any line α_0, $A \notin \alpha_0$, passing through two knots of $\mathcal{X}$. two knots we can take any two knots of $\mathcal{X}$, such that they are not collinear together with A.

The remaining cases we consider in three steps.

Step 1. Suppose that there is an irreducible conic β passing through A and at least $2n-1$ other knots of $\mathcal{X}$. Notice that there can not be more than one such conic. Indeed, if there are two such conics, then we have at least $1 + 2(2n-1) - 3 = 4n - 4$ knots in $\mathcal{X}$, where 3 stands for the possible intersection knots of the two conics different from A. On the other hand $4n - 4 > |\mathcal{X}| = 3n - k$, if $n \geq 4$, where $k \geq 1$.

Notice that according to the condition iii) the conic β contains at most $2n$ knots of $\mathcal{X}$ different from A.

Now suppose that the condition i') is not satisfied - there is a line α_A passing through A and n other knots. Then let us verify that the following three conditions are satisfied: the line α_A and the conic β intersect at two knots of $\mathcal{X}$, the conic β contains exactly $2n-1$ knots of $\mathcal{X} \setminus \{A\}$ and $k = 1$. Indeed, in this case we have that $\mathcal{X}$ contains at least $1 + n + (2n-1) - 1 = 3n - 1$ knots, where the last -1 in the left hand side of the equality means that the line and the conic intersect at another knot B, besides A.

It is easily seen that in this case as our desired line we can take any line passing through B and any other knot from β, different from A.

Finally suppose that ii') is not satisfied. Therefore there is a line $\alpha, A \notin \alpha$, passing through exactly n knots of $\mathcal{X}$. Then we can verify that α intersects β at at least one knot of $\mathcal{X}$ and all the knots of $\mathcal{X}$, except possibly a knot, lie in $\alpha \cup \beta$. Indeed, as in the previous case we have that $\mathcal{X}$ contains at least $1+n+(2n-1)-1 = 3n-1$ knots, where the last -1 in the left hand side of the equality means that α and β intersect at a knot in $\mathcal{X}$.

It is easily seen that here α is a desired line.

Step 2. Suppose that there is a line $\alpha, A \notin \alpha$, passing through exactly n knots and a line α_A is passing through at least $n-1$ knots of $\mathcal{X} \setminus \alpha$ together with A. Let us consider two cases.

First suppose that α_A passes through A and n other knots of $\mathcal{X} \setminus \alpha$. Note that outside of these two lines there are at most $n-2$ knots. Therefore if there is a second line α', $A \notin \alpha'$, passing through n knots, then it intersects the lines α and α_A at two different knots. Therefore in this case α' is a desired line. If there is no such a line α', then it is easily seen that we can take as a desired line α_0 a line passing through one knot from α_A and another from α.

Next suppose α_A passes through A and exactly $n-1$ other knots of $\mathcal{X} \setminus \alpha$. It is easily seen that α is a desired line. Note that since there are at most $n-1$ knots outside of these two lines the conditions i') and ii') are satisfied.

Step 3. Suppose that there is a line α_A passing through A and n other knots of $\mathcal{X}$. Notice that there can be at most 2 such lines. Then it is easily seen that in this case as a desired line $\alpha_0, A \notin \alpha_0$, we can take any line that intersects each line at a knot different from A. Otherwise if there is only one line α_A then we take one knot from α_A and a knot from outside of α_A.

□

3.2 On factorization of trivariate fundamental polynomials

Let us start this section by proving Theorem 1.2.10 in a more general setting.

Proposition 3.2.1 ([28]). *Let $\mathcal{X}$ be a set of knots in a plane L with $|\mathcal{X}| \leq 3n$. Then a node $A \in \mathcal{X}$ has an n-fundamental polynomial in L if and only if the following four statements hold:*

i) No $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

ii) If $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear and are lying in a line α then no n knots of $\mathcal{X} \setminus \alpha$ are collinear together with A;

iii) If $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear and are lying in a line α then no n knots of $\mathcal{X} \setminus \alpha$ are collinear together with A;

iv) If $|\mathcal{X}| = 3n$, then there are no curves $\gamma \in \Pi_3(L)$ and $\sigma_n \in \Pi_n(L)$ such that $\mathcal{X} = \sigma_n \cap \gamma$.

Proof. Let us start with the 'only if' part of Proposition. Suppose that the knot A has an n-fundamental polynomial and let us prove that the conditions of Proposition hold.

First we show that the condition i) is satisfied. Indeed, suppose by way of contradiction that a line passes through A and $n+1$ other knots of $\mathcal{X}$. Then the fundamental polynomial $p^{\star}_{A,\mathcal{X}}$ vanishes at those $n+1$ knots. Therefore, by Proposition 1.2.12, it vanishes at all the knots of the line including A, which is a contradiction.

Next we show that ii) and iii) take place. Indeed, suppose by way of contradiction that a line α passes through $n+1$ knots of $\mathcal{X}$ and another line α_A passes through A and n knots of $\mathcal{X} \setminus \alpha_A$. Then the fundamental polynomial $p^{\star}_{A,\mathcal{X}}$ vanishes at these $n+1$ and n knots. Therefore, by Lemma 1.2.13, it vanishes at all the knots of the lines α and α_A, including A, which is a contradiction. Now assume on the contrary that an irreducible conic β passes through A and $2n+1$ other knots of $\mathcal{X}$. Then the fundamental polynomial $p^{\star}_{A,\mathcal{X}}$ vanishes at those $2n+1$ knots and therefore, by Lemma 1.2.14, it vanishes at all the knots of β, including A, which is a contradiction.

Finally we show that iv) is satisfied. Indeed, suppose by way of contradiction that $|\mathcal{X}| = 3n$, and there are curves $\gamma \in \Pi_3(L)$ and $\sigma_n \in \Pi_n(L)$ such that $\mathcal{X} = \sigma_n \cap \gamma$. Then the fundamental polynomial $p^{\star}_{A,\mathcal{X}}$ vanishes at all $3n-1$ knots of $\mathcal{X}$ different from A. Therefore according to Theorem 1.2.15 it vanishes at all the knots of $\mathcal{X}$, including A, which is a contradiction.

Now let us prove the 'if' part. We are going to verify that the conditions i) and ii) of Theorem 1.2.10 are satisfied. Let us do this in two steps.

Step 1. Suppose there are $n+1$ knots, belonging to a line α and A is not in α. Then we take $p^{\star}_{A,\mathcal{X},n} = \alpha' p^{\star}_{A,\mathcal{X}',n-1}$, where $\mathcal{X}' = \mathcal{X} \setminus \alpha$. Here the existence of $p^{\star}_{A,\mathcal{X}',n-1}$ follows from Theorem

2.1.1, since $|\mathcal{X}'| \leq 2n-1 = 1+2(n-1)$ and according to condition iii) of Proposition no n knots are collinear together with A.

Step 2. Suppose there are $2n+2$ knots, belonging to an irreducible conic β' and $A \notin \beta'$. Then we take as the fundamental polynomial $p^\star_{A,\mathcal{X},n} = \beta' p^\star_{A,\mathcal{X}',n-2}$, where $\mathcal{X}' = \mathcal{X} \setminus \beta'$. Here $p^\star_{A,\mathcal{X}',n-2}$ exists because $|\mathcal{X}'| \leq n-2$.

Notice that in view of *Step 1* we may assume that there is no line passing through $n+1$ knots and not passing through A, and according to the condition i) there is no line passing through A and $n+1$ other knots. Therefore there is no reducible conic (pair of lines) passing through $2n+2$ knots.

Now in view of *Step 1* and *Step 2* we can assume that no $n+2$ knots of $\mathcal{X}$ are collinear and no $2n+2$ knots belong to a conic. Therefore according to Theorem 1.2.10 $\mathcal{X}$ is n-independent. Hence the knot A has an n-fundamental polynomial.

□

Next we present a main result of the thesis.

Proposition 3.2.2 ([28]). *Let $\mathcal{X}$ be a set of knots in $\mathbb{R}^3$ with $|\mathcal{X}| \leq 3n+1$ and $A \in \mathcal{X}$. Then the knot A has an n-fundamental polynomial, which is a product of linear factors, if and only if*

i) no $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A,

ii) if at least $2n+1$ of $\mathcal{X} \setminus \{A\}$ knots are lying in a plane L_A passing through A then all the knots of $\mathcal{X} \cap L_A$ different from A lie in n lines not passing through A.

Proof. Without lose of generality we can suppose that $|\mathcal{X}| = 3n+1$. Indeed, it suffices to verify that if $|\mathcal{X}| < 3n+1$ then we can add a knot B to $\mathcal{X}$ such that the set $\mathcal{X}' := \mathcal{X} \cup \{B\}$ satisfies the conditions i) and ii) with $\mathcal{X}$ replaced by $\mathcal{X}'$. For this purpose we can consider all the lines passing through any two knots of $\mathcal{X}$ and all the planes passing through any three non-collinear knots of $\mathcal{X}$. Note that both these sets are finite. Then it is easily seen that as a desired knot B one can choose any knot which does not belong to the considered lines and planes.

Let us start with the 'only if' part. Suppose that the knot A has an n-fundamental polynomial, which is a product of linear factors: $p^\star_{A,\mathcal{X}} = L_1 L_2 ... L_n$. Then a line passing through A

can intersect each plane L_i, $i = 1, 2, ..., n$ in at most one knot. Thus there is no line passing through A and $n + 1$ other knots of $\mathcal{X}$, since all the knots of $\mathcal{X}$ lie in the planes $L_1, L_2, ..., L_n$. Therefore the condition i) is satisfied.

Next suppose that L is a plane passing through A and at least $2n + 1$ other knots of $\mathcal{X}$. Then the linear factors of the fundamental polynomial of the knot A will intersect with L in at most n lines. Thus all the knots of $\mathcal{X} \cap L$ different from A lie in at most n lines not passing through A. Therefore the condition ii) holds.

Now we will prove the 'if' part in following steps.

Step 1. First consider the case when there are at least $2n$ knots that belong to a plane L_A together with A. Let us show that all these knots lie in n lines in L_A not passing through A. Indeed, if the number of these knots is greater than $2n$ then this follows from the condition ii). On the other hand if there are exactly $2n$ knots in the plane L_A, then this follows from Proposition 2.1.2 by taking into account the condition i) of Proposition. Thus in *Step 1* we can take $p^{\star}_{A,\mathcal{X}}$ as a product of n planes each passing through one of the considered n lines and a knot from outside L_A. Note that in this case there are at most n knots outside of L_A, i.e., $|(\mathcal{X} \setminus L_A)| \leq n$. Notice that if there are less than n knots outside of L_A, then we can take any other point from $\mathbb{R}^3 \setminus L_A$.

Step 2. Now consider the case when there is a plane L_A passing through A and exactly $2n-1$ other knots. Therefore there are exactly $n+1$ knots in $\mathcal{X} \setminus L_A$. We want to choose three non-collinear knots - a knot from $L_A \setminus A$ and two knots from outside of L_A such that the plane L passing through them is not passing through A, and for the remaining knots in L_A there are no n knots collinear together with A. Note that after we chose such three knots we will have for the knot set $\mathcal{X}' := \mathcal{X} \setminus L$, $|\mathcal{X}'| \leq 3(n-1) + 1$ and the following conditions are satisfied:

1) there are at most $2(n-1) + 1$ knots in L_A,

2) no n knots are collinear together with A,

3) there are at most $n - 1$ knots outside of L_A.

Thus by using Theorem 2.1.1, as in the *Step 1*, we can construct a fundamental polynomial of A with respect to $\mathcal{X}'$: $p^{\star}_{A,\mathcal{X}'}$ in form of product of planes. Finally notice that we can take $p^{\star}_{A,\mathcal{X}} = Lp^{\star}_{A,\mathcal{X}'}$.

Now let us describe the choice of the mentioned three knots. Notice that in view of *Step 1* we may suppose that there is at most one line passing through A and n other knots, because otherwise we would have a plane passing through A and at least $2n$ other knots. Let us consider two cases. First suppose that there is such a line l_A in L_A. In this case we chose the first knot B any knot from l_A different from A. The other two knots C and D we chose such that the knots A, B, C and D are not coplanar. This will not be possible only if all the $n+1$ knots of $\mathcal{X} \setminus L_A$ belong to a plane L' that passes through l_A. This case was considered in *Step 1,* since there are $2n+1$ knots in L' different from A.

Step 3. Now we may assume that no $2n-1$ knots belong to a plane together with A. In this case we will use induction on n.

In the case $n = 1$ we have 4 knots in $\mathcal{X}$. If they are not coplanar then we will take $p^{\star}_{A,\mathcal{X}} = L$, where L is the plane passing through the 3 knots different from A. Otherwise if the 4 knots are coplanar then according to ii) the 3 knots different from A are lying in a line. Therefore as a plane L we can take any plane passing through that line and not passing through A.

Now suppose the proposition is true for $n-1$ and let us prove it for n. Notice that it is enough to find a plane L passing through 3 knots of $\mathcal{X}$ which is not passing through A and for $\mathcal{X}' := \mathcal{X} \setminus L$ the conditions i) and ii) hold with $n-1$. Indeed, then we will take $p^{\star}_{A,\mathcal{X}} = Lp^{\star}_{A,\mathcal{X}'}$, where in view of induction hypothesis, $p^{\star}_{A,\mathcal{X}'}$ is a product of $(n-1)$ planes. Now let us describe the choice of the mentioned three knots. In view of *Step 1* and *2* we may suppose that there are no $2(n-1)+1$ knots coplanar together with A, therefore ii) holds for the set $\mathcal{X}'$ and $(n-1)$. Thus to complete the proof it remains to note that if there is a line l_A passing through A and n other knots then we can take one of these three knots from l_A different from A. Notice that the choice of the mentioned three knots will not be possible only if the whole set $\mathcal{X}$ is coplanar.

□

3.3 Independence of $3n+1$ knots in $\mathbb{R}^3$

Theorem 3.3.1 ([28]). *Let $\mathcal{X}$ be a set of non-coplanar knots in $\mathbb{R}^3$ with $|\mathcal{X}| \leq 3n+1$. Then $\mathcal{X}$ is n-independent if and only if the following three statements hold:*

i) No $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

ii) There are no $2n+2$ coplanar knots of $\mathcal{X}$, which belong to a conic(reducible or irreducible);

iii) If $3n$ knots belong to a plane L, then there are no curves $\sigma_n \in \Pi_n(L)$ and $\gamma \in \Pi_3(L)$ such that $\mathcal{X} \cap L = \sigma_n \cap \gamma$.

The statement of Theorem readily follows from the following result which covers wider setting.

Proposition 3.3.2 ([28]). *Let $\mathcal{X}$ be a set of non-coplanar knots in $\mathbb{R}^3$ with $|\mathcal{X}| \leq 3n+1$. Then a knot $A \in \mathcal{X}$ has an n-fundamental polynomial if and only if the following four statements hold:*

i) No $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear together with A;

ii) If $n+1$ knots of $\mathcal{X} \setminus \{A\}$ are collinear and are lying in a line α, $A \notin \alpha$, then no n knots of $\mathcal{X} \setminus \alpha$ are collinear together with A;

iii) If $2n+1$ knots of $\mathcal{X} \setminus \{A\}$ are coplanar together with A and belong to a plane L, then there is no conic in the plane passing through A and $2n+1$ other knots of $\mathcal{X} \cap L$;

iv) If A and $3n-1$ other knots belong to a plane L, then there are no curves $\sigma_n \in \Pi_n(L)$ and $\gamma \in \Pi_3(L)$ such that $\mathcal{X} \cap L = \sigma_n \cap \gamma$.

First let us prove the following lemma.

Lemma 3.3.3 ([28]). *Let L be plane in $\mathbb{R}^3$, and q be a curve of degree n in L: $q \in \Pi_n(L)$. Then for any point B outside of L, there exists a surface $p \in \Pi_n^3$ passing through B, such that $p|_L = q$.*

Proof. Without lose of generality assume that the curve in L is given by an equation $q(x,y) = 0$. Then we can take p in form: $p(x,y,z) = q(x,y) + cL(x,y,z)$, where the constant c is chosen such that $p(A) = 0$. □

Now let us turn to the proof of Proposition 3.3.2.

Proof. The 'only if' part is obvious. Let us prove the 'if' part. If there is no plane passing through A and at least $2n+1$ other knots, then the Proposition follows from Proposition 3.2.2. Thus assume that there is a plane L passing through A and at least $2n+1$ other knots. Assume that there are $k+1$ knots outside of L: $1 \leq k+1 \leq n-1$.

First suppose that $k = 0$. Denote by B the knot outside of L. According to Proposition 3.2.1 A has a fundamental polynomial $q \in \Pi_n(L)$. Thus in view of Lemma 3.3.3 we can take $p^\star_{A,\mathcal{X}} = p$, where $p \in \Pi_n^3$ is a surface, such that $p|_L = q$ and $p(B) = 0$.

Next suppose that $k \geq 1$, then according to Lemma 3.1.1 we have for the fundamental polynomial $p^\star_{A,\mathcal{Y}} \in \Pi_n(L)$: $p^\star_{A,\mathcal{Y}} = \alpha_1\alpha_2...\alpha_k q$, where $\alpha_i \in \Pi_1(L)$, $q \in \Pi_{n-k}(L)$ and $\mathcal{Y} = \mathcal{X} \cap L$. So here we can take $p^\star_{A,\mathcal{X}} = L_1L_2...L_kp$, where L_i is a plane passing through the lines α_i and a knot from outside of L, and p is a surface passing through q and the last knot from outside, given in Lemma 3.3.3. $\square$

The following corollary readily follows from Theorem 3.3.1

Corollary 3.3.4 ([28])**.** *Let $\mathcal{X}$ be a set of knots in $\mathbb{R}^3$ with $|\mathcal{X}| \leq 3n+1$. Then $\mathcal{X}$ is n-independent if and only if for any plane L the set $\mathcal{X} \cap L$ is n-independent.*

Part II

THE LINES IN GC SETS

Chapter 4

GEOMETRIC CHARACTERIZATION SETS: GC SETS

In this chapter we define a GC_n set and present its classification according to the number of maximal lines. Also we present the Gasca-Maeztu conjecture. Some known results are cited that will be used in the next two chapters.

4.1 GC_n sets and the Gasca-Maeztu conjecture

Definition 4.1.1. Given an n-poised set $\mathcal{X}$. We say that a node $A \in \mathcal{X}$ *uses a line* $\ell \in \Pi_1$, if $p_A^\star = \ell q,$ where $q \in \Pi_{n-1}$.

Since in Part II we frequently use Proposition 1.2.12 it is convenient to state it here as

Proposition 4.1.2. *Suppose that a polynomial $p \in \Pi_n$ vanishes at $n+1$ points of a line ℓ. Then we have that $p = \ell r,$ where $r \in \Pi_{n-1}$.*

As it follows from Proposition 4.1.2, at most $n+1$ nodes of an n-poised set $\mathcal{X}$ can be collinear.

Definition 4.1.3. [[4]] A line passing through $n+1$ nodes is called a *maximal line.*

Clearly, in view of Proposition 4.1.2, a maximal line λ is used by all the nodes in $\mathcal{X} \setminus \lambda$.

Next let us bring two corollaries due to Carnicer and Gasca:

Corollary 4.1.4 ([8], Prop. 2.3). *Let λ be a maximal line of an n-poised set $\mathcal{X}$. Then the set $\mathcal{X} \setminus \lambda$ is an $(n-1)$-poised set. Moreover, for any node $A \in \mathcal{X} \setminus \lambda$ we have that*

$$p^{\star}_{A,\mathcal{X}} = \lambda p^{\star}_{A,\{\mathcal{X}\setminus\lambda\}}. \tag{4.1.1}$$

Corollary 4.1.5 ([6], Prop. 2.1). *Let $\mathcal{X}$ be an n-poised set. Then we have that*

(i) *Any two maximal lines of $\mathcal{X}$ intersect necessarily at a node of $\mathcal{X}$*;

(ii) *Any three maximal lines of $\mathcal{X}$ cannot be concurrent.*

(iii) *$\mathcal{X}$ possesses at most $n+2$ maximal lines.*

Now let us consider a special type of n-poised sets satisfying a geometric characterization (GC) property introduced by K.C. Chung and T.H. Yao:

Definition 4.1.6 ([14]). An n-poised set $\mathcal{X}$ is called *GC_n set* (or GC set) if the n-fundamental polynomial of each node $A \in \mathcal{X}$ is a product of n linear factors.

So, GC_n sets are n-poised sets such that each of its nodes uses exactly n lines.

Next we present the Gasca-Maeztu conjecture, briefly called GM conjecture:

Conjecture 4.1.7 ([16], Sect. 5). *Any GC_n set possesses a maximal line.*

Until now, this conjecture has been confirmed to be true for the degrees $n \leq 5$ (see [5], [19]). For a generalization of the Gasca-Maeztu conjecture to maximal curves see [20].

Let us mention the following important result of Carnicer and Gasca:

Theorem 4.1.8 ([8], Thm. 4.1). *If the Gasca-Maeztu conjecture is true for all $k \leq n$, then any GC_n set possesses at least three maximal lines.*

This yields, in view of Corollary 4.1.5 (ii) and Proposition 4.1.2, that each node of a GC_n set $\mathcal{X}$ uses at least one maximal line.

Denote by $\mu = \mu(\mathcal{X})$ the number of maximal lines of a node set $\mathcal{X}$. Thus, in view of Corollary 4.1.5, (iii), we have for any GC_n set $\mathcal{X}$:

$$3 \leq \mu(\mathcal{X}) \leq n+2, \tag{4.1.2}$$

where for the first inequality it is assumed that GM conjecture is true.

In Chapter 5 we will use the following

Lemma 4.1.9 ([10], Lem. 3.4). *Suppose that the Gasca-Maeztu conjecture is true for all $k \le n$. Suppose also that $\mathcal{X}$ is a GC_n set with exactly three maximal lines and λ is a maximal line. Then the GC_{n-1} set $\mathcal{X} \setminus \lambda$ also possesses exactly three maximal lines.*

The following proposition, which covers a wider setting, will be used in Chapter 6.

Proposition 4.1.10 ([8], Crl. 3.5). *Let λ be a maximal line of a GC_n set $\mathcal{X}$ such that $\mu(\mathcal{X}\setminus\lambda) \geq 3$. Then we have that*

$$\mu(\mathcal{X} \setminus \lambda) = \mu(\mathcal{X}) \quad \text{or} \quad \mu(\mathcal{X}) - 1.$$

4.2 Classification of GC_n sets - I

Here we will consider the results of Carnicer, Gasca, and Godés, concerning the classification of GC_n sets according to the number of maximal lines the sets possess. Let us start with

Theorem 4.2.1 ([12]). *Let $\mathcal{X}$ be a GC_n set with $\mu(\mathcal{X})$ maximal lines. Suppose also that GM conjecture is true for the degrees not exceeding n. Then $\mu(\mathcal{X}) \in \{3, n-1, n, n+1, n+2\}$.*

1. Lattices with $n+2$ maximal lines - the Chung-Yao natural lattices.

Let a set $\mathcal{M}$ of $n+2$ lines be in general position, i.e., no two lines are parallel and no three lines are concurrent, $n \geq 0$. Then the Chung-Yao set is defined as the set $\mathcal{X}$ of all $\binom{n+2}{2}$ intersection points of these lines. Note that the black nodes in Fig. 4.2.1 form a Chung-Yao lattice for $n = 3$. We have that the $n+2$ lines of $\mathcal{M}$ are maximal for $\mathcal{X}$. Each fixed node here is lying in exactly 2 lines and does not belong to the remaining n lines. Observe that the product of the latter n lines gives the fundamental polynomial of the fixed node. Thus $\mathcal{X}$ is a GC_n set. Let us mention that any n-poised set $\mathcal{X}$, with $\mu(\mathcal{X}) = n+2$, clearly forms a Chung-Yao lattice. Recall that there are no n-poised sets with more maximal lines (Proposition 4.1.5, (iii)).

2. Lattices with $n+1$ maximal lines - the Carnicer-Gasca lattices.

Let a set $\mathcal{M}$ of $n+1$ lines be in general position, $n \geq 2$. Then the Carnicer-Gasca lattice $\mathcal{X}$ is defined as $\mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)}$, where $\mathcal{X}^{(2)}$ is the set of all intersection nodes of these $n+1$

lines, and $\mathcal{X}^{(1)}$ is a set of other $n+1$ non-collinear nodes, one in each line, to make the line maximal. Note that the black and white nodes in Fig. 4.2.1 form a Carnicer-Gasca lattice for $n=4$. We have that $|\mathcal{X}| = \binom{n+1}{2} + (n+1) = \binom{n+2}{2}$. It is easily seen that $\mathcal{X}$ is a GC_n set and has exactly $n+1$ maximal lines, i.e., the lines of $\mathcal{M}$. Let us mention that any n-poised set $\mathcal{X}$, with $\mu(\mathcal{X}) = n+1$, clearly forms a Carnicer-Gasca lattice (see [6], Proposition 2.4).

3. Lattices with n maximal lines.

Let a set $\mathcal{M}$ of n lines be in general position, $n \geq 3$. Then consider the lattice $\mathcal{X}$ defined as

$$\mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)} \cup \mathcal{X}^{(0)}, \tag{4.2.1}$$

where $\mathcal{X}^{(2)}$ is the set of all intersection nodes of these n lines, $\mathcal{X}^{(1)}$ is a set of other $2n$ nodes, two in each line, to make the line maximal and $\mathcal{X}^{(0)}$ consists of a single node, denoted by O, which does not belong to any line from $\mathcal{M}$ (see Fig. 4.2.2). Correspondingly, we have that $|\mathcal{X}| = \binom{n}{2} + 2n + 1 = \binom{n+2}{2}$.

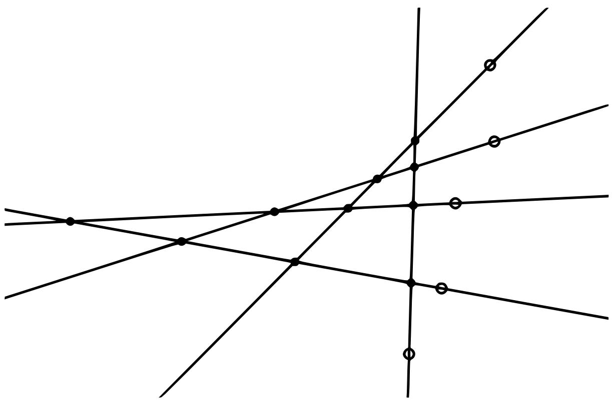

Figure 4.2.1: A GC_3 and a GC_4 sets.

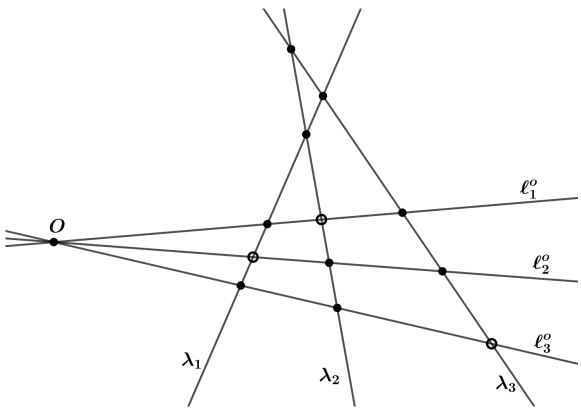

Figure 4.2.2: A lattice with n maximals

In the sequel we will need the following characterization of GC_n set $\mathcal{X}$, with $\mu(\mathcal{X}) = n$, due to Carnicer and Gasca (see Fig. 6.4.1):

Proposition 4.2.2 ([6], Prop. 2.5). *A node set $\mathcal{X}$ is a GC_n set with the set of maximal lines $\mathcal{M}$, $|\mathcal{M}| = n$, if and only if the representation* (4.2.1) *holds with the following additional properties:*

(i) *There are 3 lines $\ell_1^o, \ell_2^o, \ell_3^o$ concurrent at the node O : $O = \ell_1^o \cap \ell_2^o \cap \ell_3^o$ such that $\mathcal{X}^{(1)} \subset \ell_1^o \cup \ell_2^o \cup \ell_3^o$;*

(ii) *No line* $\ell_i^o,\ i=1,2,3,$ *contains* $n+1$ *nodes of* $\mathcal{X}$.

4.3 The sets $\mathcal{N}_\ell$ and $\mathcal{X}_\ell$

Definition 4.3.1 ([7])**.** Given an n-poised set $\mathcal{X}$ and a line ℓ. Then

(i) $\mathcal{X}_\ell$ is the subset of nodes of $\mathcal{X}$ which use the line ℓ;

(ii) $\mathcal{N}_\ell$ is the subset of nodes of $\mathcal{X}$ which do not use the line ℓ and do not lie in ℓ.

Notice that

$$\mathcal{X}_\ell \cup \mathcal{N}_\ell = \mathcal{X} \setminus \ell. \tag{4.3.1}$$

Note that the previously mentioned statement on maximal lines can be expressed as follows

$$\mathcal{X}_\ell = \mathcal{X} \setminus \ell, \text{ if } \ell \text{ is a maximal line.} \tag{4.3.2}$$

Suppose that λ is a maximal line of $\mathcal{X}$ and $\ell \neq \lambda$ is any line. Then in view of the relation (4.1.1) we have that

$$\mathcal{X}_\ell \setminus \lambda = (\mathcal{X} \setminus \lambda)_\ell. \tag{4.3.3}$$

The following proposition, due to Carnicer and Gasca, describes an important property of the set $\mathcal{N}_\ell$:

Theorem 4.3.2 ([7], Prop. 2.1)**.** *Let $\mathcal{X}$ be an n-poised set and ℓ be a line. Then the set $\mathcal{N}_\ell$ is $(n-1)$-dependent, provided that it is not empty.*

In the sequel we will use frequently the following two lemmas of Carnicer and Gasca.

Let $\mathcal{X}$ be an n-poised set and ℓ be a line with $|\ell \cap \mathcal{X}| \le n$. We call a maximal line λ *ℓ-disjoint* if

$$\lambda \cap \ell \cap \mathcal{X} = \emptyset. \tag{4.3.4}$$

Lemma 4.3.3 ([8], Lemma 4.4)**.** *Let $\mathcal{X}$ be an n-poised set and ℓ be a line with $|\ell \cap \mathcal{X}| \le n$. Suppose also that a maximal line λ is ℓ-disjoint. Then we have that*

$$\mathcal{X}_\ell = (\mathcal{X} \setminus \lambda)_\ell. \tag{4.3.5}$$

Moreover, if ℓ is an n-node line then we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\lambda \cup \ell)$, hence $\mathcal{X}_\ell$ is an $(n-2)$-poised set.

Let $\mathcal{X}$ be an n-poised set and ℓ be a line with $|\ell \cap \mathcal{X}| \leq n$. We call two maximal lines λ', λ'' *ℓ-adjacent* if

$$\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}. \tag{4.3.6}$$

Lemma 4.3.4 ([8], proof of Thm. 4.5)**.** *Let $\mathcal{X}$ be an n-poised set and ℓ be a line with $3 \leq |\ell \cap \mathcal{X}| \leq n$. Suppose also that two maximal lines λ', λ'' are ℓ-adjacent. Then we have that*

$$\mathcal{X}_\ell = (\mathcal{X} \setminus (\lambda' \cup \lambda''))_\ell. \tag{4.3.7}$$

Moreover, if ℓ is an n-node line then we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\lambda' \cup \lambda'' \cup \ell)$, hence $\mathcal{X}_\ell$ is an $(n-3)$-poised set.

Next, by the motivation of above two lemmas, let us introduce the concept of an ℓ-reduction of a GC_n set.

Definition 4.3.5. Let $\mathcal{X}$ be a GC_n set, ℓ be a k-node line, $k \geq 2$. We say that a set $\mathcal{Y} \subset \mathcal{X}$ is an ℓ-reduction of $\mathcal{X}$, and briefly denote this by $\mathcal{X} \searrow_\ell \mathcal{Y}$, if

$$\mathcal{Y} = \mathcal{X} \setminus (\mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_s),$$

where

(i) $\mathcal{C}_0$ is an ℓ-disjoint maximal line of $\mathcal{X}$, or $\mathcal{C}_0$ is the union of a pair of ℓ-adjacent maximal lines of $\mathcal{X}$;

(ii) $\mathcal{C}_i$ is an ℓ-disjoint maximal line of the GC set $\mathcal{Y}_i := \mathcal{X} \setminus (\mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{i-1})$, or $\mathcal{C}_i$ is the union of a pair of ℓ-adjacent maximal lines of $\mathcal{Y}_i,\ i = 1, \dots s$;

(iii) ℓ passes through at least 2 nodes of $\mathcal{Y}$.

We get immediately from Lemmas 4.3.3 and 4.3.4 that

$$\mathcal{X} \searrow_\ell \mathcal{Y} \Rightarrow \mathcal{X}_\ell = \mathcal{Y}_\ell. \tag{4.3.8}$$

Notice that we cannot do any further ℓ-reduction with the set $\mathcal{Y}$ if the line ℓ is a maximal line here. For this situation we have the following

Definition 4.3.6. Let $\mathcal{X}$ be a GC_n set, ℓ be a k-node line, $k \geq 2$. We say that the set $\mathcal{X}_\ell$ is an ℓ-proper GC_m subset of $\mathcal{X}$ if there is a GC_{m+1} set $\mathcal{Y}$ such that

(i) $\mathcal{X} \searrow_\ell \mathcal{Y}$;

(ii) The line ℓ is a maximal line in $\mathcal{Y}$.

In view of the relations (4.3.8) and (4.3.2) we have that

$$\mathcal{X}_\ell = \mathcal{Y} \setminus \ell = \mathcal{X} \setminus (\mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_s \cup \ell),$$

where the sets $\mathcal{C}_i$ satisfy conditions listed in Definition 4.3.5.

Let us mention that, in view of Corollary 4.1.4, the above ℓ-proper GC_m subset $\mathcal{X}_\ell$ is indeed a GC_m set, with $m = n - \sum_{i=0}^{s} \delta_i - 1$, where $\delta_i \in \{1, 2\}$ is the number of the maximal lines contained in $\mathcal{C}_i$.

Note that if ℓ is an n-node line then the node set $\mathcal{X}_\ell$ in Lemma 4.3.3 or in Lemma 4.3.4 is an ℓ-proper GC_{n-2} or GC_{n-3} subset of $\mathcal{X}$, respectively.

We immediately get from Definitions 4.3.5 and 4.3.6 the following

Proposition 4.3.7. *Suppose that $\mathcal{X}$ is a GC_n set. If $\mathcal{X} \searrow_\ell \mathcal{Y}$ and $\mathcal{Y}_\ell$ is an ℓ-proper GC_m subset of $\mathcal{Y}$ then $\mathcal{X}_\ell$ is an ℓ-proper GC_m subset of $\mathcal{X}$.*

Let us formulate the following useful (cf. [1], Corollary 3.4)

Corollary 4.3.8. *Let $\mathcal{X}$ be an n-poised set and ℓ be an n-node line.*

(i) *Suppose that a maximal line λ_0 satisfies the condition* (4.3.4)*. Then all other maximal lines λ, $\lambda \neq \lambda_0$, intersect the line ℓ at distinct nodes and $|\lambda \cap \mathcal{X}_\ell| = n - 1$;*

(ii) *Suppose that maximal lines λ' and λ'' satisfy the condition* (4.3.6)*. Then all other other maximal lines λ, $\lambda \neq \lambda', \lambda \neq \lambda''$, intersect the line ℓ at distinct nodes and $|\lambda \cap \mathcal{X}_\ell| = n - 2$.*

Proof. The cases $n = 1, 2$, are evident. Thus suppose that $n \geq 3$. Notice that the conditions (4.3.4) and (4.3.6) can not hold simultaneously, since then, by Lemmas 4.3.3 and 4.3.4, the conditions (4.3.5) and (4.3.7) hold simultaneously, which is a contradiction.

(i) We get from Lemma 4.3.3 that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda_0)$. The maximal line λ intersects the lines ℓ and λ_0 at two distinct nodes. Therefore we obtain $|\lambda \cap \mathcal{X}_\ell| = |\lambda \cap [\mathcal{X} \setminus (\ell \cup \lambda_0)]| = (n+1) - 2 = n-1$.

(ii) In this case, according to Lemma 4.3.4, we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda' \cup \lambda'')$. The maximal line λ intersects the lines ℓ, λ' and λ'' at three distinct nodes. Therefore we obtain $|\lambda \cap \mathcal{X}_\ell| = |\lambda \cap [\mathcal{X} \setminus (\ell \cup \lambda' \cup \lambda'')]| = (n+1) - 3 = n-2$. $\square$

Finally, let us bring a result on n-node lines we are going to use in the next Chapter.

Proposition 4.3.9 ([1], Prop. 2.1). *Let $\mathcal{X}$ be an n-poised set and ℓ be an n-node line. Then the following hold:*

(i) $|\mathcal{X}_\ell| \le \binom{n}{2}$;

(ii) *If* $|\mathcal{X}_\ell| \ge \binom{n-1}{2} + 1$ *then* $|\mathcal{X}_\ell| = \binom{n}{2}$. *Moreover,* $\mathcal{X}_\ell$ *is an* $(n-2)$*-poised set and* $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda)$, *where* λ *is a maximal line such that* $\lambda \cap \ell \cap \mathcal{X} = \emptyset$;

(iii) *If* $\binom{n-1}{2} \ge |\mathcal{X}_\ell| \ge \binom{n-2}{2} + 2$ *then* $|\mathcal{X}_\ell| = \binom{n-1}{2}$. *Moreover,* $\mathcal{X}_\ell$ *is an* $(n-3)$*-poised set and* $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \beta)$, *where* $\beta \in \Pi_2$ *is a conic such that* $\mathcal{N}_\ell = (\beta \setminus \ell) \cap \mathcal{X}$, *and* $|\mathcal{N}_\ell| = 2n$. *Besides these* $2n$ *nodes the conic may contain at most one extra node, which necessarily belongs to* ℓ. *Furthermore, if the conic* β *is reducible:* $\beta = \ell_1 \ell_2$ *then we have that* $|\ell_i \cap (\mathcal{X} \setminus \ell)| = n, \ i = 1, 2$.

4.4 The Gasca-Maeztu conjecture for $n = 4$

In this Section we present a simple, short, and clear proof of the Gasca-Maeztu conjecture for the case $n = 4$. The Conjecture was proposed in 1981 by Gasca and Maeztu [16]. Until now, this has been confirmed only for the values $n \le 5$. The case $n = 5$ was proven in 2014 by Hakopian, Jetter, and Zimmermann, in [19]. So far that is the only proof for $n = 5$. In addition, it is very long and complicated. In our opinion a simple proof of the Gasca-Maeztu conjecture for smaller values of n greatly simplifies its generalization for higher values. We believe that this is a way in trying to prove the Conjecture for the values $n \ge 6$.

The following formulation of the Theorem 4.3.2 we will use in this section.

Theorem 4.4.1 ([7]). *Suppose, that we have a line ℓ and an n-poised set $\mathcal{X}$. Then the following hold:*

(i) *If the set* $\mathcal{N}_\ell$ *is nonempty, then it is* $(n-1)$*-dependent and for no node* $A \in \mathcal{N}_\ell$*, there exists a fundamental polynomial* $p^\star_{A,\mathcal{N}_\ell}$ *in* Π_{n-1}.

(ii) $\mathcal{N}_\ell = \emptyset$ *if and only if* ℓ *passes through* $n+1$ *nodes in* $\mathcal{X}$.

Since the fundamental polynomial of an n-poised set is unique we get (see e.g. [18], Lemma 2.5)

Lemma 4.4.2 ([18]). *Suppose* $\mathcal{X}$ *is a poised set and a node* $A \in \mathcal{X}$ *uses a line* $\ell:\ p^\star_A = \ell q, q \in \Pi_{n-1}$. *Then* ℓ *passes through at least two nodes from* $\mathcal{X}$*, at which* q *does not vanish.*

Next we formulate the Gasca-Maeztu conjecture for $n = 4$ as:

Theorem 4.4.3 ([29]). *Any* GC_4*-set* $\mathcal{X}$ *of* 15 *nodes possesses a maximal line, i.e., a line passing through* 5 *nodes.*

To prove the theorem assume by way of contradiction the following.

Assumption 4.4.4. *The set* $\mathcal{X}$ *is a* GC_4*-set without any maximal line.*

We call a line k-node line if it passes through exactly k nodes of the set $\mathcal{X}$. In the next subsection we discuss the problem: Given a $2, 3$ or 4-node line. By how many nodes in $\mathcal{X}$ it can be used at most.

The following lemma is in ([18], Lemma 4.1). We bring it here for the sake of completeness.

Lemma 4.4.5. *Any* 2 *or* 3*-node line can be used by at most one node of* $\mathcal{X}$.

Proof. Assume by contradiction that ℓ is a 2 or 3-node line used by two points $A, B \in \mathcal{X}$. Consider the fundamental polynomial $p^\star_A$. The node A uses the line ℓ and three more lines, which contain the remaining ≥ 11 nodes of $\mathcal{X} \setminus (\ell \cup \{A\})$, including B. Since there is no 5-node line, we get

$$p^\star_A = \ell \ell_{=4} \ell'_{=4} \ell_{\geq 3}.$$

Here the subscript $= 4$ means that the corresponding line is a 4-node line, while the subscript ≥ 3 means that except the 3 nodes the corresponding line may also pass through some nodes belonging to the other lines. First suppose that B belongs to one of the 4-node lines, say to $\ell'_{=4}$. We also have

$$p^\star_B = \ell q, \text{ where } q \in \Pi_3.$$

Notice that q vanishes at 4 nodes of $\ell_{=4}$ and 3 nodes of $\ell'_{=4}$ (i.e., except B). Therefore by using Proposition 4.1.2 twice we get that $q = \ell_{=4}r,\ r \in \Pi_2$ and $r = \ell'_{=4}s,\ s \in \Pi_1$. Thus $p_B^\star = \ell\ell_{=4}\ell'_{=4}s$. Hence $p_B^\star$ vanishes at B ($B \in \ell'_{=4}$), which is a contradiction.

Now assume that B belongs to the line $\ell_{\geq 3}$. Then q vanishes at 4 nodes of $\ell_{=4}$, 4 (≥ 3) nodes of $\ell'_{=4}$ and at least 2 nodes of $\ell_{\geq 3}$. Therefore again, as above, by consecutive usage of Proposition 4.1.2 we get that $p_B^\star = \ell\ell_{=4}\ell'_{=4}\ell_{\geq 3}$. Hence again $p_B^\star$ vanishes at B ($B \in \ell_{\geq 3}$), which is a contradiction. □

The following lemma is in ([2], Lemma 2.6). Here we bring a very brief proof of it.

Lemma 4.4.6 ([29])**.** *Any* 4*-node line can be used by at most three nodes of* $\mathcal{X}$*.*

Proof. Assume by contradiction that ℓ is a 4-node line used by four points from $\mathcal{X}$. Therefore we have $|\mathcal{N}_\ell| \leq 15 - 4 - 4 = 7$. In view of Theorem 4.4.1 $\mathcal{N}_\ell \neq \emptyset$ is (essentially) 3-dependent. According to Theorem 1.2.9 a set of $\leq 2 \times 3 + 1 = 7$ nodes is 3-dependent if and only if there is a 5-node line, which contradicts Assumption 4.4.4. □

Now we are in a position to prove the Gasca-Maeztu conjecture for $n = 4$.

Let us start with an observation from ([19], Section 3.2). Fix any node $A \in \mathcal{X}$, and consider all the lines through the node A and some other node(s) of $\mathcal{X}$. Denote this set of lines by $\mathcal{L}_A$. Let $n_m(A)$ be the number of m-node lines from $\mathcal{L}_A$. In view of Assumption 4.4.4 we have

$$1n_2(A) + 2n_3(A) + 3n_4(A) = |(\mathcal{X} \setminus \{A\})| = 14. \tag{4.4.1}$$

Denote by $M(A)$ the total number of uses of the lines passing through A. By Lemma 4.4.2 each of 14 nodes of $\mathcal{X} \setminus \{A\}$ uses at least one line from $\mathcal{L}_A$. On the other hand, we get from Lemmas 4.4.5 and 4.4.6 that

$$14 \leq M(A) \leq 1n_2(A) + 1n_3(A) + 3n_4(A).$$

Comparing this with (4.4.1), we conclude that necessarily $M(A) = 14$ and $n_3(A) = 0$, i.e., there is no 3-node line in $\mathcal{L}_A$.

Thus we have

$$n_2(A) + 3n_4(A) = 14. \tag{4.4.2}$$

Therefore each 4-node line in $\mathcal{L}_A$ is used exactly three times and each 2-node line is used exactly once. From here we conclude easily that $n_2(A) \geq 2$. Next we show that actually $n_2(A) = 2$.

Consider two 2-node lines passing through A. Suppose that besides A they pass through B and C, respectively. Denote these two lines by ℓ_B and ℓ_C, respectively (see Fig 6.2.1).

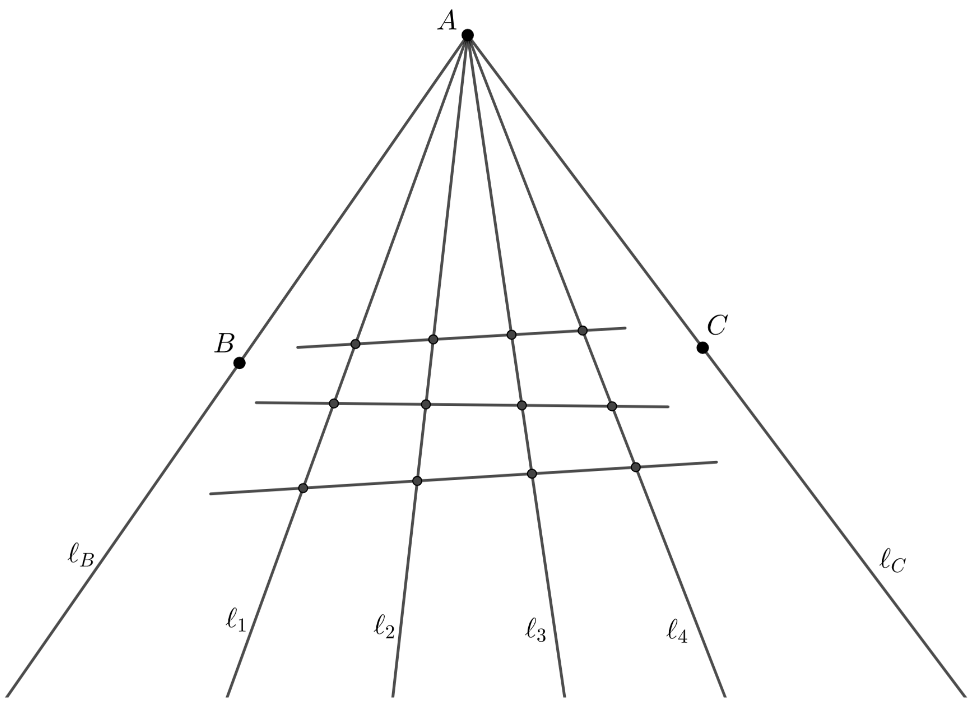

Figure 4.4.1: The lines of $\mathcal{L}_A$

Next, we will prove that B uses ℓ_C. Let us verify that in this case the node C uses ℓ_B. Indeed, if B uses ℓ_C we have $p_B^\star = \ell_C q$, where q is a product of three lines. Notice that the polynomial $\ell_B q$ is the fundamental polynomial of the node C, which means that C uses ℓ_B. Now, suppose by way of contradiction that B does not use ℓ_C. Therefore C does not use ℓ_B.

Thus, there are two nodes D and E in the 12 nodes of $\mathcal{X} \setminus \{A, B, C\}$ using the lines ℓ_B and ℓ_C respectively. In this case, we have $p_D^\star = \ell_B q_1$ and $p_E^\star = \ell_C q_2$, where q_1 and q_2 are polynomials of degree 3.

Since q_1 and q_2 have 10 common nodes we get from the Bezout theorem that they have common linear factor α, passing through at most 4 nodes. So we can write $q_1 = \alpha\beta_1$ and

$q_2 = \alpha\beta_2$, where β_1 and β_2 have at least 6 common nodes. Therefore, β_1 and β_2 have common linear factor α_1, passing through at most 4 nodes.

Now, we have for the following presentations of the fundamental polynomials: $p_D^\star = \ell_B\alpha\alpha_1\alpha_2$ and $p_E^\star = \ell_C\alpha\alpha_1{\alpha_2}'$. Therefore α_2 and ${\alpha_2}'$ have at least two common nodes, which means that they coincide. We have that $E \in \alpha \cup \alpha_1 \cup \alpha_2$ and thus come to a contradiction, which proves that B uses ℓ_C.

Note that ℓ_C was an arbitrary 2-node line, which means that B uses all 2-node lines different from ℓ_B. It is easy to see that any node from $\mathcal{X}$ can use at most one 2-node line, since otherwise if some node uses two 2-node lines the remaining ≥ 10 nodes have to lie on two lines. Therefore, we conclude that there are no 2-node lines other than ℓ_B and ℓ_C, i.e., $n_2(A) = 2$. From here and the equality (4.4.2) we get $n_4(A) = 4$.

Thus, the 12 nodes of $\mathcal{X} \setminus \{A, B, C\}$ lie on four 4-node lines passing through A. We denote these lines by $\ell_1, ..., \ell_4$.

Finally, by taking $p(x, y) = \ell_1\ell_2\ell_3\ell_4$, in the Lagrange formula (1.2.2), we obtain

$$\ell_1\ell_2\ell_3\ell_4 = \lambda_1 p_B^\star + \lambda_2 p_C^\star, \tag{4.4.3}$$

since $\ell_1\ell_2\ell_3\ell_4$ vanishes in $\mathcal{X} \setminus \{B, C\}$. Now recall that $p_B^\star = \ell_C q$ and $p_C^\star = \ell_B q$, where q is a product of three 4-node lines passing through the 12 nodes of $\mathcal{X} \setminus \{A, B, C\}$. Thus we get

$$\ell_1\ell_2\ell_3\ell_4 = q(\lambda_1\ell_C + \lambda_2\ell_B).$$

Clearly none of the lines ℓ_i here are factors of q. Hence this leads to a contradiction, which proves Theorem 4.4.3.

Chapter 5

ON A CORRECTION OF A PROPERTY OF GC SETS

In this chapter we consider the main result of the paper [1] by V. Bayramyan and H. Hakopian., stating that any n-node line of GC_n set is used either by exactly $\binom{n}{2}$ nodes or by exactly $\binom{n-1}{2}$ nodes, provided that the Gasca-Maeztu conjecture is true.

Here we show that this result is not correct in the case $n = 3$. Namely, in Subsection 5.1.1, we bring an example of a GC_3 set and a 3-node line there which is not used at all. The proof of the result in [1] is inductive and based on the case $n = 3$. For this reason we needed to consider a new proof of the result. Fortunately, we were able to establish that the above mentioned result is true for all $n \geq 4$ (see the forthcoming Theorem 5.1.1).

We also characterize the exclusive case $n = 3$ (Proposition 5.1.3) and present some new results on the maximal lines and the usage of n-node lines in GC_n sets. Let ℓ be an n-node line in a GC_n set $\mathcal{X}$, where $n \geq 4$. Namely, we prove that if there are n maximal lines passing through n distinct nodes in ℓ then there is at least one more maximal line in $\mathcal{X}$ (Proposition 5.2.2). We also prove that if $\mathcal{X}$ has exactly three maximal lines then there are exactly three n-node lines and each is used by exactly $\binom{n}{2}$ nodes (Corollary 5.2.4).

5.1 On n-node lines in GC_n sets

First of all let us present the corrected version of the main result of the paper [1] by V. Bayramyan and H. Hakopian:

Theorem 5.1.1 ([25]). *Assume that Conjecture 4.1.7 holds for all degrees up to n. Let $\mathcal{X}$ be a GC_n set, $n \geq 4$, and ℓ be an n-node line. Then we have that*

$$|\mathcal{X}_\ell| = \binom{n}{2} \quad or \quad \binom{n-1}{2}. \tag{5.1.1}$$

Moreover, the following hold:

(i) $|\mathcal{X}_\ell| = \binom{n}{2}$ *if and only if there is a maximal line λ_0 such that $\lambda_0 \cap \ell \cap \mathcal{X} = \emptyset$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda_0)$. Hence it is a GC_{n-2} set;*

(ii) $|\mathcal{X}_\ell| = \binom{n-1}{2}$ *if and only if there are two maximal lines λ', λ'', such that $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda' \cup \lambda'')$. Hence it is a GC_{n-3} set.*

In [1] this result is stated for all $n \geq 1$. Note that the cases $n = 1, 2$, are obvious (see [1]). In the next subsection we bring a counterexample showing that Theorem 5.1.1 is not correct in the case $n = 3$.

Let us mention now that the converse implications in the assertions (i) and (ii) of Theorem 5.1.1 follow from Lemmas 4.3.3 and 4.3.4, respectively. The same is true in the case of the forthcoming Proposition 5.1.3 (the case $n = 3$).

Remark 5.1.2. Note that the conclusions in the statements (i) and (ii) of Corollary 4.3.8 can be subjoined to the statements (i) and (ii) of Theorem 5.1.1, respectively. The same is true with Theorem 5.1.1 replaced by Proposition 5.1.3 (the case $n = 3$).

5.1.1 A counterexample

Let us start with a counterexample. Consider a GC_3 set $\mathcal{X}^\star$ with exactly three maximal lines: $\lambda_1, \lambda_2, \lambda_3$ (see Fig. 5.1.1). We have that in such sets nine nodes are lying in the maximal lines and one - O is outside of them. Also there are three 3-node lines ℓ_1, ℓ_2, ℓ_3 passing through the node O (see Fig. 6.4.1 and Case 2 of the proof of Proposition 5.1.3, below). In the set $\mathcal{X}^\star$ we have a fourth 3-node line: $\ell^\star$ which is not passing through the node O.

As we will see in the next proposition such a line cannot be used by any node in $\mathcal{X}$. It is worth mentioning that this could also be verified directly.

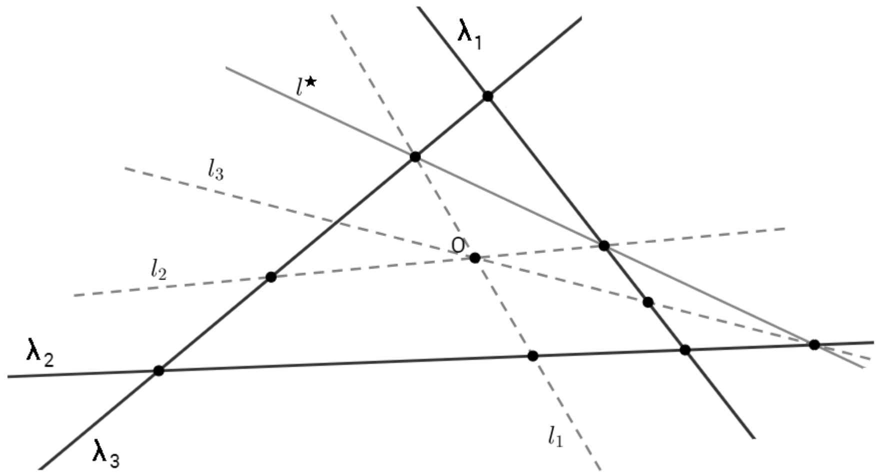

Figure 5.1.1: A non-used 3-node line $\ell^\star$ in the GC_3 set $\mathcal{X}^\star$.

Before starting the proof of Theorem 5.1.1 let us characterize the exclusive case $n = 3$.

5.1.2 On 3-node lines in GC_3 sets

Proposition 5.1.3 ([25]). *Let $\mathcal{X}$ be a GC_3 set and ℓ be a 3-node line. Then we have that*

$$|\mathcal{X}_\ell| = 3, \quad 1, \quad or \quad 0. \tag{5.1.2}$$

Moreover, the following hold:

(i) $|\mathcal{X}_\ell| = 3$ *if and only if there is a maximal line λ_0 such that $\lambda_0 \cap \ell \cap \mathcal{X} = \emptyset$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda_0)$. Hence it is a GC_1 set.*

(ii) $|\mathcal{X}_\ell| = 1$ *if and only if there are two maximal lines λ', λ'', such that $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda' \cup \lambda'')$;*

(iii) $|\mathcal{X}_\ell| = 0$ *if and only if there are exactly three maximal lines in $\mathcal{X}$ and they intersect ℓ at three distinct nodes.*

Furthermore, if the node set $\mathcal{X}$ possesses exactly three maximal lines then any 3-node line ℓ is either used by exactly three nodes or is not used at all:

$$|\mathcal{X}_\ell| = 3 \quad or \quad 0. \tag{5.1.3}$$

Proof. The proofs of the assertion (iii) and the direct implications (i) and (ii) are divided into cases, depending on the number $\mu(\mathcal{X})$. Recall that, according to the relation (4.1.2), we have that $3 \le \mu(\mathcal{X}) \le 5$.

Case 1. Suppose that $\mu(\mathcal{X}) = 4$, or 5. The line ℓ is a 3-node line. Therefore either there is a maximal line λ such that $\lambda \cap \ell \cap \mathcal{X} = \emptyset$ or there are two maximal lines λ', λ'', such that $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. Thus the result holds in this case since the converse implications in the assertions (i) and (ii) are valid.

Case 2. Suppose that there are exactly 3 maximal lines in $\mathcal{X}$: $\lambda_1, \lambda_2, \lambda_3$ (see Fig. 6.4.1). By Corollary 4.1.5 these lines form a triangle and the vertices A, B, C, are nodes in $\mathcal{X}$. There

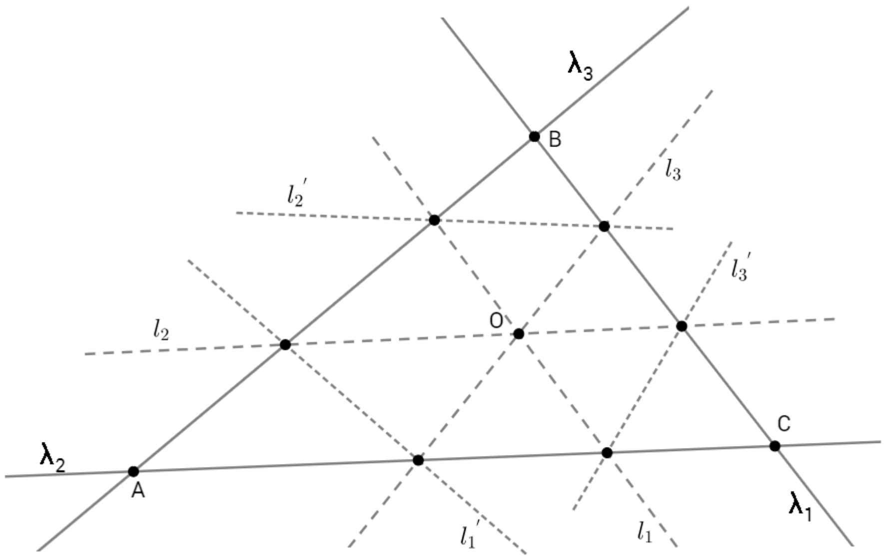

Figure 5.1.2: The case of GC_3 set with exactly three maximal lines.

are 6 more nodes, called "free", 2 in each maximal line. The tenth node - O is outside of the maximal lines. We find readily that the 6 "free" nodes are located also in 3 lines: ℓ_1, ℓ_2, ℓ_3, passing through O, 2 in each line (see Fig. 6.4.1).

To prove the result in this case it suffices to verify the following assertions:

(a) The lines ℓ_1, ℓ_2, ℓ_3, are 3-node lines;

(b) For each line ℓ_1, ℓ_2, ℓ_3, there is a maximal line in $\mathcal{X}$ which does not intersect it at a node;

(c) Each of the lines ℓ_1, ℓ_2, ℓ_3, is used by exactly 3 nodes in $\mathcal{X}$;

(d) Except of the lines ℓ_1, ℓ_2, ℓ_3, there is no other used 3-node line in $\mathcal{X}$.

(e) Through each node of any non-used 3-node line there pass a maximal line.

Let us start the verification.

(a) Note that the lines ℓ_1, ℓ_2, ℓ_3, are 3-node lines, i.e., they do not contain any more nodes, except O and intersection nodes with two maximal lines. Indeed, otherwise they would become a maximal line and make the number of maximal lines of $\mathcal{X}$ more than three.

(b) In view of (a) we get readily that the maximal line λ_i does not intersect the line ℓ_i at a node, $i = 1, 2, 3$.

(c) In view of (b) and converse implication of the statement (i) we get readily that the line ℓ_i is used by the 3-nodes of the set $\mathcal{X} \setminus (\ell_i \cup \lambda_i),\ i = 1, 2, 3$.

(d) To verify this item let us specify all the used lines in $\mathcal{X}$ and see that except of the lines ℓ_1, ℓ_2, ℓ_3, there is no other used 3-node line in $\mathcal{X}$.

First notice that all the nodes of the set $\mathcal{X}$, except the vertices A, B, C, use only the maximal lines and the three 3-node lines ℓ_1, ℓ_2, ℓ_3 (see Fig. 6.4.1). Then observe that each of the vertices A, B, C, uses, except a maximal line and a 3-node line ℓ_1, ℓ_2, ℓ_3 also a 2-node line. Namely, these vertices use the lines $\ell_1', \ell_2', \ell_3'$, respectively (see Fig. 6.4.1). Note that the latter lines infact are 2-node lines. Indeed, say the line ℓ_1' obviously does not pass through any more nodes from the two maximal lines that pass through the vertex A, since each of these maximal lines intersects ℓ_1' at one of its two nodes. The line ℓ_1' does not pass also through the remaining three nodes. Indeed, they belong to the lines ℓ_2, ℓ_3 that intersect the line ℓ_1' at one of its two nodes.

(e) Suppose that there is a 3-node line different from ℓ_1, ℓ_2, ℓ_3 (as $\ell^\star$ in Fig. 5.1.1). Then clearly it is not passing through the vertices or the node O. Thus the three nodes of ℓ belong to the maximal lines $\lambda_i,\ i = 1, 2, 3$, one node to each. Note that such a 3-node line is not used since it is not among the used lines we specified in the item (d).

Finally, for the part "Furthermore", it suffices to observe that the case (ii), i.e., $|\mathcal{X}_\ell| = 1$, cannot happen if the node set $\mathcal{X}$ has exactly three maximal lines. Indeed, as it was mentioned in the item (e), it is easily seen that there cannot be a 3-node line passing through any intersection node of maximal lines, i.e., through A, B, C (see Fig. 6.4.1). □

Let us mention that the first statement of Proposition 5.1.3 (without "Moreover" and "Furthermore" parts) for 3-node lines in n-poised sets was proven in [21], Corollary 6.1.

5.2 The proof of the main theorem

The original version of Theorem 5.1.1 was proven in [1] by induction on n. As the first step of the induction the case $n = 3$ was used. We have already verified that in this case the statement is not valid. Thus we start with the special case $n = 4$ of Theorem 5.1.1:

Proposition 5.2.1 ([25]). *Let $\mathcal{X}$ be a GC_4 set, hence $|\mathcal{X}| = 15$, and ℓ be a 4-node line. Then we have that*

$$|\mathcal{X}_\ell| = 6 \quad or \quad 3. \tag{5.2.1}$$

Moreover, the following hold:

(i) $|\mathcal{X}_\ell| = 6$ *if and only if there is a maximal line λ_0 such that $\lambda_0 \cap \ell \cap \mathcal{X} = \emptyset$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda_0)$. Hence it is a GC_2 set;*

(ii) $|\mathcal{X}_\ell| = 3$ *if and only if there are two maximal lines λ', λ'', such that $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. In this case we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell \cup \lambda' \cup \lambda'')$. Hence it is a GC_1 set.*

First we will prove the following

Proposition 5.2.2 ([25]). *Let $\mathcal{X}$ be a GC_n set and ℓ be an n-node line, where $n \geq 4$. Suppose that there are n maximal lines passing through n distinct nodes in ℓ. Then there exists at least one more maximal line in $\mathcal{X}$.*

Proof. Suppose by way of contradiction that the node set $\mathcal{X}$ possesses exactly n maximal lines denoted by $\lambda_1, \ldots, \lambda_n$. Then the characterization of Proposition 4.2.2 with (4.2.1) holds. Now notice that the line ℓ does not pass through an intersection node of two maximal lines. Indeed, in this case two maximal lines intersect ℓ at a node and there remain only $n-2$ maximal lines to intersect ℓ at other $n-1$ nodes. Thus clearly the hypothesis of the Proposition cannot be satisfied. Therefore the line ℓ may pass through only the "free" nodes $A_i, A_i',\ i=1,\ldots,n$, and the outside node O (see (4.2.1)). Thus, in view of Proposition 4.2.2, the n nodes of the line ℓ are lying in the three lines $\ell_1^*, \ell_2^*, \ell_3^*$. Since $n\geq 4$ we deduce that ℓ coincides with a line $\ell_i^*, i=1,2,3$. On the other hand none of these three lines intersects n maximal lines at nodes of $\mathcal{X}$, since otherwise it would become a maximal line. This contradiction completes the proof. $\square$

Remark 5.2.3. It is worth mentioning that Proposition 5.2.2 is not valid in the case $n=3$. Indeed, the GC_3 set $\mathcal{X}^\star$ and the 3-node line $\ell^\star$ (see Fig. 5.1.1) give us a counterexample for this.

Now we are in a position to start

The proof of Proposition 5.2.1. The proof of the direct implications is divided into cases, depending on the number $\mu(\mathcal{X})$. Recall that, according to the relation (4.1.2), $3\leq\mu(\mathcal{X})\leq 6$.

Case 1. Assume that $\mu(\mathcal{X})\geq 5$. The line ℓ is a 4-node line. Therefore either there is a maximal line λ such that $\lambda\cap\ell\cap\mathcal{X}=\emptyset$ or there are two maximal lines λ',λ'', such that $\lambda'\cap\lambda''\cap\ell\in\mathcal{X}$. Therefore the result holds in this case since the converse implications in the assertions (i) and (ii) are valid.

Case 2. Assume that there are exactly 4 maximal lines in $\mathcal{X}$. In view of Case 1 we may suppose that the four maximal lines intersect ℓ in four distinct nodes. Now, in view of the case $n=4$ of Proposition 5.2.2, we conclude that there is a fifth maximal line for $\mathcal{X}$, which contradicts our assumption.

Case 3. Assume that there are exactly 3 maximal lines in $\mathcal{X}$.

Notice that as above, in view of the converse implications in the assertions (i) and (ii), we may assume that 3 maximal lines intersect ℓ at 3 distinct nodes, say first three: A_1, A_2, A_3 (see Fig.6.2.2). Let us prove that this case is impossible. To this end it is enough to prove the following statements in this case.

(a) There is a node that uses the line ℓ, i.e $|\mathcal{X}_\ell|\geq 1$;

(b) If a node uses the line ℓ, then there are at least 5 nodes using it;

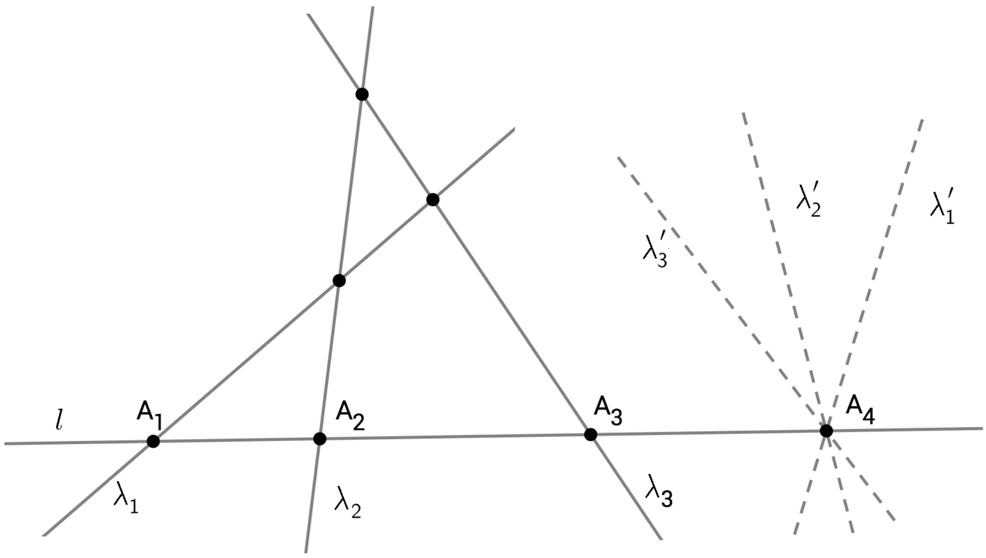

Figure 5.2.1: The case of 3 maximal lines.

(c) If five nodes use the line ℓ, then there is a forth maximal of the node set $\mathcal{X}$.

Obviously the statement (c) contradicts our assumption.

Now let us start with the statement (a). Assume, by way of contradiction, that ℓ is not used by any node of $\mathcal{X}$. Let us consider the set $\mathcal{X}_1 = \mathcal{X} \setminus \lambda_1$. We have, in view of Lemma 4.1.9, that there are exactly 3 maximal lines in $\mathcal{X}_1$. Namely, λ_2, λ_3, and a third maximal line denoted by λ_1', which clearly does not intersect λ_1 at a node. Indeed, otherwise, we would have 4 maximal lines in the node set $\mathcal{X}$. Now since ℓ is not used also in $\mathcal{X}_1$ we obtain, in view of Proposition 5.1.3, (iii), that in the node set $\mathcal{X}_1$ the third maximal line λ_1' passes through the fourth node A_4 of ℓ (see Fig.6.2.2). In a similar way the third maximal lines: λ_2', λ_3' in the sets $\mathcal{X}_2 = \mathcal{X} \setminus \lambda_2$ and $\mathcal{X}_3 = \mathcal{X} \setminus \lambda_3$ pass through the node A_4 and do not intersect the maximal lines λ_2, λ_3 at nodes, respectively. Next consider the GC_1 set $\mathcal{X} \setminus (\lambda_1 \cup \lambda_2 \cup \lambda_3)$. Here the lines λ_1', λ_2' and λ_3' have each two nodes and thus are maximal, which contradicts Corollary 4.1.5 (ii).

Now let us prove the statement (b). Denote by A the node that uses ℓ. Since the 3 maximal lines are not concurrent we can choose a maximal line not passing through A. Suppose, without loss of generality, that it is the line λ_1. Consider the GC_3 set $\mathcal{X}_1$. As we mentioned above there are exactly 3 maximal lines in $\mathcal{X}_1$. Note that the node $A \in \mathcal{X}_1$ uses the line ℓ. On the other hand, by Proposition 5.1.3, part “Furthermore", the line ℓ can be used here either by 3 or by no node in $\mathcal{X}_1$. Thus we conclude that $(\mathcal{X}_1)_\ell$ consists of three noncollinear nodes. On the other hand, we get from Corollary 4.3.8, (i), (or Remark 5.1.2) that there are two nodes in the maximal line λ_2 (as well as in λ_3) that use the line ℓ. Next, there is a node in the node set $\mathcal{X}_2$ that uses the line ℓ, since the three nodes of $(\mathcal{X}_1)_\ell$ are noncollinear. Thus we may repeat discussion of the node set $\mathcal{X}_1$ with $\mathcal{X}_2$ and obtain that $|(\mathcal{X}_2)_\ell| = 3$. Now we have that

$$|\mathcal{X}_\ell| = |(\mathcal{X}_2)_\ell| + |\lambda_2 \cap \mathcal{X}_\ell| \geq 3 + 2 = 5.$$

The first equality above follows from the relation (4.3.3).

Next let us prove the statement (c). Consider the set $\mathcal{N}_\ell$. In view of the relation (4.3.1) we get

$$|\mathcal{N}_\ell| \leq 15 - (4 + 5) = 6.$$

By Theorem 4.3.2 we have that the set $\mathcal{N}_\ell$ is 3-dependent. Now, Theorem 1.2.9 implies that 5 points from $\mathcal{N}_\ell$ are collinear, i.e., they are in a maximal line. This maximal line cannot coincide with the three maximal lines of $\mathcal{X}$, since each of them intersects ℓ at a node and hence has only 4 nodes in the set $\mathcal{N}_\ell$. Thus we get a fourth maximal line. $\square$

Now we are in a position to prove the theorem for $n \geq 5$. Let us mention that the proof here is similar to one from [1], Section 3.4. But it is much shorter due to the fact that the first step of the induction here is the case $n = 4$.

Thus let us prove Theorem 5.1.1 by induction on n. Assume that Theorem is true for all degrees less than n and let us prove that it is true for the degree n, where $n \geq 5$.

First assume that $|\mathcal{X}_\ell| \geq \binom{n-1}{2} + 1$. Then by Proposition 4.3.9, (ii), we get that $|\mathcal{X}_\ell| = \binom{n}{2}$ and the direct implication in the assertion (i) holds.

Thus to prove Theorem 5.1.1 it suffices to assume that

$$|\mathcal{X}_\ell| \leq \binom{n-1}{2} \tag{5.2.2}$$

and to prove that $|\mathcal{X}_\ell| = \binom{n-1}{2}$ and the direct implication in the assertion (ii) holds, i.e., there are two maximal lines λ', λ'', such that $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. Indeed, this will complete the proof in view of Lemma 4.3.4.

Now let us show that there is a maximal line λ such that

$$|\lambda \cap \mathcal{X}_\ell| \geq 2. \tag{5.2.3}$$

Indeed, we have at least three maximal lines, denoted by $\lambda_1, \lambda_2, \lambda_3$ for the node set $\mathcal{X}$. In view of Lemmas 4.3.3 and 4.3.4 we may suppose that they intersect the line ℓ at three distinct nodes. Now consider the GC_{n-1} set $\mathcal{X}_1 := \mathcal{X} \setminus \lambda_1$. Here, the maximal lines λ_2, λ_3, intersect ℓ at two distinct nodes. Therefore, in view of Corollary 4.3.8 and the induction hypothesis, for one of them, denoted by λ, we have $|\lambda \cap (\mathcal{X}_1)_\ell| = (n-1)-1$, or $(n-1)-2$. Since $n \geq 5$ hence the inequality (5.2.3) holds.

Now notice that, in view of (4.3.3), we have that

$$|\mathcal{X}_\ell| = |(\mathcal{X} \setminus \lambda)_\ell| + |\lambda \cap \mathcal{X}_\ell|.$$

Hence, by making use of (5.2.3) and the induction hypothesis applied to the GC_{n-1} set $\mathcal{X} \setminus \lambda$, we obtain that

$$|\mathcal{X}_\ell| \geq |(\mathcal{X} \setminus \lambda)_\ell| + 2 \geq \binom{n-2}{2} + 2. \tag{5.2.4}$$

Therefore, in view of the condition (5.2.2) and Proposition 4.3.9, (iii), we conclude that

$$|\mathcal{X}_\ell| = \binom{n-1}{2} \text{ and } \mathcal{N}_\ell \subset \beta \in \Pi_2,\ |\mathcal{N}_\ell| = 2n.$$

Let us use the induction hypothesis. By taking into account the first equality above and (5.2.3), we deduce that

$$|(\mathcal{X} \setminus \lambda)_\ell| = \binom{n-2}{2}.$$

Then we get that $2(n-1)$ nodes in $\mathcal{N}_\ell \cap (\mathcal{X} \setminus \lambda)$ are located in two maximal lines denoted by λ' and λ'', which intersect at a node $A \in \ell$. Since $n \geq 5$ each of these two maximal lines passes through 4 nodes of $\mathcal{N}_\ell \subset \beta$. Thus each of them divides β and we get $\beta = \lambda'\lambda''$. Finally, according to Proposition 4.3.9, (iii), each of these lines passes through exactly n nodes of $\mathcal{X} \setminus \ell$. Therefore, since $A \in \lambda' \cap \lambda''$, we get that each of these lines is maximal also for the set $\mathcal{X}$.

Hence the direct implication in the assertion (ii) holds. □

At the end let us present

Corollary 5.2.4 ([25]). *Assume that Conjecture 4.1.7 holds for all degrees up to n. Let $\mathcal{X}$ be a GC_n set with exactly three maximal lines, where $n \geq 4$. Then there are exactly three n-node lines in $\mathcal{X}$ and each of them is used by exactly $\binom{n}{2}$ nodes from $\mathcal{X}$.*

Proof. Suppose that λ_1, λ_2 and λ_3 are the three maximal lines of $\mathcal{X}$ and ℓ is any n-node line. Let us call the intersection nodes

$$A := \lambda_1 \cap \lambda_2,\ B := \lambda_2 \cap \lambda_3,\ C := \lambda_3 \cap \lambda_1,$$

vertices. Let us prove first that the case (i) of Theorem 5.1.1 takes place here, in particular we have that $|\mathcal{X}_\ell| = \binom{n}{2}$. Assume by way of contradiction that the case (ii) holds, i.e., ℓ passes through a vertex, say A. Observe first that ℓ intersects also the maximal line λ_3 at a node. Indeed, otherwise in the GC_{n-1} set $\mathcal{X}_3 := \mathcal{X} \setminus \lambda_3$ the maximal lines λ_1, λ_2 and ℓ are concurrent at A.

Now let us consider the set $\mathcal{X}_1 := \mathcal{X} \setminus \lambda_1$. We have, in view of Lemma 4.1.9, that there are exactly 3 maximal lines in $\mathcal{X}_1$. Namely, λ_2, λ_3, and a third maximal line denoted by λ_1'. Of course λ_1' intersects λ_2 and λ_3 at nodes different from vertices. Also λ_1' does not intersect λ_1 at a node. Indeed, otherwise, λ_1' would be a fourth maximal line in the node set $\mathcal{X}$.

In a similar way the third maximal lines λ_2' and λ_3' in the sets $\mathcal{X}_2 := \mathcal{X} \setminus \lambda_2$ and $\mathcal{X}_3$ do not intersect the maximal lines λ_2 and λ_3 at nodes, respectively. Also λ_2' intersects λ_1 and λ_3 at nodes different from the vertices and λ_3' intersects λ_1 and λ_2 at nodes different from the vertices. From these intersection properties we conclude that the lines $\lambda_1', \lambda_2', \lambda_3'$ are distinct n-node lines in $\mathcal{X}$. We have also that the lines $\lambda_1', \lambda_2', \lambda_3'$ are different from ℓ, since only ℓ from these lines passes through a vertex.

Next consider the GC_{n-3} set $\mathcal{X} \setminus (\lambda_1 \cup \lambda_2 \cup \lambda_3)$. Observe that here we have four maximal lines: $\lambda_1', \lambda_2', \lambda_3'$, and ℓ, which contradicts Lemma 4.1.9.

Consequently the case (i) of Theorem 5.1.1 takes place. Hence, for a maximal line $\lambda_i,\ 1 \leq i \leq 3$, we have that $\lambda_i \cap \ell \cap \mathcal{X} = \emptyset$. Therefore ℓ is a maximal line in the node set $\mathcal{X}_i = \mathcal{X} \setminus \lambda_i$

and clearly it coincides with the n-node line λ_i' there. Thus the lines $\lambda_1', \lambda_2', \lambda_3'$, are the only n-node lines in $\mathcal{X}$. $\square$

Remark 5.2.5. It is worth mentioning that Corollary 5.2.4 is not valid in the case $n = 3$. Indeed, the GC_3 set $\mathcal{X}^\star$ and the 3-node lines ℓ_1, ℓ_2, ℓ_3 and $\ell^\star$ (see Fig. 5.1.1) give us a counterexample for this.

Chapter 6

ON THE USAGE OF LINES IN GC SETS

It was proven in previous Chapter (and in [1] and [25]) that an n-node line in a GC_n set $\mathcal{X}$, where $n \geq 4$, is used either by exactly $\binom{n}{2}$ or by $\binom{n-1}{2}$ nodes. Also a conjecture was proposed in the paper [1] by V. Bayramyan and H. Hakopian, concerning the usage of any k-node line in GC_n sets, $2 \leq k \leq n+1$. In this Chapter we make an adjustment in the mentioned conjecture and then prove it. Namely, by assuming that the Gasca-Maeztu conjecture is true, we prove that for any GC_n set $\mathcal{X}$ and any k-node line ℓ the following statement holds:
The line ℓ is not used at all, or it is used by exactly $\binom{s}{2}$ nodes of $\mathcal{X}$, where s satisfies the condition $k - \delta \leq s \leq k,\ \delta = n + 1 - k$. If in addition $k - \delta \geq 3$ and $\mu(\mathcal{X}) > 3$ then the first case here is excluded, i.e., the line ℓ is necessarily a used line. Here $\mu(\mathcal{X})$ denotes the number of maximal lines of $\mathcal{X}$. We also prove that the subset of nodes of $\mathcal{X}$ that use the line ℓ forms a GC_{s-2} set if it is not an empty set. Moreover, we prove that actually it is an ℓ-proper subset of $\mathcal{X}$, meaning that it can be obtained from $\mathcal{X}$ by removing the nodes in subsequent maximal lines, which do not intersect the line ℓ at a node of $\mathcal{X}$ or the nodes in pairs of maximal lines intersecting ℓ at the same node of $\mathcal{X}$. At the last step, when the line ℓ becomes maximal, the nodes in ℓ are removed (see the forthcoming Definition 4.3.6).

At the end, we bring a characterization for the usage of k-node lines in GC_n sets when $k - \delta = 2$ and $\mu(\mathcal{X}) > 3$.

Let us mention that earlier Carnicer and Gasca proved that a k-node line ℓ can be used by at most $\binom{k}{2}$ nodes of a GC_n set $\mathcal{X}$ and in addition there are no k collinear nodes that use ℓ, provided that GM conjecture is true (see [8], Theorem 4.5).

6.1 Classification of GC_n sets - II

In this section, we present the characterization of GC_n sets according to the number of maximal lines, in the cases $\mu(\mathcal{X}) = n-1$ and $\mu(\mathcal{X}) = 3$.

1. Lattices with $n-1$ maximal lines.

Let a set $\mathcal{M} = \{\lambda_1, \ldots, \lambda_{n-1}\}$ of $n-1$ lines be in general position, $n \geq 4$. Then consider the lattice $\mathcal{X}$ defined as

$$\mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)} \cup \mathcal{X}^{(0)}, \tag{6.1.1}$$

where $\mathcal{X}^{(2)}$ is the set of all intersection nodes of these $n-1$ lines, $\mathcal{X}^{(1)}$ is a set of other $3(n-1)$ nodes, three in each line, to make the line maximal and $\mathcal{X}^{(0)}$ consists of exactly three nodes, denoted by O_1, O_2, O_3, which do not belong to any line from $\mathcal{M}$. Correspondingly, we have that $|\mathcal{X}| = \binom{n-1}{2} + 3(n-1) + 3 = \binom{n+2}{2}$. Note that all the nodes of $\mathcal{X}^{(k)}$ belong to exactly k maximal lines and are called k_m-nodes, $k = 0, 1, 2$.

Clearly, the three 0_m-nodes O_1, O_2, O_3 are non-collinear. Indeed, otherwise the set $\mathcal{X}$ is lying in n lines, which are the $n-1$ lines of $\mathcal{M}$ and the line passing through the three nodes. This, in view of Proposition 1.2.3, contradicts the n-poisedness of $\mathcal{X}$.

Denote by ℓ_i^{oo}, $1 \leq i \leq 3$, the line passing through the two 0_m-nodes in $\{O_1, O_2, O_3\} \setminus \{O_i\}$. We call this lines OO lines. Suppose that $\mathcal{X}^{(1)} = \{A_i^1, A_i^2, A_i^3 \in \lambda_i : \ 1 \leq i \leq n-1\}$ (see Fig. 6.1.1).

In the sequel we will need the following characterization of GC_n set $\mathcal{X}$, with $\mu(\mathcal{X}) = n-1$, due to Carnicer and Godés, see Fig. 6.2.1, 6.2.4 (see [13], Section 5, Case d=3, for a proof detail):

Proposition 6.1.1 ([11], Thm. 3.2). *A set $\mathcal{X}$ is a GC_n set with exactly $n-1$ maximal lines $\lambda_1, \ldots, \lambda_{n-1}$, where $n \geq 4$, if and only if, with some permutation of the indexes of the maximal lines and 1_m-nodes, the representation* (6.1.1) *holds with the following additional properties:*

(i) $\mathcal{X}^{(1)} \setminus \{A_1^1, A_2^2, A_3^3\} \subset \ell_1^{oo} \cup \ell_2^{oo} \cup \ell_3^{oo}$;

(ii) *Each line* $\ell_i^{oo}, i = 1, 2, 3,$ *passes through exactly* $(n-2)$ 1_m*-nodes (and through two* 0_m*-nodes). Moreover,* $\ell_i^{oo} \cap \lambda_i \notin \mathcal{X},\ i = 1, 2, 3;$

(iii) *The triples* $\{O_1, A_2^2, A_3^3\},\ \{O_2, A_1^1, A_3^3\},\ \{O_3, A_1^1, A_2^2\}$ *are collinear.*

2. Lattices with 3 *maximal lines - generalized principal lattices.*

A principal lattice is defined as an affine image of the set (see Fig. 6.1.2)

$$PL_n := \left\{(i,j) \in \mathbb{N}_0^2 : \quad i + j \leq n\right\}.$$

Let us set $I = \{0, 1, \ldots, n+1\}$. Observe that the following 3 set of $n+1$ lines, namely $\{x = i :\ i \in I\}$, $\{y = j :\ j \in I\}$, and $\{x + y = k :\ k \in I\}$, intersect at PL_n. We have that PL_n is a GC_n set. Moreover, the following is the fundamental polynomial of the node $(i_0, j_0) \in PL_n$:

$$p^{\star}_{i_0 j_0}(x,y) = \prod_{0 \leq i < i_0,\ 0 \leq j < j_0,\ 0 \leq k < k_0} (x-i)(y-j)(x+y-n+k), \tag{6.1.2}$$

where $k_0 = n - i_0 - j_0$.

Next let us bring the definition of the generalized principal lattice due to Carnicer, Gasca and Godés (see [9], [10]):

Definition 6.1.2 ([10])**.** A node set $\mathcal{X}$ is called a generalized principal lattice, briefly GPL_n, if there are 3 sets of lines each containing $n+1$ lines

$$\ell_i^j(\mathcal{X})_{i \in \{0,1,\ldots,n\}},\ j = 0, 1, 2, \tag{6.1.3}$$

such that the $3n+3$ lines are distinct,

$$\ell_i^0(\mathcal{X}) \cap \ell_j^1(\mathcal{X}) \cap \ell_k^2(\mathcal{X}) \cap \mathcal{X} \neq \emptyset \iff i + j + k = n$$

and

$$\mathcal{X} = \left\{x_{ijk} \mid x_{ijk} := \ell_i^0(\mathcal{X}) \cap \ell_j^1(\mathcal{X}) \cap \ell_k^2(\mathcal{X}), 0 \leq i, j, k \leq n, i + j + k = n\right\}.$$

Observe that if $0 \leq i, j, k \leq n,\ i+j+k = n$ then the three lines $\ell_i^0(\mathcal{X}), \ell_j^1(\mathcal{X}), \ell_k^2(\mathcal{X})$ intersect at a node $x_{ijk} \in \mathcal{X}$. This implies that each node of $\mathcal{X}$ belongs to only one line of each of the three sets of $n+1$ lines. Therefore $|\mathcal{X}| = (n+1)(n+2)/2$.

One can find readily, as in the case of PL_n, the fundamental polynomial of each node $x_{ijk} \in \mathcal{X},\ i+j+k = n$:

$$p^\star_{i_0 j_0 k_0} = \prod_{0 \leq i < i_0,\ 0 \leq j < j_0,\ 0 \leq k < k_0} \ell_i^0(\mathcal{X}) \ell_j^1(\mathcal{X}) \ell_k^2(\mathcal{X}). \tag{6.1.4}$$

Thus $\mathcal{X}$ is a GC_n set.

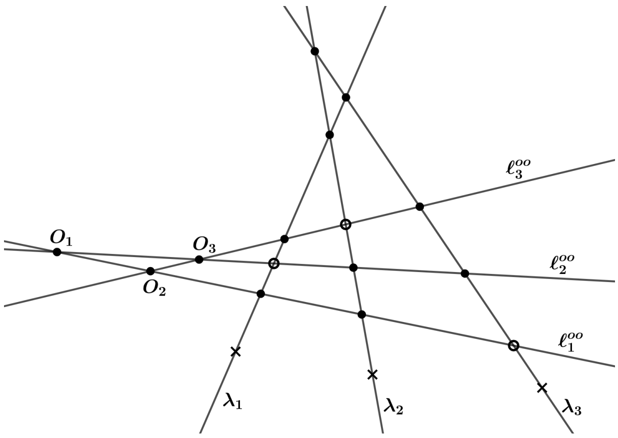

Figure 6.1.1: A lattice with $(n-1)maximals$

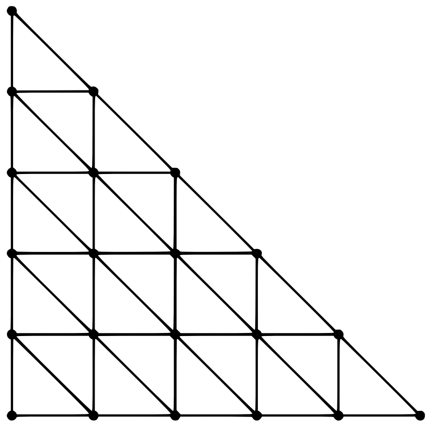

Figure 6.1.2: A principal lattice PL_5.

Let us bring a characterization for GPL_n set due to Carnicer and Godés:

Theorem 6.1.3 ([10], Thm. 3.6). *Assume that GM Conjecture holds for all degrees up to $n-3$. Then the following statements are equivalent:*

(i) $\mathcal{X}$ *is generalized principal lattice of degree n;*

(ii) $\mathcal{X}$ *is a GC_n set with $\mu(\mathcal{X}) = 3$.*

6.2 The main result

In this Section we formulate an adjusted version of the conjecture proposed by V. Bayramyan and H. Hakopian in ([1], Conj. 3.7) as:

Theorem 6.2.1 ([26], [31]). *Let $\mathcal{X}$ be a GC_n set, and ℓ be a k-node line, $k \geq 2$. Assume that GM Conjecture holds for all degrees up to n. Then we have that*

$$\mathcal{X}_\ell = \emptyset, \ \ or \tag{6.2.1}$$

$$\mathcal{X}_\ell \ is \ an \ \ell\text{-}proper \ GC_{s-2} \ subset \ of \ \mathcal{X}, \ hence \ |\mathcal{X}_\ell| = \binom{s}{2}, \tag{6.2.2}$$

for some $k - \delta \leq s \leq k$ and $\delta = n + 1 - k$.
Moreover, if $k - \delta \geq 3$ and $\mu(\mathcal{X}) > 3$ then $\mathcal{X}_\ell \neq \emptyset$, i.e., (6.2.2) holds with $s \geq 2$. Furthermore, in the case $\mathcal{X}_\ell \neq \emptyset$ we have for any maximal line λ :
$|\lambda \cap \mathcal{X}_\ell| = 0$ or $|\lambda \cap \mathcal{X}_\ell| = s - 1$.

Let us mention that (6.2.1) was missed in the original conjecture in [1] and the possibility that the set $\mathcal{X}_\ell$ may be empty was associated only with the case $k - \delta \leq 1$, i.e., with the possibility of the equality $|\mathcal{X}_\ell| = \binom{1}{2} = 0$ in (6.2.2). Thus we assume that a GC_{s-2} subset with $s < 2$ is empty set.

Note that we added here the statement that $\mathcal{X}_\ell$ is an ℓ-proper GC subset.

In the last subsection we characterize constructions of GC_n sets for which there is a non-used k-node line with $k - \delta = 2$ and $\mu(\mathcal{X}) > 3$.

6.2.1 Some preliminaries for the proof of the main theorem

First, let us mention that, in view of the relation (4.3.2), Theorem 6.2.1 is true if the line ℓ is a maximal line ($\delta = 0$).

Theorem 6.2.1 is also true in the case when GC_n set $\mathcal{X}$ is a Chung -Yao lattice. Indeed, in this lattice the only used lines are the maximal lines. Next, for any k-node line ℓ with $k \leq n$ we have that $2k \leq \mu(\mathcal{X}) = n + 2$, since through any node there pass two maximal lines. Thus for ℓ we have $k - \delta \leq 1$ (see [1]) and the fact $\mathcal{X}_\ell = \emptyset$ is in accordance with Theorem 6.2.1.

Next proposition reveals a rich structure of the Carnicer-Gasca lattice.

Proposition 6.2.2 ([1], Prop. 3.8). *Let $\mathcal{X}$ be a Carnicer-Gasca lattice of degree n and ℓ be a*

k-node line, $k \geq 2$. Then we have that

$$\mathcal{X}_\ell \text{ is an } \ell\text{-proper } GC_{s-2} \text{ subset of } \mathcal{X}, \text{ hence } |\mathcal{X}_\ell| = \binom{s}{2}, \tag{6.2.3}$$

where $k - \delta \leq s \leq k$ and $\delta = n + 1 - k$.

Moreover, in the case $\mathcal{X}_\ell \neq \emptyset$ we have for any maximal line λ :

$|\lambda \cap \mathcal{X}_\ell| = 0$ or $|\lambda \cap \mathcal{X}_\ell| = s - 1$.

Furthermore, for each $n, k,$ and s, with $k - \delta \leq s \leq k$ there is a Carnicer-Gasca lattice $\mathcal{X}$ of degree n and a k-node line ℓ such that (6.2.3) *is satisfied.*

Note that the phrase "ℓ-proper" is not present in the formulation of Proposition in [1] but it follows readily from the proof there.

The following result is due to Carnicer and Gasca (see also [1], eq. (1.4)).

Proposition 6.2.3 ([8], Prop. 4.2)**.** *Let $\mathcal{X}$ be a GC_n set and ℓ be a 2-node line, then $|\mathcal{X}_\ell| = 1$ or 0.*

Let us complement this with the following

Lemma 6.2.4. *Let $\mathcal{X}$ be a GC_n set, ℓ be a 2-node line, and $|\mathcal{X}_\ell| = 1$. Assume that GM Conjecture holds for all degrees up to n. Then $\mathcal{X}_\ell$ is an ℓ-proper GC_0 subset.*

Proof. Indeed, suppose that $\mathcal{X}_\ell = \{A\}$ and ℓ passes through the nodes $B, C \in \mathcal{X}$. The node A uses a maximal $(n+1)$-node line in $\mathcal{X}$ which we denote by λ_0. Next, A uses a maximal n-node line in $\mathcal{X} \setminus \lambda_0$ which we denote by λ_1. Continuing this way we find consecutively the lines $\lambda_2, \lambda_3, \ldots, \lambda_{n-1}$ and obtain that

$$\{A\} = \mathcal{X} \setminus (\lambda_0 \cup \lambda_1 \cup \cdots \cup \lambda_{n-1}).$$

To finish the proof it suffices to show that $\lambda_{n-1} = \ell$ and the remaining lines $\lambda_i, i = 0, \ldots, n-2$ are ℓ-disjoint. Indeed, the node A uses ℓ and since it is a 2-node line it may coincide only with the last maximal line λ_{n-1}. Now, suppose conversely that a maximal line $\lambda_k,\ 0 \leq k \leq n-2,$

intersects ℓ at a node, say B. Then consider the polynomial of degree n :

$$p = \ell_{A,C} \prod_{i\in\{0,\dots,n-1\}\setminus\{k\}} \lambda_i,$$

where $\ell_{A,C}$ is the line through A and C. Clearly p passes through all the nodes of $\mathcal{X}$ which contradicts Proposition 1.2.3. $\square$

Now, in view of Proposition 6.2.3 and Lemma 6.2.4, we conclude that Theorem 6.2.1 is true for the case of 2-node lines in any GC_n sets.

Now let us show that Theorem 6.2.1 is true for the node set $\mathcal{X}$ if $\mu(\mathcal{X}) = 3$.

Proposition 6.2.5 ([26], [31]). *Let $\mathcal{X}$ be a GC_n set with $\mu(\mathcal{X}) = 3$ and ℓ be a k-node line, $k \geq 2$. Assume that GM Conjecture holds for all degrees up to $n-3$. Then we have that*

$$\mathcal{X}_\ell = \emptyset, \text{ or } \mathcal{X}_\ell \text{ is an } \ell\text{-proper } GC_{k-2} \text{ subset of } \mathcal{X}, \text{ hence } |\mathcal{X}_\ell| = \binom{k}{2}. \tag{6.2.4}$$

Moreover, if $k \leq n$ and $\mathcal{X}_\ell \neq \emptyset$ then for a maximal line λ_1 of $\mathcal{X}$ we have that $\lambda_1 \cap \ell \notin \mathcal{X}$ and $|\lambda_1 \cap \mathcal{X}_\ell| = 0$.

For the remaining two maximal lines we have that $|\lambda \cap \mathcal{X}_\ell| = k-1$.

Furthermore, if the line ℓ intersects each maximal line at a node then $\mathcal{X}_\ell = \emptyset$.

Proof. According to Theorem 6.1.3 the set $\mathcal{X}$ is a generalized principal lattice of degree n with some three sets of $n+1$ lines: $\ell_i^j(\mathcal{X})_{i\in\{0,1,\dots,n\}},\ j = 0,1,2,$ (6.1.3). Then we obtain from (6.1.4) that the only used lines in $\mathcal{X}$ are the lines $\ell_s^r(\mathcal{X})$, where $0 \leq s < n,\ r = 0,1,2$. Therefore the only used k-node lines are the lines $\ell_{n-k+1}^r(\mathcal{X}),\ r = 0,1,2$. Consider the line, say with $r = 0$, i.e., $\ell \equiv \ell_{n-k+1}^0(\mathcal{X})$. It is used by all the nodes $x_{ijl} \in \mathcal{X}$ with $i > n-k+1$, i.e., $i = n-k+2, n-k+3, \dots, n$. Thus, ℓ is used by exactly $\binom{k}{2} = (k-1)+(k-2)+\cdots+1$ nodes. This implies also that $\mathcal{X}_\ell = \mathcal{X} \setminus (\ell_0^0 \cup \ell_1^0 \cup \cdots \cup \ell_{n-k+1}^0)$. Hence $\mathcal{X}_\ell$ is an ℓ-proper GC_{k-2} subset of $\mathcal{X}$. The part "Moreover" also follows readily from here. Now it remains to notice that the part "Furthermore" is a straightforward consequence of the part "Moreover". $\square$

Next statement is on the presence and usage of $(n-1)$-node lines in GC_n sets with $\mu(\mathcal{X}) = n-1$ (cf. Proposition 4.2, [25]).

Proposition 6.2.6 ([26], [31]). *Let $\mathcal{X}$ be a GC_n set with $\mu(\mathcal{X}) = n-1$, and ℓ be an $(n-1)$-node line, where $n \geq 4$. Assume also that through each node of ℓ there passes exactly one maximal line. Then we have that either $n = 4$ or $n = 5$. Moreover, in both these cases we have that $\mathcal{X}_\ell = \emptyset$.*

Proof. Consider a GC_n set with $\mu(\mathcal{X}) = n - 1$. In this case we have the representation (6.1.1), i.e., $\mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)} \cup \mathcal{X}^{(0)}$ satisfying the conditions of Proposition 6.1.1. Here $\mathcal{X}^{(k)}$ is the set of all k_m-nodes, i.e., nodes belonging exactly to k maximal lines. Recall that $\mathcal{X}^{(0)}$ consists of three non-collinear nodes: $\mathcal{X}^{(0)} = \{O_1, O_2, O_3\}$ outside the maximal lines.

Let ℓ be an $(n-1)$-node line. First notice that, according to the hypothesis of Proposition, all the nodes of the line ℓ are 1_m-nodes. Therefore ℓ does not coincide with any OO line, i.e., line passing through two 0_m-nodes.

From Proposition 6.1.1, (i), we have that all the nodes of $\mathcal{X}^{(1)}$, except the three nodes A_1^1, A_2^2, A_3^3, which are called here special nodes, belong to the three OO lines. We have also, in view of Proposition 6.1.1, (iii), that the nodes A_1^1, A_2^2, A_3^3 are not collinear. Therefore there are three possible cases:

(i) ℓ does not pass through any special node,

(ii) ℓ passes through two special nodes,

(iii) ℓ passes through one special node.

In the first case ℓ may pass only through nodes lying in three OO lines. Then it may pass through at most three nodes, i.e., $n \leq 4$. Therefore, in view of the hypothesis $n \geq 4$, we get that $n = 4$ and $\mu(\mathcal{X}) = 3$. Now, in view of Proposition 6.2.5, part “Furthermore", we get that $\mathcal{X}_\ell = \emptyset$.

Next, consider the case when ℓ passes through two special nodes. Then, according to Proposition 6.1.1, (iii), it passes through an 0_m-node. Recall that this case is excluded since ℓ passes through 1_m-nodes only.

Finally, consider the third case when ℓ passes through exactly one special node. Then it may pass through at most three other 1_m-nodes lying in OO lines. Therefore ℓ may pass through at most four nodes.

First suppose that ℓ passes through exactly 3 nodes. Then again we obtain that $n = 4$, $\mu(\mathcal{X}) = 3$ and $\mathcal{X}_\ell = \emptyset$.

Next suppose that ℓ passes through exactly 4 nodes. Then we have that $n = 5$. Without loss of generality we may assume that the special node ℓ passes through is, say, A_1^1. Next let us show first that $|\mathcal{X}_\ell| \leq 1$. Here we have exactly $4 = n - 1$ maximal lines. Consider the maximal line λ_4, for which, in view of Proposition 6.1.1, the intersection with each OO line is a node in $\mathcal{X}$ (see Fig. 6.2.1). Denote $B := \ell \cap \lambda_4$. Assume that the node B belongs to the line $\in \ell_i^{oo}$, $1 \leq i \leq 3$, i.e., the line passing through $\{O_1, O_2, O_3\} \setminus \{O_i\}$ ($i = 2$ in Fig. 6.2.1). According to the condition (ii) of Proposition 6.1.1 we have that $\ell_i^{oo} \cap \lambda_i \notin \mathcal{X}$. Denote $C := \lambda_i \cap \lambda_4$. Now let us prove that $\mathcal{X}_\ell \subset \{C\}$ which implies $|\mathcal{X}_\ell| \leq 1$.

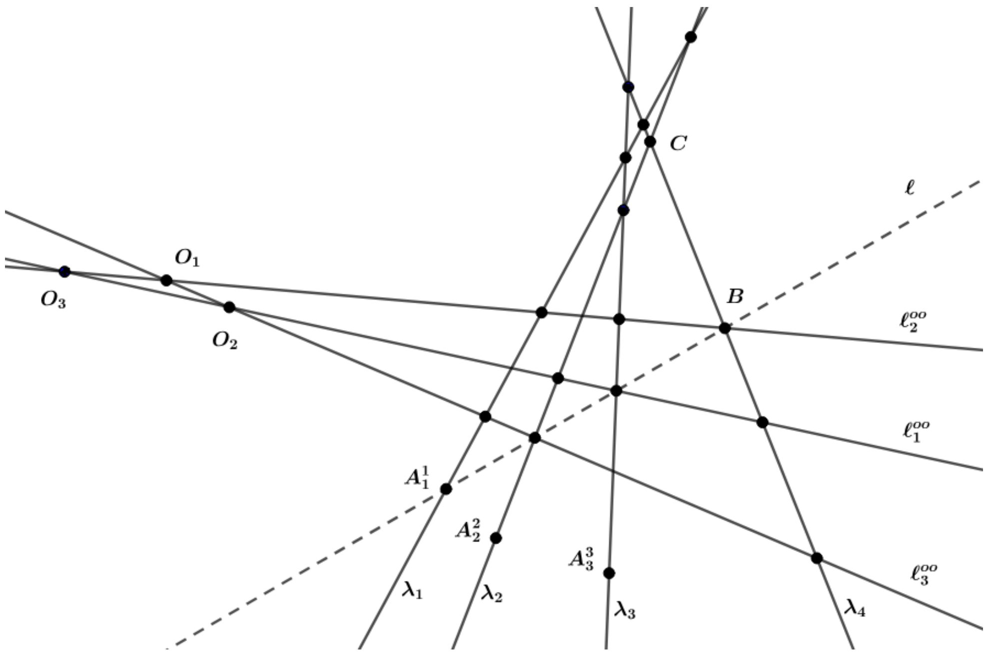

Figure 6.2.1: A GC_n set with $n-1$ maximals for $n = 5$ and a 4-node line

Consider the GC_4 set $\mathcal{X}_i = \mathcal{X} \setminus \lambda_i$. Here we have two maximal lines intersecting at the node $B \in \ell$, i.e., λ_4 and ℓ_i^{oo}. Therefore we conclude from Lemma 4.3.4 that no node from these two maximal lines uses ℓ in $\mathcal{X}_i$. Thus, in view of (4.3.3), no node from λ_4, except possibly C, uses ℓ in $\mathcal{X}$. Now consider the GC_4 set $\mathcal{X}_4 = \mathcal{X} \setminus \lambda_4$. Observe, on the basis of the characterization of Proposition 6.1.1, that $\mathcal{X}_4$ has exactly 3 maximal lines. On the other hand here the line ℓ intersects each maximal line at a node. Therefore, in view of Proposition 6.2.5, part "Furthermore", we have that $(\mathcal{X}_4)_\ell = \emptyset$. Hence, in view of (4.3.3), we conclude that $\mathcal{X}_\ell \subset \{C\}$.

Now, to complete the proof it suffices to show that the node C does not use ℓ. Let us determine the lines the node C uses. Since $C = \lambda_i \cap \lambda_4$ first of all it uses the two maxmal lines in $\{\lambda_1, \lambda_2, \lambda_3\} \setminus \{\lambda_i\}$. It is easily seen that the next two lines C uses are OO lines: $\{\ell_1^{oo}, \ell_2^{oo}, \ell_3^{oo}\} \setminus \{\ell_i^{oo}\}$. Now notice that, the two nodes, except C, which do not belong to the four used lines are B and the special node A_i^i. Hence the fifth line used by C is the line passing through the latter two nodes. Now observe that this line coincides with ℓ if and only if $i = 1$. Note that Fig. 6.2.1 depicts the case $\mathcal{X}_\ell = \emptyset$ with $i \neq 1$ ($n = 2$) and is not valid for the later discussion.

In the final and most interesting part of the proof we will show that in the case $i = 1$, i.e., when the special node ℓ passes is A_1^1 and $B = \ell \cap \lambda_4 \cap \ell_1^{oo}$, the node C can not use the line ℓ.

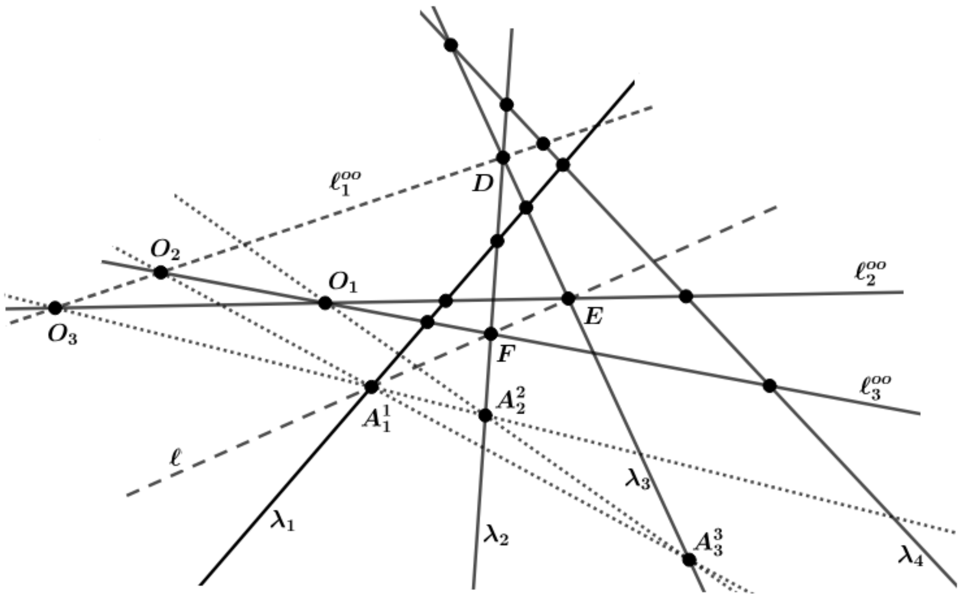

Figure 6.2.2: The case $i = 1$.

More precisely, we will do the following. By assuming that (see Fig. 6.2.2)

(i) the maximal lines $\lambda_i,\ i = 1, \ldots, 4$, and the three O_1, O_2, O_3 are given, $D := \lambda_2 \cap \lambda_3$, hence $D \in \mathcal{X}^{(2)}$,

(ii) the two OO lines ℓ_2^{oo}, ℓ_3^{oo} do not pass through the nodes of $\mathcal{X}^{(2)}$,
$E := \ell \cap \lambda_3 \cap \ell_2^{oo}, \quad F := \ell \cap \lambda_2 \cap \ell_3^{oo}$, and

(iii) the conditions in Proposition 6.1.1, (iii), are satisfied, i.e., the line through the two special nodes in $\{A_1^1, A_2^2, A_3^3\} \setminus \{A_i^i\}$ passes through the node O_i for each $i = 1, 2, 3$, we will prove that the third OO line ℓ_1^{oo} passes necessarily through the node $D \in \mathcal{X}^{(2)}$, which contradicts Proposition 6.1.1, (ii).

To this end, we simplify Fig. 6.2.2 by deleting from it the maximal lines λ_1 and λ_4 to obtain the following Fig. 6.2.3. Let us now apply the well-known Pappus hexagon theorem for the

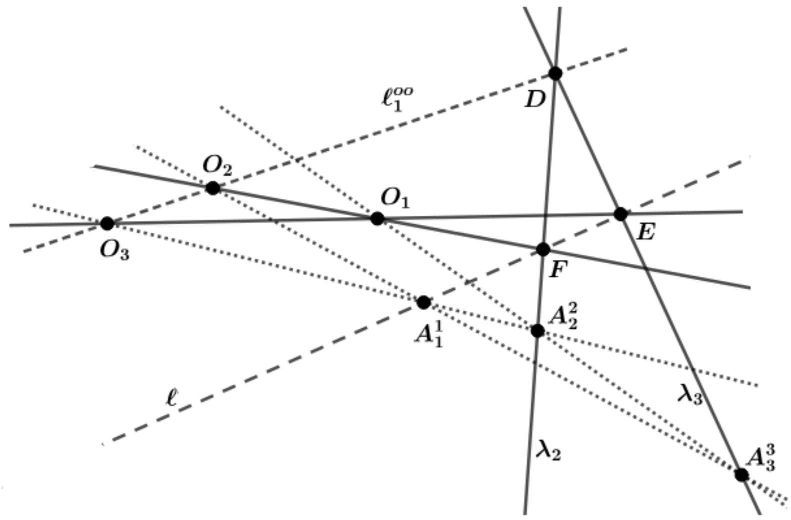

Figure 6.2.3: The set $\mathcal{X}$ without the maximal lines λ_1 and λ_4.

pair of triple collinear nodes here

$$A_1^1, \quad E, \quad F;$$

$$O_1, \quad A_2^2, \quad A_3^3.$$

Now observe that

$$\ell(A_1^1, A_2^2) \cap \ell(E, O_1) = O_3, \;\; \ell(E, A_3^3) \cap \ell(F, A_2^2) = D, \;\; \ell(A_1^1, A_3^3) \cap \ell(F, O_1) = O_2,$$

where $\ell(A, B)$ denotes the line passing through the points A and B. Thus, according to the Pappus theorem we get that the triple of nodes D, O_2, O_3 is collinear, leading to contradiction.

$\square$

Remark 6.2.7. Let us show that the case of non-used 4-node line in Figure 6.2.1 is possible nevertheless. The problem with this is that we have to confirm that the three conditions in Proposition 6.1.1, are satisfied. More precicely:

(i) $\ell_i^{oo} \cap \lambda_i = \emptyset$ for each $i = 1, 2, 3$;

(ii) The line through the two special nodes in $\{A_1^1, A_2^2, A_3^3\} \setminus \{A_i^i\}$ passes through the outside node O_i for each $i = 1, 2, 3$.

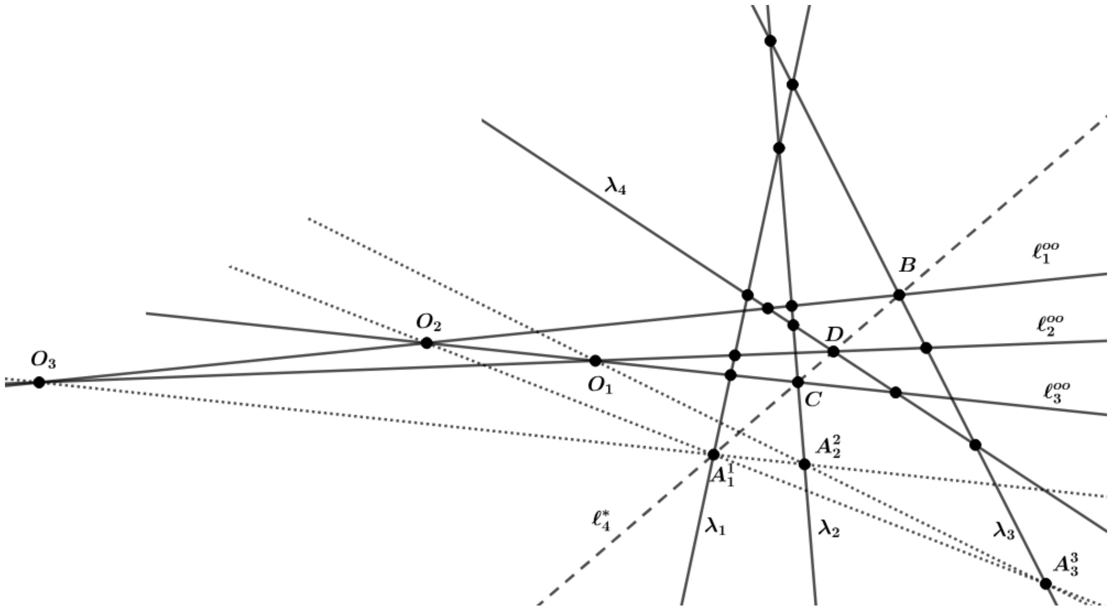

Figure 6.2.4: A non-used 4-node line ℓ_4^* in a GC_5 set $\mathcal{X}^*$

Let us outline how one can get a desired figure (see Fig. 6.2.4). Let us start the figure with the three maximal lines (non-concurrent) $\lambda_1, \lambda_2, \lambda_3$. Then we choose two OO lines ℓ_1^{oo}, ℓ_3^{oo} through the outside node O_2, which intersect the three maximal lines at 6 distinct points. Next we get the line ℓ_4^* which passes through the points B and C, where $B = \ell_1^{oo} \cap \lambda_3$ and $C = \ell_3^{oo} \cap \lambda_2$. Now we find the special node $A_1^1 := \ell_4^* \cap \lambda_1$. By intersecting the line through O_2 and A_1^1 with the maximal line λ_3 we get the special node A_3^3. Then we choose a node on the maximal line λ_2 as A_2^2. This enables to determine the two remaining 0_m-nodes: O_1, O_3. Namely we have that O_1 is the intersection point of the line through A_2^2 and A_3^3 with the line ℓ_3^{oo} and O_3 is the intersection point of the line through A_2^2 and A_1^1 with the line ℓ_1^{oo}. Thus we get the line ℓ_2^{oo} which passes through the nodes O_1, O_3. Next we get the points of intersection of the line ℓ_2^{oo} with the three maximal lines as well as the point of intersection with the line ℓ_4^* denoted by D. Now, we choose the maximal line λ_4 passing through D and intersecting the remaining maximal lines and the three OO lines at 6 new distinct nodes. Finally, all the specified intersection points in Fig. 6.2.4 we declare as the nodes of $\mathcal{X}^*$.

6.3 Proof of the main theorem

Let us start the proof with a list of the major cases in which Theorem 6.2.1 is true.

Step 1. Theorem 6.2.1 is true in the following cases:

(i) The line ℓ is a maximal line.

Indeed, as we have mentioned already, in this case we have $\mathcal{X}_\ell = \mathcal{X} \setminus \ell$ and all the conclusions of Theorem can be readily verified.

(ii) The line ℓ is an n-node line, $n \in \mathbb{N}$.

In this case Theorem 6.2.1 is valid by virtue of Theorem 5.1.1 (for $n \in \mathbb{N} \setminus \{3\}$) and Proposition 5.1.3 (for $n = 3$).

(iii) The line ℓ is a 2-node line.

In this case Theorem 6.2.1 follows from Proposition 6.2.3 and Lemma 6.2.4.

Now, let us prove Theorem by complete induction on n - the degree of the node set $\mathcal{X}$. Obviously Theorem is true in the cases $n = 1, 2$. Note that this follows also from Step 1 (i) and (ii).

Assume that Theorem is true for any node set of degree not exceeding $n-1$. Then let us prove that it is true for the node set $\mathcal{X}$ of degree n. Suppose that we have a k-node line ℓ.

Step 2: Suppose additionally that there is an ℓ-disjoint maximal line λ. Then we get from Lemma 4.3.3 that

$$\mathcal{X}_\ell = (\mathcal{X} \setminus \lambda)_\ell. \tag{6.3.1}$$

Therefore by using the induction hypothesis for the GC_{n-1} set $\mathcal{X}' := \mathcal{X} \setminus \lambda$ we get the relation (6.2.2), i.e., $k - \delta' \le s \le k$ and $\delta' = \delta(\mathcal{X}', \ell) = (n-1) + 1 - k = n - k = \delta - 1$. Thus we get $k - \delta + 1 \le s \le k$. Next we use Proposition 4.3.7 in checking that $\mathcal{X}_\ell$ is an ℓ-proper subset of $\mathcal{X}$.

Now let us verify the part "Moreover". Suppose that $k - \delta = 2k - n - 1 \ge 3$, i.e. $2k \ge n+4$, and $\mu(\mathcal{X}) > 3$. For the line ℓ in the GC_{n-1} set $\mathcal{X}'$ we have $k - \delta' = k - \delta + 1 \ge 4$. Thus if $\mu(\mathcal{X}') > 3$ then, by the induction hypothesis, we have that $(\mathcal{X}')_\ell \neq \emptyset$. Therefore we get, in view of (6.3.1), that $\mathcal{X}_\ell \neq \emptyset$. It remains to consider the case $\mu(\mathcal{X}') = 3$. In this case, in view of Proposition 4.1.10, we have that $\mu(\mathcal{X}) = 4$, which, in view of Theorem 4.2.1, implies that $4 \in \{n-1, n, n+1, n+2\}$, i.e., $2 \le n \le 5$.

The case $n = 2$ was verified already. Now, since $2k \ge n+4$ we deduce that either $k \ge 4$ if $n = 3, 4$, or $k \ge 5$ if $n = 5$. These cases follow from Step 1 (i) or (ii). The part "Furthermore" follows readily from the relation (6.3.1).

Step 3: Suppose additionally that there is a pair of ℓ-adjacent maximal lines λ', λ''. Then we get from Lemma 4.3.4 that

$$\mathcal{X}_\ell = (\mathcal{X} \setminus (\lambda' \cup \lambda''))_\ell. \tag{6.3.2}$$

Therefore by using the induction hypothesis for the GC_{n-2} set $\mathcal{X}'' := \mathcal{X} \setminus (\lambda' \cup \lambda''))$ we get the relation (6.2.2), i.e., $k-1-\delta'' \le s \le k-1$ and $\delta'' = \delta(\mathcal{X}'', \ell) = (n-2)+1-(k-1) = n-k = \delta-1$. Thus we get $k - \delta \le s \le k-1$. Next we use Proposition 4.3.7 to check that $\mathcal{X}_\ell$ is an ℓ-proper subset of $\mathcal{X}$.

Now let us verify the part "Moreover". Suppose that $k - \delta = 2k - n - 1 \ge 3$, i.e. $2k \ge n+4$, and $\mu(\mathcal{X}) > 3$. The line ℓ is $(k-1)$-node line in the GC_{n-2} set $\mathcal{X}''$ and we have that $k - 1 - \delta'' = k - \delta \ge 3$. Thus if $\mu(\mathcal{X}'') > 3$ then, by the induction hypothesis, we have that $(\mathcal{X}'')_\ell \ne \emptyset$ and therefore we get, in view of (6.3.2), that $\mathcal{X}_\ell \ne \emptyset$. It remains to consider the case $\mu(\mathcal{X}'') = 3$. Then, in view of Proposition 4.1.10, we have that $\mu(\mathcal{X}) = 4$ or 5, which, in view of Theorem 4.2.1, implies that 4 or $5 \in \{n-1, n, n+1, n+2,\}$ i.e., $2 \le n \le 6$.

The cases $2 \le n \le 5$ were considered in the previous step. Thus suppose that $n = 6$. Then, since $2k \ge n+4$, we deduce that $k \ge 5$. In view of Step 1, (ii), we may suppose that $k = 5$.

Now the set $\mathcal{X}''$ is a GC_4 and the line ℓ is a 4-node line there. Thus, in view of Step 1 (ii) we have that $(\mathcal{X}'')_\ell \ne \emptyset$. Therefore we get, in view of (6.3.2), that $\mathcal{X}_\ell \ne \emptyset$. The part "Furthermore" follows readily from the relation (6.3.2).

Step 4. Now consider any k-node line ℓ in a GC_n set $\mathcal{X}$. In view of Step 1 (iii) we may assume that $k \ge 3$. In view of Theorem 4.2.1 and Propositions 5.1.3 and 6.2.5 we may assume also that $\mu(\mathcal{X}) \ge n-1$ and $n \ge 4$.

Next suppose that $k \le n-2$. Since then $\mu(\mathcal{X}) > k$ we necessarily have either the situation of Step 2 or Step 3.

Thus we may assume that $k \geq n-1$. Then, in view of Step 1 (i) and (ii), it remains to consider the case $k = n-1$, i.e., ℓ is an $(n-1)$-node line. Again if $\mu(\mathcal{X}) \geq n$ then we necessarily have either the situation of Step 2 or Step 3. Therefore we may assume also that $\mu(\mathcal{X}) = n-1$. By the same argument we may assume that each of the $n-1$ nodes of the line ℓ is an intersection node with one of the $n-1$ maximal lines.

Therefore the conditions of Proposition 6.2.6 are satisfied and we arrive to the two cases: $n = 4, k = 3, k - \delta = 1$ or $n = 5, k = 4, k - \delta = 2$. In both cases we have that $\mathcal{X}_\ell = \emptyset$ and $k - \delta \leq 2$. Thus in this case Theorem is true. $\square$

From the above proof we get immediately

Corollary 6.3.1. *Let $\mathcal{X}_0$ be a GC_n set, with $\mu(\mathcal{X}_0) > 3$ and $n \geq 6$. Let also ℓ be a k-node line, $2 \leq k \leq n$. Assume that GM Conjecture holds for all degrees up to n. Then there is either ℓ-disjoint maximal line or a pair of ℓ-adjoint maximal lines.*

Indeed, assume conversely that the conclusion here is not true. Then in the above proof for the set $\mathcal{X}_0$ and the k-node line ℓ we arrive necessarily to Step 4. Next, in Step 4, as above, we conclude that $n = 4$, or $n = 5$, which contradict the condition $n \geq 6$.

6.4 The characterization of the case $k - \delta = 2,\ \mu(\mathcal{X}) > 3$

Here, for each n and k, with $k - \delta = 2k - n - 1 = 2$, we bring two constructions of GC_n sets and a non-used k-node line in each case. At the end (see forthcoming Proposition 6.4.1) we prove that these are the only constructions with the mentioned property.

Let us start with a counterexample in the case $n = k = 3$ (see [25], Section 3.1). Consider a GC_3 set $\mathcal{Y}^*$ of 10 nodes with exactly three maximal lines: $\lambda_1, \lambda_2, \lambda_3$ (see Fig. 6.4.1). This set is of the form (4.2.1) and satisfies the conditions listed in Proposition 4.2.2. Now observe that the 3-node line ℓ_3^* here intersects all the three maximal lines at nodes. Therefore, in view of Proposition 5.1.3, (iii), the line ℓ_3^* is non-used, i.e., $(\mathcal{Y}^*)_{\ell_3^*} = \emptyset$.

Let us outline how one can get Fig. 6.4.1. We start the mentioned figure with the three lines $\ell_1^o, \ell_2^o, \ell_3^o$ through O, i.e., the 0_m-node. Then we choose the maximal lines λ_1, λ_2, intersecting ℓ_1^o, ℓ_2^o at 4 distinct points. Let $A_i := \lambda_i \cap \ell_i^o,\ i = 1, 2$. We choose the points A_1 and A_2 such that

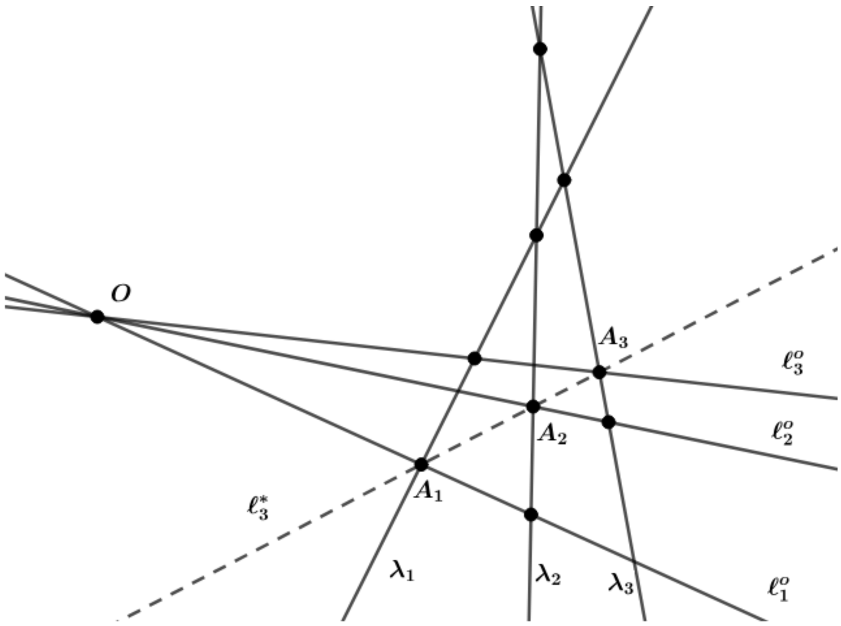

Figure 6.4.1: A GC_n set $\mathcal{Y}^*$ with n maximals for $n = 3$ and a 3-node line ℓ_3^*

the line through them: ℓ_3^* intersects the line ℓ_3^o at a point A_3. Next we choose a third maximal line λ_3 passing through A_3. Let us mention that we choose the maximal lines such that they are not concurrent and intersect the three lines through O at nine distinct points. Finally, all the specified intersection points in Fig. 6.4.1 we declare as the nodes of $\mathcal{Y}^*$.

In the general case of $k - \delta = 2k - n - 1 = 2$ we set $k = m + 3$ and obtain $n = 2m + 3$, where $m = 0, 1, 2, \ldots$. Let us describe how the previous GC_3 node set $\mathcal{Y}^*$ together with the 3-node line ℓ_3^* can be modified to GC_n node set $\bar{\mathcal{X}}^*$ with a k-node line $\bar{\ell}_3^*$ such that $(\bar{\mathcal{Y}}^*)_{\bar{\ell}_3} = \emptyset$.

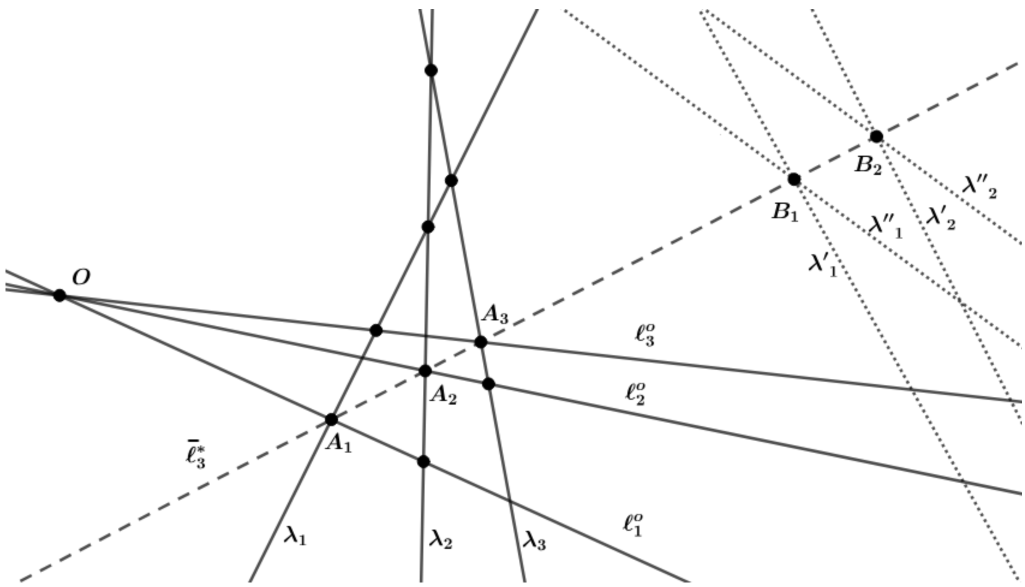

Figure 6.4.2: A non-used k-node line $\bar{\ell}_3^*$ in a GC_n set $\bar{\mathcal{Y}}^*$ with $k - \delta = 2$ $(m = 2)$

For this end we just leave the line ℓ_3^* unchanged, i.e., $\bar{\ell}_3^* \equiv \ell_3^*$ and extend the set $\mathcal{Y}^*$ to a GC_n set $\bar{\mathcal{Y}}^*$ in the following way (see Fig. 6.4.2). We fix m points: $B_i, i = 1, \ldots, m$, in ℓ_3^* different from A_1, A_2, A_3 ($m = 2$ in Fig. 6.4.2). Then we add m pairs of (maximal) lines $\lambda_i', \lambda_i'', i = 1, \ldots, m$, intersecting at these m points, respectively: $\lambda_i' \cap \lambda_i'' = B_i, \ i = 1, \ldots, m$.

We assume that the following condition is satisfied:

(i) The $2m$ lines $\lambda_i', \lambda_i'', i = 1, \ldots, m$, together with $\lambda_1, \lambda_2, \lambda_3$ are in general position, i.e., no two lines are parallel and no three lines are concurrent;

(ii) The mentioned $2m + 3$ lines intersect the lines $\ell_1^o, \ell_2^o, \ell_3^o$ at distinct $3(2m + 3)$ points.

Now all the points of the intersections of the $2m + 3$ lines $\lambda_i', \lambda_i'', i = 1, \ldots, m$, together with $\lambda_1, \lambda_2, \lambda_3$ are declared as the nodes of the set $\bar{\mathcal{Y}}^*$. Next for each of the lines $\lambda_i', \lambda_i'', i = 1, \ldots, m$, also two from the three intersection points with the lines ℓ_1, ℓ_2, ℓ_3, are declared as (1_m-) nodes. After this the lines $\lambda_1, \lambda_2, \lambda_3$ and the lines $\lambda_i', \lambda_i'', i = 1, \ldots, m$, become $(2m+4)$-node lines, i.e., maximal lines.

Now one can verify readily that $\bar{\mathcal{Y}}^*$ is a GC_n set of form (4.2.1) and satisfies the conditions in Proposition 4.2.2 with $n = 2m + 3$ maximal lines: $\lambda_i', \lambda_i'', i = 1, \ldots, m$, together with $\lambda_1, \lambda_2, \lambda_3$. Finally, in view of Lemma 4.3.4 and the relation (4.3.7), applied m times with respect to the pairs $\lambda_i', \lambda_i'', i = 1, \ldots, m$, gives: $(\bar{\mathcal{Y}}^*)_{\bar{\ell}_3^*} = (\mathcal{Y}^*)_{\ell_3^*} = \emptyset$.

Let us call the set $\bar{\mathcal{Y}}^*$ an m-modification of the set $\mathcal{Y}^*$. In the same way we could define $\bar{\mathcal{X}}^*$ as an m-modification of the set $\mathcal{X}^*$ from Fig. 6.2.1, with the 4-node non-used line ℓ_4^* (see Remark 6.2.7). The only differences from the previous case here are:

(i) Now $k = m + 4, \ n = 2m + 5, m = 0, 1, 2, \ldots$ (again $k - \delta = 2$);

(ii) We have $2m + 4$ maximal lines: $\lambda_i', \lambda_i'', i = 1, \ldots, m$, and the lines $\lambda_1, \lambda_2, \lambda_3, \lambda_4$;

(iii) The lines $\ell_1^o, \ell_2^o, \ell_3^o$ are replaced with the lines $\ell_1^{oo}, \ell_2^{oo}, \ell_3^{oo}$;

(iv) For each of the lines $\lambda_i', \lambda_i'', i = 1, \ldots, m$, all three intersection points with the lines $\ell_1^{oo}, \ell_2^{oo}, \ell_3^{oo}$, are declared as ($1_m$-) nodes.

Now one can verify readily that the set $\bar{\mathcal{X}}^*$ is a GC_n set of the form (6.1.1) and satisfies the conditions in Proposition 6.1.1 with $n - 1 = 2m + 4$ maximal lines. In the same way as above we get: $(\bar{\mathcal{X}}^*)_{\bar{\ell}_4^*} = (\mathcal{X}^*)_{\ell_4^*} = \emptyset$. Thus we obtain another construction of GC_n sets, with non-used k-node lines, where $k = m+4, n = 2m+5$ and $k - \delta = 2$.

At the end let us prove the following

Proposition 6.4.1 ([26], [31]). *Let $\mathcal{X}$ be a GC_n set and ℓ be a k-node line with $k - \delta := 2k - n - 1 = 2$ and $\mu(\mathcal{X}) > 3$. Suppose that the line ℓ is a non-used line. Then we have that either $\mathcal{X} = \bar{\mathcal{X}}^*,\ \ell = \bar{\ell}_4^*$, or $\mathcal{X} = \bar{\mathcal{Y}}^*,\ \ell = \bar{\ell}_3^*$.*

Proof. Notice that n is an odd number and $n \geq 3$. In the case $n = 3$ Proposition 6.4.1 follows from Proposition 5.1.3.

Thus suppose that $n \geq 5$. Since $\mu(\mathcal{X}) > 3$, we get, in view of Theorem 4.2.1, that

$$\mu(\mathcal{X}) \geq n - 1. \tag{6.4.1}$$

Now, let us prove that there is no ℓ-disjoint maximal line λ in $\mathcal{X}$.

Suppose conversely that λ is a maximal line with $\lambda \cap \ell \notin \mathcal{X}$. Denote by $\mathcal{X}' := \mathcal{X} \setminus \lambda$. Since $\mathcal{X}_\ell = \emptyset$ therefore, by virtue of the relation (4.3.3), we obtain that $(\mathcal{X}')_\ell = \emptyset$. Then we have that $k - \delta' := k - \delta(\mathcal{X}', \ell) = k - [(n-1) + 1 - k) = 2k - n = 3$. By taking into account the latter two facts, i.e., $(\mathcal{X}')_\ell = \emptyset$ and $k - \delta' = 3$, we conclude from Theorem 6.2.1, part "Moreover", that $\mu(\mathcal{X}') = 3$. Next, by using (6.4.1) and Proposition 4.1.10, we obtain that that $\mu(\mathcal{X}) = 4$. By applying again (6.4.1) we get that $4 \geq n - 1$, i.e., $n \leq 5$. Therefore we arrive to the case: $n = 5$. Since $k - \delta = 2$ we conclude that $k = 4$. Then observe that the line ℓ is 4-node line in the GC_4 set $\mathcal{X}'$. By using Theorem 5.1.1 we get that $(\mathcal{X}')_\ell \neq \emptyset$, which is a contradiction.

Next let us prove Proposition in the case $n = 5$. As we mentioned above then $k = 4$. We have that there is no ℓ-disjoint maximal line. Suppose also that there is no pair of ℓ-adjacent maximal lines. Then, in view of (6.4.1), we readily get that $\mu(\mathcal{X}) = 4$ and through each of the four nodes of the line ℓ there passes a maximal line. Therefore, Proposition 6.1.1 yields that $\mathcal{X}$ coincides with $\mathcal{X}^*$ (or, in other words, $\mathcal{X}$ is a 0-modification of $\mathcal{X}^*$) and ℓ coincides with ℓ_4^*.

Next suppose that there is a pair of ℓ-adjacent maximal lines: λ', λ''. Denote by $\mathcal{X}'' := \mathcal{X} \setminus (\lambda' \cup \lambda'')$. Then we have that ℓ is a 3-node non-used line in $\mathcal{X}''$. Thus we conclude readily that $\mathcal{X}$ coincides with $\bar{\mathcal{Y}}^*$ and ℓ coincides with $\bar{\ell}_3^*$ (with $m = 1$).

Now let us continue by using induction on n. Assume that Proposition is valid for all degrees up to $n-1$. Let us prove it in the case of the degree n. We may suppose that $n \geq 7$. It suffices to prove that there is a pair of ℓ-adjacent maximal lines: λ', λ''. Indeed, in this case we can complete the proof just as in the above case $n = 5$.

Suppose by way of contradiction that there is no pair of ℓ-adjacent maximal lines. Also we have that there is no ℓ-disjoint maximal line. Therefore we have that $\mu(\mathcal{X}) \leq k$. Now, by using (6.4.1), we get that $k \geq n-1$. Therefore $2 = k - \delta = 2k - n - 1 \geq 2n - 2 - n - 1 = n - 3$. This implies that $n - 3 \leq 2$, i.e., $n \leq 5$, which is a contradiction. $\square$

SUMMARY

VAHAGN VARDANYAN

ON THE USAGE OF LINES IN GC SETS

The following main results are obtained in the thesis:

- Necessary and sufficient conditions are provided for the factorization of fundamental polynomials of node sets in $\mathbb{R}^2$ of cardinality not exceeding $2n + [n/2] + 1$ as a product of factors of at most second degree.

- Independence of node sets with $3n+1$ nodes in $\mathbb{R}^3$. Necessary and sufficient conditions are provided for the factorization of fundamental polynomials of such node sets as a product of linear factors.

- A correction of a property on the usage of n-node lines in GC_n sets established by V. Bayramyan and H. Hakopian.

- An adjustment of the formulation of a conjecture proposed by V. Bayramyan and H. Hakopian on the usage of k-node lines in GC_n sets, $2 \leq k \leq n+2$, is provided and then proved. Namely, by assuming that the Gasca-Maeztu conjecture is true, we prove that any k-node line ℓ is not used at all, or it is used by exactly $\binom{s}{2}$ nodes, where s satisfies the condition $2k - n - 1 \leq s \leq k$.

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